ON EDGE'S CORRESPONDENCE ASSOCIATED WITH $\cdot 222$ COMPUTATIONAL DATA

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1. Introduction

This note explains the contents of the computational data for the paper [3], which is available as the text file `"compdataEdge.txt"` from the author's website

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/lattice.html`

These data were made by `GAP` [2]. We use the notation defined in [3].

2. Conventions

- Each vector of Λ_{24} is written with respect to the standard basis of $\mathbb{Z}^M$ (*not* with respect to the basis $\mathbf{b}_1, \dots, \mathbf{b}_{24}$ given in `BLambda`).
- Suppose that two finite sets S and T of the same cardinality N are expressed as lists `S` $= [a_1, \dots, a_N]$ and `T` $= [b_1, \dots, b_N]$, respectively. Then a bijection $F\colon S \to T$ is given by a list $[f(1), \dots, f(N)]$ of positive integers $\le N$, which indicates that $a_i^F = b_{f(i)}$ for $i = 1, \dots, N$. In particular, a permutation σ on S is given by a list $[s(1), \dots, s(N)]$, which indicates that $a_i^\sigma = a_{s(i)}$ for $i = 1, \dots, N$.
- A list is always sorted in the standard way.
- The group $\mathrm{P\Gamma U}(6,4)$ acts on $\mathbb{P}^5$ from the *right*. The orthogonal group $\mathrm{O}(L)$ of a lattice L acts on L from the *right*.
- The root $\omega \in \mathbb{F}_4$ of $x^2 + x + 1 = 0$ is written as `omega`.

3. The data

3.1. The Fermat cubic 4-fold X.

- `XF4` is the list $X(\mathbb{F}_4)$ of $\mathbb{F}_4$-rational points on X. Each point is expressed by a row vector of length 6 with respect to the homogeneous coordinates $(x_1 : \cdots : x_6)$ of $\mathbb{P}^5$. If an $\mathbb{F}_4$-rational point P of X appears at the ith position of `XF4`, then we put $\nu(P) := i$.
- `PX` is the list $\mathcal{P}_X$ of planes contained in X. Each plane $\Pi \in \mathcal{P}_X$ is expressed by the list consisting 21 positive integers $\nu(P)$, where P runs through $\Pi(\mathbb{F}_4)$.

3.2. The Leech lattice Λ_{24}.

- `C24` is the list of codewords of the extended binary Golay code $\mathcal{C}_{24}$. Each codeword is expressed by a subset of the set M of the positions $[1, \dots, 24]$ of MOG.
- `BLambda` is the matrix B_Λ in Figure 4.12 of [1], with the scalar multiplication $1/\sqrt{8}$ removed. The row vectors $\mathbf{b}_1, \dots, \mathbf{b}_{24}$ of `BLambda` considered as vectors of $\mathbb{Z}^M$ form a basis of Λ_{24}.
- `GramLeech` is the Gram matrix of Λ_{24} with respect to the basis $\mathbf{b}_1, \dots, \mathbf{b}_{24}$; that is, `GramLeech` is the symmetric matrix $(1/8)\, B_\Lambda \cdot {}^{\mathrm{T}} B_\Lambda$.
- `A` is the point $A = (0^{21}, 4, 0, -4)$.
- `B` is the point $B = (0^{21}, 0, 4, -4)$.
- `TAB` is the list $\mathcal{T}_{AB}$.

3.3. The group $\mathrm{P\Gamma U}(6, 4)$.

- `alpha` is the 6×6 matrix $\alpha \in \mathrm{GU}(6, 4)$.
- `beta` is the 6×6 matrix $\beta \in \mathrm{GU}(6, 4)$.
- `alphaXF4perm` is the permutation of $X(\mathbb{F}_4) =$ `XF4` induced by the action of α on X.
- `betaXF4perm` is the permutation of $X(\mathbb{F}_4) =$ `XF4` induced by the action of β on X.
- `gammaXF4perm` is the permutation of $X(\mathbb{F}_4) =$ `XF4` induced by the Frobenius action γ on X.
- `alphaPXperm` is the permutation of $\mathcal{P}_X =$ `PX` induced by the action of α on X.
- `betaPXperm` is the permutation of $\mathcal{P}_X =$ `PX` induced by the action of β on X.
- `gammaPXperm` is the permutation of $\mathcal{P}_X =$ `PX` induced by the Frobenius action γ on X.

3.4. The Edge correspondence ϕ_0.

- `Pi0` is the plane $\Pi_0 \in \mathcal{P}_X$, expressed in the same way as in `PX`.
- `Piinf` is the plane $\Pi_\infty \in \mathcal{P}_X$, expressed in the same way as in `PX`.
- `the21Pis` is the list $S = \{\Pi_1, \dots, \Pi_{21}\}$ of 21 planes $\Pi_s \in \mathcal{P}_X$ that satisfy $\dim(\Pi_0 \cap \Pi_s) = 1$ and $\dim(\Pi_\infty \cap \Pi_s) = -1$. Each member of `the21Pis` is expressed in the same way as in `PX`.
- `wt5wordsHX` is the list of codewords of weight 5 in $\mathcal{H}_X \subset 2^S$. A member $[i_1, \dots, i_5]$ of `wt5wordsHX` expresses the word consisting of the i_νth element of $S =$ `the21Pis` $(\nu = 1, \dots, 5)$.
- `T0` is the lattice point $T_0 \in \mathcal{T}_{AB}$.
- `Tinf` is the lattice point $T_\infty \in \mathcal{T}_{AB}$.
- `the21Ts` is the list M' consisting of $T \in \mathcal{T}_{AB}$ satisfying $\langle T_0, T \rangle = 0$ and $\langle T_\infty, T \rangle = 1$.
- `wt5wordsC21` is the list of codewords of weight 5 in $\mathcal{C}_{21} \subset 2^{M'}$. A member $[j_1, \dots, j_5]$ of `wt5wordsC21` expresses the word consisting of the j_νth element of $M' =$ `the21Ts` $(\nu = 1, \dots, 5)$.
- `varphi0` is the list expressing the bijection $\varphi_0 \colon S \xrightarrow{\sim} M'$ from $S =$ `the21Pis` to $M' =$ `the21Ts` that induces an isomorphism $\mathcal{H}_X \cong \mathcal{C}_{21}$ of binary codes.

- `phi0` is the list expressing the Edge correspondence $\phi_0 \colon (\mathcal{P}_X, \nu_{\mathcal{P}}) \xrightarrow{\sim} (\mathcal{T}_{AB}, \nu_{\mathcal{T}})$ from $\mathcal{P}_X =$ `PX` to $\mathcal{T}_{AB} =$ `TAB`.

Let $\phi_0' \colon \mathrm{P\Gamma U}(6,4) \to \mathrm{Aut}(\mathcal{T}_{AB}, \nu_{\mathcal{T}})$ denote the composite of $\rho_{\mathcal{P}} \colon \mathrm{P\Gamma U}(6,4) \to \mathrm{Aut}(\mathcal{P}_X, \nu_{\mathcal{P}})$ and the isomorphism $\mathrm{Aut}(\mathcal{P}_X, \nu_{\mathcal{P}}) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{T}_{AB}, \nu_{\mathcal{T}})$ induced by ϕ_0.

- `alphaTABperm` is the permutation of $\mathcal{T}_{AB} =$ `TAB` induced by $\phi_0'(\alpha)$.
- `betaTABperm` is the permutation of $\mathcal{T}_{AB} =$ `TAB` induced by $\phi_0'(\beta)$.
- `gammaTABperm` is the permutation of $\mathcal{T}_{AB} =$ `TAB` induced by $\phi_0'(\gamma)$.
- `alphatilde` is the matrix $\tilde{\alpha} = \Psi_0(\alpha) \in \circ 222_{AB}$.
- `betatilde` is the matrix $\tilde{\beta} = \Psi_0(\alpha) \in \circ 222_{AB}$.
- `gammatilde` is the matrix $\tilde{\gamma} = \Psi_0(\alpha) \in \circ 222_{AB}$.

References

[1] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, third edition, 1999.

[2] The GAP Group. GAP - Groups, Algorithms, and Programming. Version 4.7.9; 2015 (http://www.gap-system.org).

[3] Ichiro Shimada. On Edge's correspondence associated with ·222. `http://www.math.sci.hiroshima-u.ac.jp/~shimada/preprints`, 2017.

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