有理関数体上の楕円曲線の有理点について

岡山–広島代数学シンポジウム

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Problem

Give $\mathbb{C}(s)$-rational points of the following elliptic curve.

$$E : y^2 = x^3 - \frac{1}{4}\sqrt[3]{76771008 + 44330496\sqrt{3}}\ s^4 x + s^5(s^2+1)$$

Theorem (Kuwata & U.)

$$E(\mathbb{C}(s)) \simeq \mathbb{Z}S_1 \oplus \mathbb{Z}S_2$$

$$S_1 = \left[\frac{1}{144} \frac{1}{(-1+s)^2} \Big(\big(606569375222602654749016417 + 350202992066952759103881512\sqrt{3} \right.$$

$$+ 238575801875078451092408154\sqrt{3+2\sqrt{3}} + 137741803434707366958250154\sqrt{3}\sqrt{3+2\sqrt{3}} \big)^{1/3} \big(-2017 - 616\sqrt{3}$$

$$- 1250\sqrt{3+2\sqrt{3}} + 1422\sqrt{3}\sqrt{3+2\sqrt{3}} \big) \big(1 + 168\,s\sqrt{3} + 294\,s + s^6 + 168\,s^5\sqrt{3} + 294\,s^5 + 78960\,s^3\sqrt{3} + 136820\,s^3$$

$$+ 7392\sqrt{3}\,s^4 + 12831\,s^4 + 7392\sqrt{3}\,s^2 + 12831\,s^2 \big) \Big), -\frac{1}{(-1+s)^3} \Big(\frac{1}{1728} I \big(56\sqrt{3} - 97 \big) \big(1 + 252\,s\sqrt{3} + 440\,s + s^8$$

$$+ 252\sqrt{3}\,s^7 + 440\,s^7 + 47880\sqrt{3}\,s^6 + 82972\,s^6 - 294588\,s^5\sqrt{3} - 510136\,s^5 - 294588\,s^3\sqrt{3} - 510136\,s^3 - 26376336\sqrt{3}\,s^4$$

$$\left. - 45684794\,s^4 + 47880\sqrt{3}\,s^2 + 82972\,s^2 \big) (1+s) \Big) \right]$$

$$S_2 = \left[\frac{1}{144} \frac{1}{(1+s)^2} \Big(\big(606569375222602654749016417 + 350202992066952759103881512\sqrt{3} \right.$$

$$+ 238575801875078451092408154\sqrt{3+2\sqrt{3}} + 137741803434707366958250154\sqrt{3}\sqrt{3+2\sqrt{3}} \big)^{1/3} \big(-2017 - 616\sqrt{3}$$

$$- 1250\sqrt{3+2\sqrt{3}} + 1422\sqrt{3}\sqrt{3+2\sqrt{3}} \big) \big(168\,s^5\sqrt{3} - s^6 - 7392\sqrt{3}\,s^4 + 294\,s^5 + 78960\,s^3\sqrt{3} - 12831\,s^4 - 7392\sqrt{3}\,s^2$$

$$+ 136820\,s^3 + 168\,s\sqrt{3} - 12831\,s^2 + 294\,s - 1 \big) \Big), \frac{1}{1728} \frac{1}{(1+s)^3} \Big(\big(56\sqrt{3} - 97 \big) (-1+s) \big(252\sqrt{3}\,s^7 - s^8$$

$$- 47880\sqrt{3}\,s^6 + 440\,s^7 - 294588\,s^5\sqrt{3} - 82972\,s^6 + 26376336\sqrt{3}\,s^4 - 510136\,s^5 - 294588\,s^3\sqrt{3} + 45684794\,s^4$$

$$\left. - 47880\sqrt{3}\,s^2 - 510136\,s^3 + 252\,s\sqrt{3} - 82972\,s^2 + 440\,s - 1 \big) \Big) \right]$$

Example

$$\Big[\Big(\frac{1}{288} I \Big(606569375222602654749016417 + 350202992066952759103881512\sqrt{3} + 238575801875078451092408154\sqrt{3+2\sqrt{3}}$$
$$+ 137741803434707366958250154\sqrt{3}\sqrt{3+2\sqrt{3}}\Big)^{1/3}\Big(-2017 - 616\sqrt{3} - 1250\sqrt{3+2\sqrt{3}} + 1422\sqrt{3}\sqrt{3+2\sqrt{3}}\Big)\big(s^{12}$$
$$- 30095280 I s^3 - 12525304032 I s^7 - 94368\sqrt{3}\, s^{10} - 163590 s^{10} - 12525304032 I s^5 - 7231488048 I s^5\sqrt{3} + 646065792\sqrt{3}\, s^8$$
$$+ 1119019407 s^8 - 17375256 I s^9\sqrt{3} - 17375256 I s^3\sqrt{3} - 16530119616\sqrt{3}\, s^6 - 28631008532 s^6 + 144 I s + 72 I s\sqrt{3}$$
$$+ 646065792\sqrt{3}\, s^4 + 1119019407 s^4 + 144 I s^{11} + 72 I s^{11}\sqrt{3} - 94368\sqrt{3}\, s^2 - 163590 s^2 - 7231488048 I s^7\sqrt{3}$$
$$- 30095280 I s^9 + 1\big)\Big)\Big/\big(48 I s^3\sqrt{3} + 76 I s^3 - s^4 + 48 I s\sqrt{3} - 216\sqrt{3}\, s^2 + 76 I s - 366 s^2 - 1\big)^2, -\frac{1}{6912}\Big(\Big(56 I\sqrt{3}$$
$$- 97 - 97 I + 56\sqrt{3}\Big)\Big(1 + 16605906912\sqrt{3}\, s^4 - 149328\sqrt{3}\, s^2 - 6488174914992\sqrt{3}\, s^6 + s^{16} - 149328\sqrt{3}\, s^{14}$$
$$+ 16605906912 s^{12}\sqrt{3} + 2147762616984 I s^{11} - 1035416728522296 I s^9 - 1035416728522296 I s^7 + 2147762616984 I s^5$$
$$- 77491512 I s^3 + 216 I s + 216 I s^{15} - 77491512 I s^{13} - 258992 s^{14} + 28762282252 s^{12} - 11237848589072 s^{10}$$
$$- 6488174914992\sqrt{3}\, s^{10} - 3121650593998272\sqrt{3}\, s^8 - 11237848589072 s^6 + 28762282252 s^4 - 258992 s^2$$
$$- 5406857432335514 s^8 + 1240011318348 I s^{11}\sqrt{3} - 597798126976284 I s^9\sqrt{3} - 597798126976284 I s^7\sqrt{3}$$
$$+ 1240011318348 I s^5\sqrt{3} - 44738460 I s^3\sqrt{3} + 108 I s\sqrt{3} + 108 I\sqrt{3}\, s^{15} - 44738460 I\sqrt{3}\, s^{13}\Big)(1+s)(-1+s)\Big)\Big/$$
$$\big(48 I s^3\sqrt{3} + 76 I s^3 - s^4 + 48 I s\sqrt{3} - 216\sqrt{3}\, s^2 + 76 I s - 366 s^2 - 1\big)^3\Big]$$

$$= S_1 + S_2$$

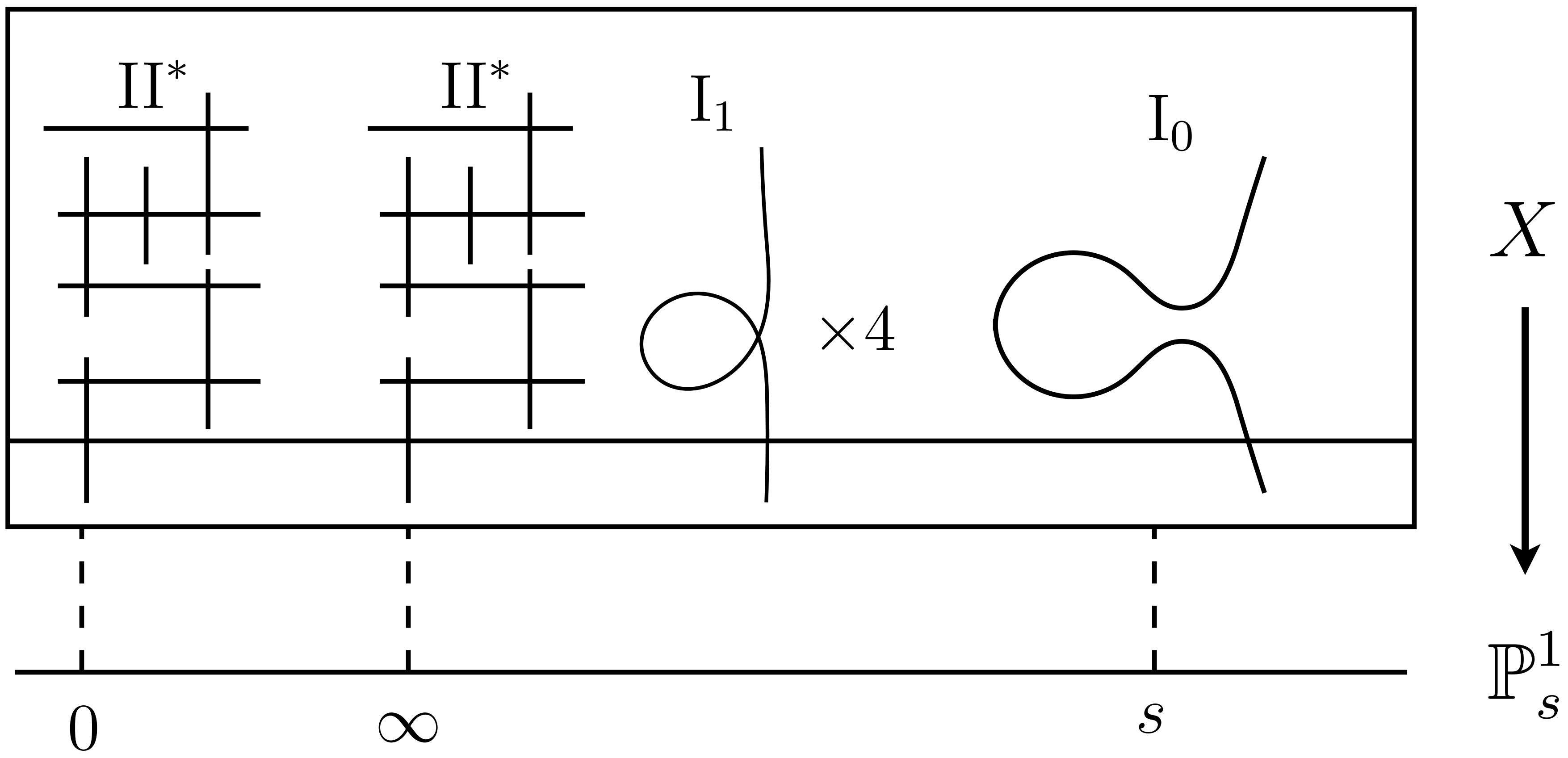
§1 What?

§2 How?

§1 What?

X : singular $K3$ surface with $T_X \simeq \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

$X \to \mathbb{P}^1$: Jacobian fibration with $\mathrm{II}^* \times 2$



§1 What?

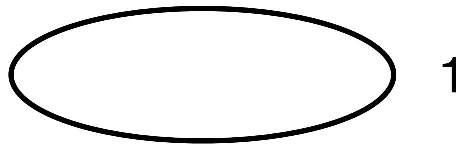
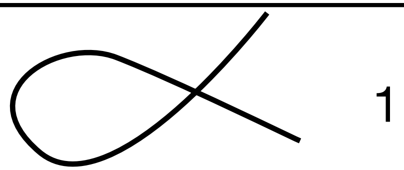
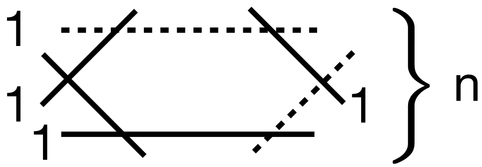
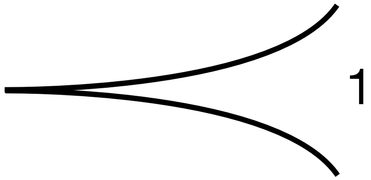
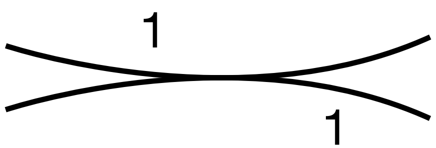
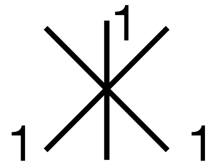
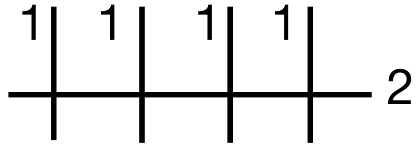
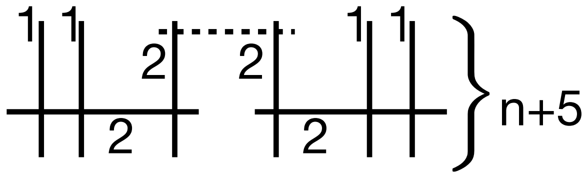
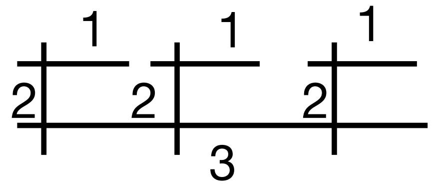
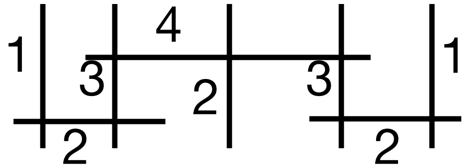
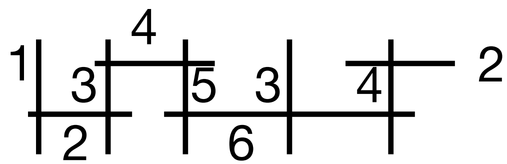
$$X : \text{singular } K3 \text{ surface with } T_X \simeq \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$X \to \mathbb{P}^1 : \text{Jacobian fibration with } \mathrm{II}^* \times 2$$

is given by the following Weierstrass equation.

$$y^2 = x^3 - \frac{1}{4}\sqrt[3]{76771008 + 44330496\sqrt{3}}\, s^4 x + s^5(s^2+1)$$

Singular fibers (Kodaira's classification)

Type	Arrangement (with multiplicity)
I_0	
I_1	
I_n	
II	
III	
IV	
I_0^*	
I_n^*	
IV^*	
III^*	
II^*	

Example (Elliptic K3 surface with $2\mathrm{II}^*$)

$$X \longrightarrow \mathbb{P}^1$$

$$(x, y, s) \mapsto s$$

Elliptic parameter

$$y^2 = x^3 - 3\alpha s^4 x + s^5(s^2 + s - 2\beta), \quad \alpha, \beta \in \mathbb{C}$$

Weierstrass equation

$$\left(y^2 = x^3 - 3\alpha x + s + \frac{1}{s} - 2\beta \right)$$

Properties

$$U \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

X : $K3$ surface

$$H^2(X,\mathbb{Z}) \simeq E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$$

$$\cup$$

$$NS(X) := \{\text{divisors}\}/ \overset{\text{alg. equiv.}}{\approx}$$

Néron-Severi lattice

$$\simeq \{\text{divisors}\}/ \overset{\text{lin. equiv.}}{\sim} = \mathrm{Pic}(X)$$

$$T_X := NS(X)^{\perp} \subset H^2(X,\mathbb{Z})$$

Transcendental lattice

algebraic

X : $K3$ surface

$\rho(X)$: Picard number

$\|$

$$1 \leq \mathrm{rk}\ NS(X) \leq 20$$

singular $K3$ surface $\Leftrightarrow \mathrm{rk}\ T_X = 2$

singular $K3$ surface

<u>Theorem (T. Shioda & H. Inose)</u>

$$Q_1 \sim Q_2 \Leftrightarrow \exists \gamma \in \mathrm{SL}_2(\mathbb{Z}), Q_1 = {}^t\gamma Q_2 \gamma$$

$$\{\text{singular } K3 \text{ surface}\} \overset{1:1}{\longleftrightarrow} \mathcal{Q}/\mathrm{SL}_2(\mathbb{Z})$$

$$X_{[a,b,c]} := X \quad \mapsto \quad T_X$$

$$\mathcal{Q} := \left\{ \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \mid a, b, c \in \mathbb{Z}, a, c > 0, b^2 - 4ac < 0 \right\}$$

(positive definite integral even lattice)

Theorem (D. Morrison)

$\forall$singular $K3$ surface admits a Shioda-Inose structure

$(\rho(X) \geq 19 \Rightarrow)$

Definition

A $K3$ surface X admits a **Shioda-Inose structure**.

$\overset{\text{def}}{\Leftrightarrow}$ $\exists \iota \in \mathrm{Aut}(X)$: symplectic involution

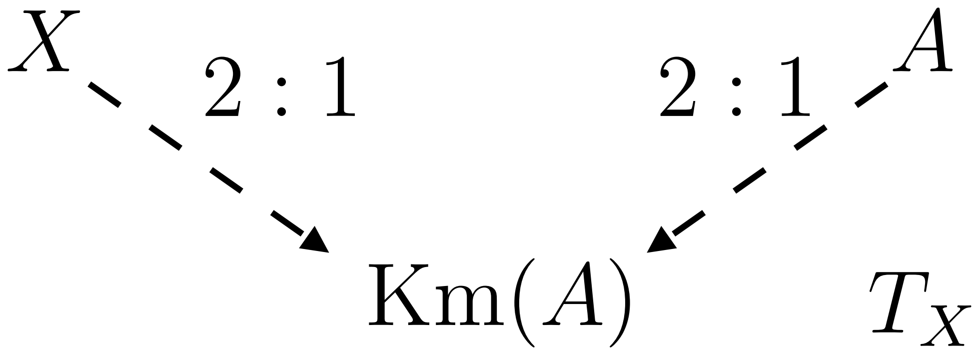
with rational quotient map $\pi : X \to \widetilde{X/\iota} \simeq \mathrm{Km}(A)$

&

$\pi_* : H^2(X) \to H^2(\mathrm{Km}(A))$ induces a Hodge isometry

$$T_X(2) \simeq T_{\mathrm{Km}(A)}$$

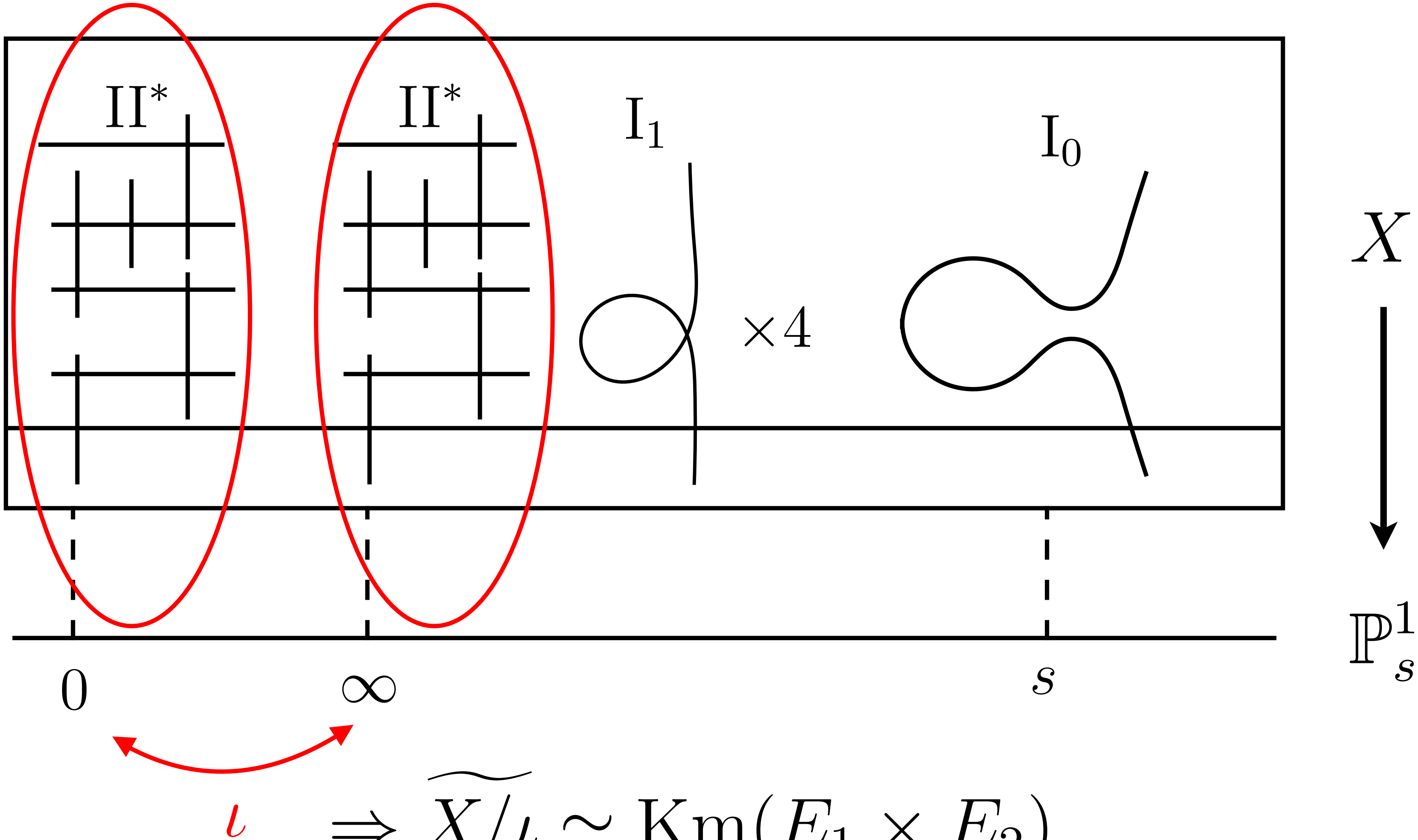
Shioda-Inose structure



$$T_X \simeq T_A$$

X is a double cover of a Kummer surface

Symplectic involution on elliptic K3 surface with $2\mathrm{II}^*$



$$\Rightarrow \widetilde{X/\iota} \simeq \mathrm{Km}(\underline{E_1 \times E_2})$$

product type

Elliptic K3 surface with $2\mathrm{II}^*$

$$X \xrightarrow{2:1} \mathrm{Km}(E_1 \times E_2)$$

$$X \ : \ y^2 = x^3 - 3\alpha s^4 x + s^5(s^2 + s - 2\beta)$$

$$\alpha = \sqrt[3]{J(E_1)J(E_2)}, \ \beta = \sqrt{(1 - J(E_1))(1 - J(E_2))}$$

$$(J(E_i) := j(E_i)/1728)$$

Elliptic K3 surface with $2\mathrm{II}^*$

The case of $X_{[a,b,c]}$ $\left(\text{singular } K3 \text{ surface with } T_X \simeq \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\right)$

$$E_i \simeq \mathbb{C}/\mathbb{Z} + \tau_i\mathbb{Z}$$

$$\tau_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \ \tau_2 = \frac{b + \sqrt{b^2 - 4ac}}{2}$$

X : singular $K3$ surface with $T_X \simeq \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

$\|$

$$X_{[3,0,3]} \quad \Rightarrow \quad E_1 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$$

$$E_2 = \mathbb{C}/\mathbb{Z} + 3i\mathbb{Z}$$

§2 How?

Based on

Some Shioda's papers :

- "Correspondence of elliptic curves and Mordell- Weil lattices of certain elliptic K3 surfaces"
- "Kummer sandwich theorem of certain elliptic K3 surfaces"
- …

A. Kumar & M. Kuwata :

- "Elliptic K3 surfaces associated with the product of two elliptic curves: Mordell-Weil lattices and their fields of definition", arXiv:1409.2931.

Kummer sandwich theorem (T. Shioda)

X : Elliptic $K3$ surface with $\mathrm{II}^* \times 2$

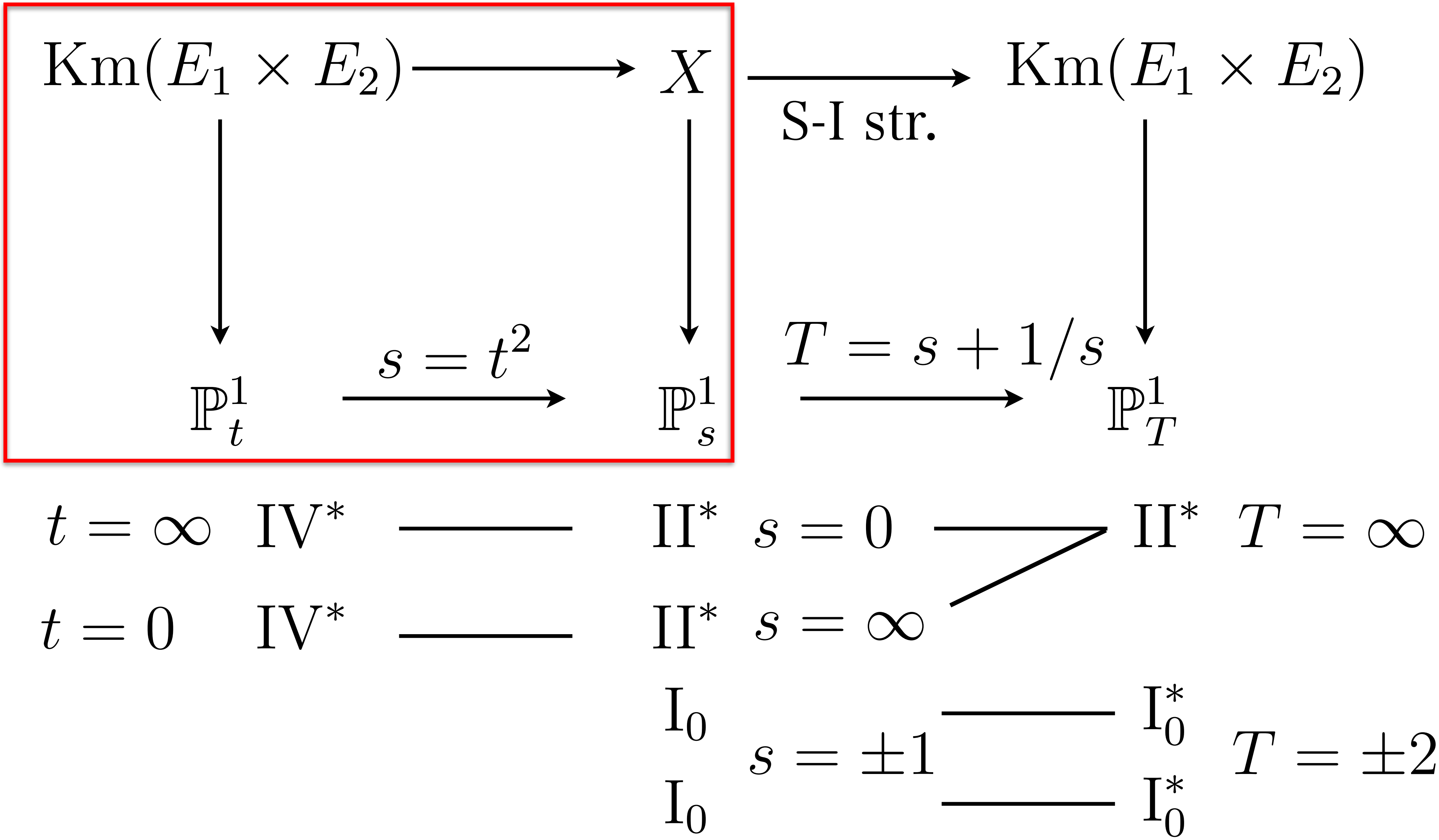
$\exists \sigma,\ \tau \in \mathrm{Aut}(X)$: symplectic involutions

$\exists E_1, E_2$: elliptic curves

$$X/\langle \sigma \rangle \sim \mathrm{Km}(E_1 \times E_2)$$

$$X/\langle \sigma, \tau \rangle \sim X$$

Kummer sandwich theorem (T. Shioda)



$$F^{(2)}(E_1, E_2) : Y^2 = X^3 - 3\alpha t^4 X + t^4(t^4 - 2\beta t^2 + 1)$$

$$\begin{array}{ccc} \mathrm{Km}(E_1 \times E_2) & \longrightarrow & \mathbb{P}^1_t \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{P}^1_s \end{array}$$

$$s = t^2$$

$$Y = y/t^3$$

$$X = x/t^2$$

$$F^{(1)}(E_1, E_2) : \ y^2 = x^3 - 3\alpha s^4 x + s^5(s^2 - 2\beta s + 1)$$

$$F^{(2)}(E_1, E_2) : Y^2 = X^3 - 3\alpha t^4 X + t^4(t^4 - 2\beta t^2 + 1)$$

$$F^{(2)}(\mathbb{C}(t)) \simeq \mathrm{Hom}(E_1, E_2) \oplus (\mathbb{Z}/2\mathbb{Z})^2$$

$$\cup$$

$$F^{(1)}(\mathbb{C}(s)) \simeq \mathrm{Hom}(E_1, E_2)[2]$$

$$F^{(1)}(E_1, E_2) : \ y^2 = x^3 - 3\alpha s^4 x + s^5(s^2 - 2\beta s + 1)$$

$$\mathrm{rk}\ \mathrm{Hom}(E_1, E_2) = \begin{cases} 0 & E_1 \overset{\text{isog.}}{\not\sim} E_2 \\ 1 & E_1 \overset{\text{isog.}}{\sim} E_2 \\ 2 & E_1 \overset{\text{isog.}}{\sim} E_2 : \text{CM type} \end{cases}$$

X : singular $K3$ surface

$$X : \text{singular } K3 \text{ surface with } T_X \simeq \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$\parallel$$

$$X_{[3,0,3]} \quad \Rightarrow \quad E_1 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$$

$$E_2 = \mathbb{C}/\mathbb{Z} + 3i\mathbb{Z}$$

$$X : \text{singular } K3 \text{ surface with } T_X \simeq \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$\|$$

$$X_{[3,0,3]} \Rightarrow E_1 = \mathbb{C}/\mathbb{Z} + 3i\mathbb{Z}$$

$$j(E_2) = 76771008 + 44330496\sqrt{3}$$

$$E_1 : y_1^2 = x_1(x_1 - 1)(x_1 - \lambda_1)$$

$$\left(\lambda_1 = -193 - 112\sqrt{3} - 44\sqrt{9 + 6\sqrt{3}} - \frac{76}{\sqrt{3}}\sqrt{9 + 6\sqrt{3}}\right)$$

$$E_2 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$$

$$j(E_2) = 1728$$

$$E_2 : y_2^2 = x_2(x_2 - 1)(x_2 + 1)$$

$$\mathrm{Hom}(E_1, E_2) = \langle \Phi_1, \Phi_2 \rangle$$

$$
\begin{aligned}
\Phi_1 = \Big\{ x_2 &= \Big(9\big(-3 - 2\sqrt{3} + 2\sqrt{-9+6\sqrt{3}} + 2\sqrt{-9+6\sqrt{3}}\,\sqrt{3}\big)\,x_1\,\big(30 + 18\sqrt{3} + 26\sqrt{-9+6\sqrt{3}} \\
&\quad + 15\sqrt{-9+6\sqrt{3}}\,\sqrt{3} + x_1\big)^2\Big) \Big/ \big(18 + 6\sqrt{3} + 12\sqrt{-9+6\sqrt{3}} + 7\sqrt{-9+6\sqrt{3}}\,\sqrt{3} - 9x_1\big)^2,\ y_2 \\
&= -\Big(9\,I\,\big(-3\sqrt{3} - 18 + 11\sqrt{-9+6\sqrt{3}} + 5\sqrt{-9+6\sqrt{3}}\,\sqrt{3}\big)\,y_1\,\big(3x_1 + 3\sqrt{3} + 3 \\
&\quad + 3\sqrt{-9+6\sqrt{3}} + 2\sqrt{-9+6\sqrt{3}}\,\sqrt{3}\big)\big(30 + 18\sqrt{3} + 26\sqrt{-9+6\sqrt{3}} + 15\sqrt{-9+6\sqrt{3}}\,\sqrt{3} \\
&\quad + x_1\big)\big(37 - x_1 + 18\sqrt{-9+6\sqrt{3}}\,\sqrt{3} + 21\sqrt{3} + 31\sqrt{-9+6\sqrt{3}}\big)\Big) \Big/ \big(18 + 6\sqrt{3} \\
&\quad + 12\sqrt{-9+6\sqrt{3}} + 7\sqrt{-9+6\sqrt{3}}\,\sqrt{3} - 9x_1\big)^3 \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Phi_2 = \Big\{ x_2 &= -\Big(9\big(-3 - 2\sqrt{3} + 2\sqrt{-9+6\sqrt{3}} + 2\sqrt{-9+6\sqrt{3}}\,\sqrt{3}\big)\,x_1\,\big(30 + 18\sqrt{3} + 26\sqrt{-9+6\sqrt{3}} \\
&\quad + 15\sqrt{-9+6\sqrt{3}}\,\sqrt{3} + x_1\big)^2\Big) \Big/ \big(18 + 6\sqrt{3} + 12\sqrt{-9+6\sqrt{3}} + 7\sqrt{-9+6\sqrt{3}}\,\sqrt{3} - 9x_1\big)^2,\ y_2 \\
&= \Big(9\big(-3\sqrt{3} - 18 + 11\sqrt{-9+6\sqrt{3}} + 5\sqrt{-9+6\sqrt{3}}\,\sqrt{3}\big)\,y_1\,\big(3x_1 + 3\sqrt{3} + 3 \\
&\quad + 3\sqrt{-9+6\sqrt{3}} + 2\sqrt{-9+6\sqrt{3}}\,\sqrt{3}\big)\big(30 + 18\sqrt{3} + 26\sqrt{-9+6\sqrt{3}} + 15\sqrt{-9+6\sqrt{3}}\,\sqrt{3} \\
&\quad + x_1\big)\big(37 - x_1 + 18\sqrt{-9+6\sqrt{3}}\,\sqrt{3} + 21\sqrt{3} + 31\sqrt{-9+6\sqrt{3}}\big)\Big) \Big/ \big(18 + 6\sqrt{3} \\
&\quad + 12\sqrt{-9+6\sqrt{3}} + 7\sqrt{-9+6\sqrt{3}}\,\sqrt{3} - 9x_1\big)^3 \Big\}
\end{aligned}
$$

$$\Phi_1 : E_1 \xrightarrow{3:1} E_2 \qquad \Phi_2 : E_1 \xrightarrow{\Phi_1} E_2 \xrightarrow{\times i} E_2$$

singular affine model of $\mathrm{Km}(E_1 \times E_2)$

$$x_2(x_2-1)(x_2+1) = t^2 x_1(x_1-1)(x_1-\lambda_1),\ t = \frac{y_2}{y_1}$$

$$F^{(2)} : Y^2 = X^3 - \frac{1}{4}\sqrt[3]{\underbrace{76771008 + 44330496\sqrt{3}}_{=j(E_1)}}\, t^4 X + t^4(t^4+1)$$

$$\begin{pmatrix} Y^2 = X^3 - 3\alpha t^4 X + t^4(t^4 - 2\beta t^2 + 1) \\ \alpha = \sqrt[3]{J(E_1)J(E_2)},\ \beta = \sqrt{(1-J(E_1))(1-J(E_2))} \end{pmatrix}$$

How to find rational points of $F^{(2)}(E_1, E_2)$

$$F^{(2)}(\mathbb{C}(t)) \simeq NS(X)/T(X)$$

$$T(X) = \langle \text{ irr. comps. of red. fibs., zero-section, gen. fib.} \rangle$$

$$\simeq E_8(-1) \oplus E_8(-1) \oplus U \quad \text{: Trivial lattice}$$

$$NS(X) \to F^{(2)}(\mathbb{C}(t))$$

$$D \mapsto sum(\underline{D|_{F^{(2)}}})$$

$$\in F^{(2)}(\overline{\mathbb{C}(t)})$$

$$\varphi \in \mathrm{Hom}(E_1, E_2)$$

$$\varphi : (x_1, y_1) \mapsto (x_2, y_2) = (\varphi_x(x_1), \varphi_y(x_1)y_1)$$

$$\begin{cases} x_2(x_2 - 1)(x_2 - \lambda_2) = t^2 x_1(x_1 - 1)(x_1 - \lambda_1) \\ x_2 = \varphi_x(x_1) \end{cases}$$

gives the divisor on $\mathrm{Km}(E_1 \times E_2)$.

$$x_2(x_2 - 1)(x_2 - \lambda_2) = t^2 x_1(x_1 - 1)(x_1 - \lambda_1)$$

$$\longrightarrow (\varphi_y(x_1) - t)(\varphi_y(x_1) + t)x_1(x_1 - 1)(x_1 - \lambda_1) = 0$$

$$(\varphi_y(x_1)-t)(\varphi_y(x_1)+t)x_1(x_1-1)(x_1-\lambda_1)=0$$

<u>Proposition</u>

$$D_\varphi^\pm \in \mathrm{div}(\mathrm{Km}(E_1\times E_2)) \text{ is defined by } \varphi_y(x_1)=\pm t.$$

$$(1)\ \ D_\varphi^\pm = Q_1^\pm + \cdots Q_r^\pm : \text{ irr. decomp. } /\overline{\mathbb{C}(t)}$$

$$\Rightarrow P_\varphi^\pm := \sum Q_i^\pm \in F^{(2)}(\mathbb{C}(t))$$

$$(2)\ \ P_\varphi^+ - P_\varphi^- \in \mathrm{Im}\left(F^{(1)}(\mathbb{C}(s)) \to F^{(2)}(\mathbb{C}(t))\right)$$

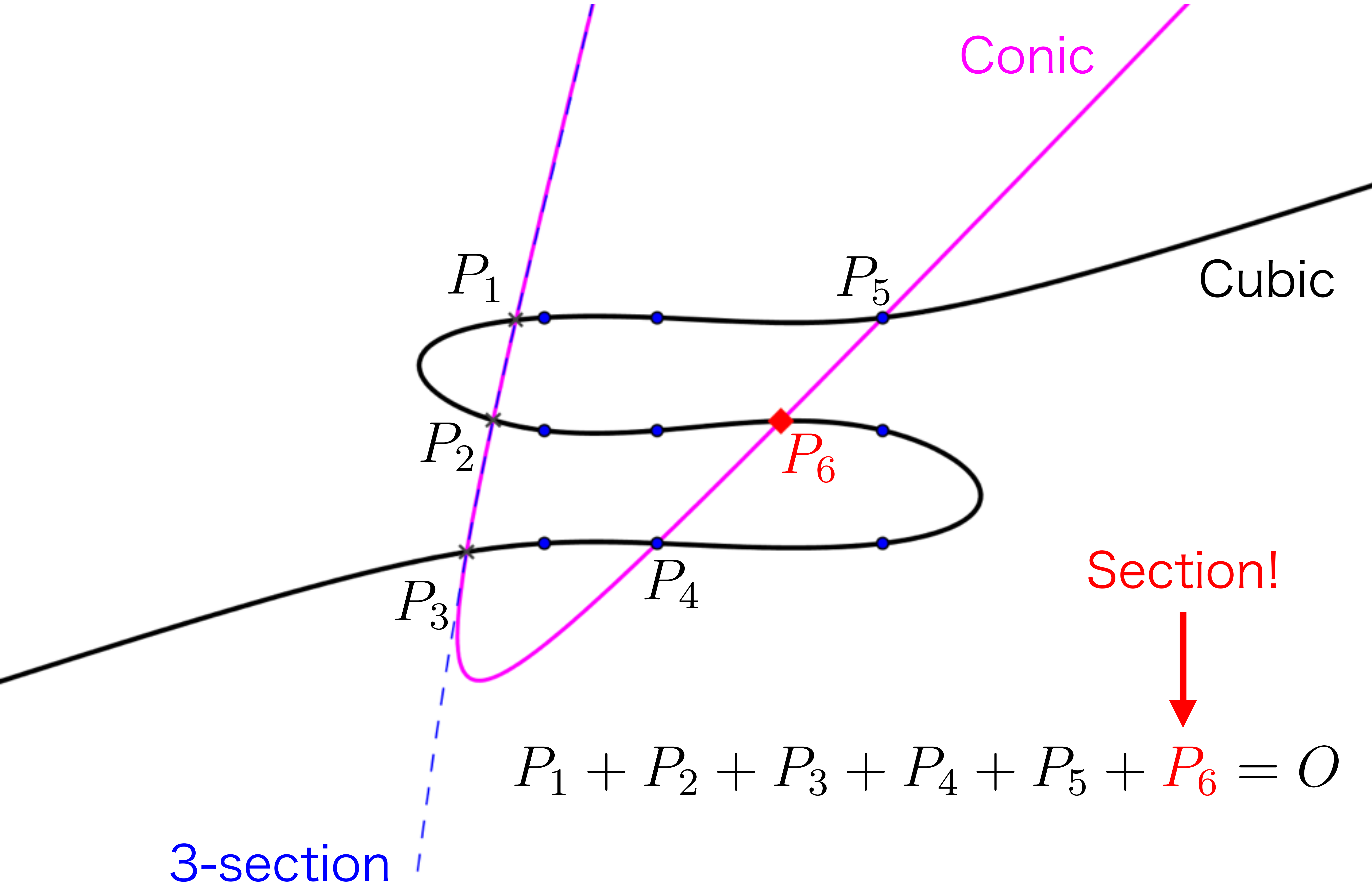
For our $\Phi_1, \Phi_2 \cdots$

- $Q_i^{\pm}$: solution of cubic equation over $\mathbb{C}(t)$
- $\sum Q_i^{\pm}$: hard to compute

$D_{\Phi_i}^{\pm}$: 3-sections on $F^{(2)}(\mathbb{C}(t))$ intersect with

$$x_2(x_2-1)(x_2-\lambda_2) = t^2 x_1(x_1-1)(x_1-\lambda_1)$$

at 3 points (over $\mathbb{C}(t)$).



Conic
Cubic
P_1
P_5
P_2
P_6
P_3
P_4
Section!
$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = O$
3-section

How to get such a conic

Case of $D^+_{\Phi_1}$

- $p(x_1) :=$ numerator of $\Phi_{1y}(x_1) - t$ $(\deg p(x_1) = 3)$
- $x_2 = \Phi_{1x}(x_1) =: ax_1^2 + bx_1 + c \in \mathbb{C}(t)[x_1]/(p(x_1))$
- give a matrix A s.t.

$$\begin{pmatrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_1^2 \end{pmatrix} A$$

- for $\ker A =: \langle v_1, v_2, v_3 \rangle$ give q_1, q_2, q_3 s.t.

$$(q_1v_1 + q_2v_2 + q_3v_3) \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1x_2 \\ x_1^2 \\ x_2^2 \end{pmatrix} = 0 \quad \text{passes } (0,0), (0,-1).$$

$$\mathrm{Resultant}(\mathrm{Cubic}, \mathrm{Conic}, x_2) = (x_1 - \alpha) p(x_1) x_1^2 q(t)$$