On the profinite regular inverse Galois problem

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Abstract

Given a field k and a (pro)finite group G, consider the following weak version of the regular inverse Galois problem: (WRIGP$/G/k$) *there exists a smooth geometrically irreducible curve X_G/k and a Galois extension $E/k(X_G)$ regular over k with group G.* (the regular inverse Galois problem (RIGP$/G/k$) corresponding to the case $X_G = \mathbb{P}^1_k$). A standard descent argument shows that for a finite group G the (WRIGP$/G/k$) can be deduced from the (RIGP$/G/k((T))$). For profinite groups G, the (WRIGP$/G/k((T))$) has been proved for lots of fields (including the cyclotomic closure of characteristic 0 fields) but the descent argument no longer works.

Let $p \geq 2$ be a prime, then a profinite group G is said to be *p-obstructed* if it fits in a profinite group extension

$$1 \to K \to G \to G_0 \to 1$$

with G_0 a finite group and $K \twoheadrightarrow \mathbb{Z}_p$. Typical examples of such profinite groups G are universal p-Frattini covers of finite p-perfect groups or pronilpotent projective groups.

I will show that the (WRIGP$/G/k$) - even under its weaker formulation: (WWRIGP$/G/k$) *there exists a smooth geometrically irreducible curve X_G/k and a Galois extension $E/k(X_G).\overline{k}$ with group G and field of moduli k.* - fails for the whole class of p-obstructed profinite groups G and any field k which is either a finitely generated field of characteristic 0 or a finite field of characteristic $\neq p$.

The proof uses a profinite generalization of the cohomological obstruction for a G-cover to be defined over its field of moduli and an analysis of the constrainsts imposed on a smooth geometrically irreducible curve X by a degree p^n cyclic G-cover $X_n \to X$, constrainsts which are too rigid to allow the existence of projective systems $(X_n \to X_G)_{n\geq 0}$ of degree p^n cyclic G-covers defined over k. I will also discuss other implicsations of these constrainsts for the (RIGP).