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History-dependent phase slips and rectification of a few coupled oscillators system

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Abstract

History dependence of the phase slippage is investigated using coupled oscillator systems containing two and three degrees of freedom with repulsive interaction. In some situations, these systems possess two values of the minimal external force needed to realize phase slip, with that actually realized being determined by the direction of the slippage relative to the direction of the previous slippage. In particular, the minimal external force for the phase slip is smaller when the previous slippage was in the same direction than when it was in the opposite direction. Owing to this property, a particle can continue to move in a given direction even in the case that it is subject to an external force whose direction changes in time as in a ratchet system which is regarded as one of the simplest model of molecular motors.

Keywords: Phase slips; Oscillator system; Simulation

1. Introduction

Many systems in physics, chemistry, and biology can be described as populations of coupled oscillators [1,2]. Examples are charge-density waves [3], Josephson junction arrays [4], some chemical reaction systems [1], circadian rhythms, heart beat generation [2], and so on. Therefore, understanding the cooperative dynamical properties of such coupled oscillators systems is of considerable theoretical and experimental interest. Recently, some remarkable features of such oscillator systems, collective synchronization, phase lock, phase slip and so on, have been extensively studied [1–7].

In this paper, we investigate the existence and characteristics of history dependence of the phase slip in coupled oscillator systems. First, a coupled oscillator system containing only two degrees of freedom with repulsive interaction and an external force is studied. In this system, the minimum value of the external force necessary to start the phase slippage depends on the direction of the previous slippage. We show that this property can cause the system to exhibit rectification behavior (Sections 2 and 3). We also show that similar phenomena are observed in a similar system with three degrees of freedom (Section 4).

2. Two oscillators model and simulation

We study a system containing two over-damped pendulums with repulsive interactions in which both pendulums are subject to a uniform field and one is subject to an external force F_{ex} (Fig. 1). The mo-

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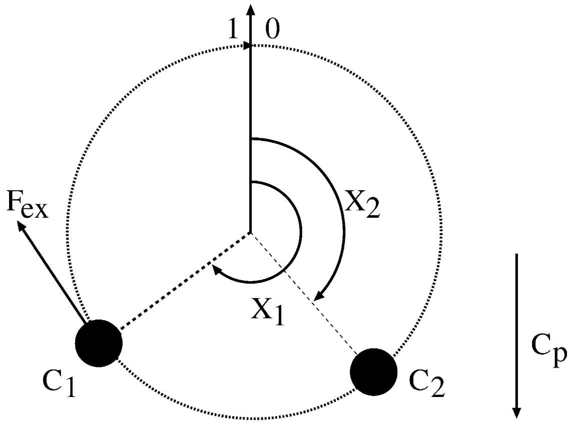

Fig. 1. Schematic depiction of the system described by (2.1) and (2.2), consisting of two pendulums in a uniform field.

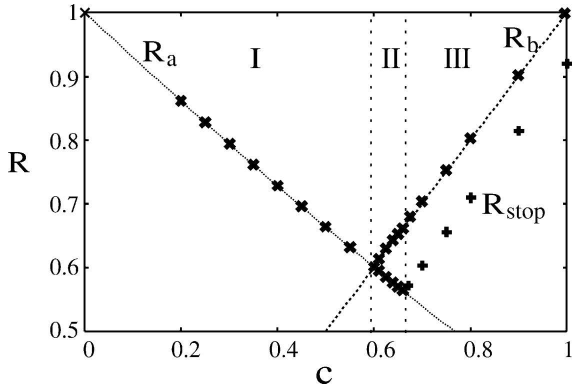

Fig. 2. The (✖) points and dotted lines represent the critical force R as a function of c for two conditions, $x_1^b > x_2^b$ and $x_1^b < x_2^b$, before slippage. The (✖) points are the results of our simulation, and dotted lines were obtained analytically. The (✚) points represent R_{stop} as a function of c in case III.

tion of these two oscillators' (pendulums') phase are described by the following equations:

$$\dot{x}_1 = c_1c_2 \sin(2\pi(x_1 - x_2)) + c_1c_p \sin(2\pi x_1) + F_{\text{ex}}, \tag{2.1}$$

$$\dot{x}_2 = c_1c_2 \sin(2\pi(x_2 - x_1)) + c_2c_p \sin(2\pi x_2), \tag{2.2}$$

where x_i is the ith oscillator's phase ($x_i \ldots \bmod 1$) and $c_i > 0$.

In the following, we report the results of simulations employing the above system. The purpose is to determine the relationships between phase slip behaviors and observed microscopic states. We set $c_1 = c_p = 1$ and $c_2 = c$, and use $F_{\text{ex}} = (1 - \cos(t/T))/2$ with large T so that the external force varies smoothly and very slowly. (The initial value of F_{ex} is 0.) The data points (✖) in Fig. 2 indicate values of the critical force R, plotted as a function of c ($0 \le c \le 1$) for two cases, $x_1^b > x_2^b$ and $x_1^b < x_2^b$. Here, the critical force R is defined as the minimum value of $|F_{\text{ex}}|$ for which the first oscillator starts the slippage as $|F_{\text{ex}}|$ increases, and x_i^b is the ith oscillator's phase at the most recent time at which $F_{\text{ex}} = 0$. Slippage is here defined as the state where the oscillator repeatedly crosses the other oscillator or the potential barrier formed by the uniform field. This c–R relationship is divided into three regions: (I) for $c < \hat{c}_{\text{crit}^1} \approx 0.59$, R is a decreasing function of c for all x_1^b, x_2^b, (II) for $\hat{c}_{\text{crit}^1} < c < \hat{c}_{\text{crit}^2} \approx 0.66$, R has two possible values, one realized for $x_1^b > x_2^b$ and one for $x_1^b < x_2^b$, and (III) for $c > \hat{c}_{\text{crit}^2}$, R is an increasing function of c for all x_1^b, x_2^b.

When $|F_{\text{ex}}(t)|$ decreases from some initial value greater than R, the first oscillator ceases slipping at a value of $|F_{\text{ex}}|$. Through our simulations, we found relations between such $|F_{\text{ex}}|$ and R for each of the above described cases: The slippage is stopped at $|F_{\text{ex}}| < R$ in case I, at $|F_{\text{ex}}| < R_{\text{stop}} < R$ in case III as shown in Fig. 2, and at $|F_{\text{ex}}| < R_{\text{smaller}}$ in case II, where R_{smaller} is the smaller of the two values of R.

Fig. 3 displays typical temporal evolutions of each oscillator's velocity and phase for cases I and III, with F_{ex} as given above. Here, in (a) $c = 0.2$ and $x_1^b < x_2^b$, in (b) $c = 0.2$ and $x_1^b > x_2^b$, in (c) $c = 0.8$ and $x_1^b < x_2^b$, and in (d) $c = 0.8$ and $x_1^b > x_2^b$. In this figure, the gray curves represent phase and velocity of the first oscillator, and the black curves represent those of the second. In the situation depicted in Fig. 3(a) and (b), $x_1 > x_2$ always holds just before the slippage of the first oscillator, independently of x_1^b and x_2^b. Contrastingly, in Figs. 3(c) and (d), $x_1 < x_2$ always holds just before the slippage of the first oscillator, independently of x_1^b and x_2^b. Due to these facts, R takes only a single value for each c value in cases I and III.

In case II, the critical force R is determined by the history. It depends on the direction of the first oscillator's previous slip. This can be understood as

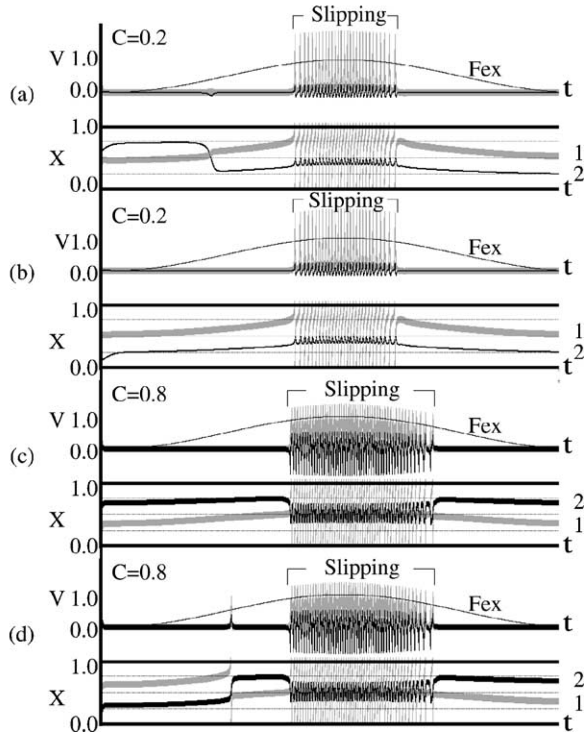

Fig. 3. Typical temporal evolutions of the velocity and phase of each oscillator for case I in (a) and (b) and case III in (c) and (d) with slowly changing of F_{ex}. In (a) $c = 0.2$ and $x_1^b < x_2^b$, in (b) $c = 0.2$ and $x_1^b > x_2^b$, in (c) $c = 0.8$ and $x_1^b < x_2^b$, and in (d) $c = 0.8$ and $x_1^b > x_2^b$. The gray curves represent the first oscillator, and the black curves represent the second oscillator. The thickness of each curve is proportional to the value of c_i for the oscillator to which it corresponds. The thin sinusoidal curve represents F_{ex}. The numbers at the right of the figures are the oscillator label.

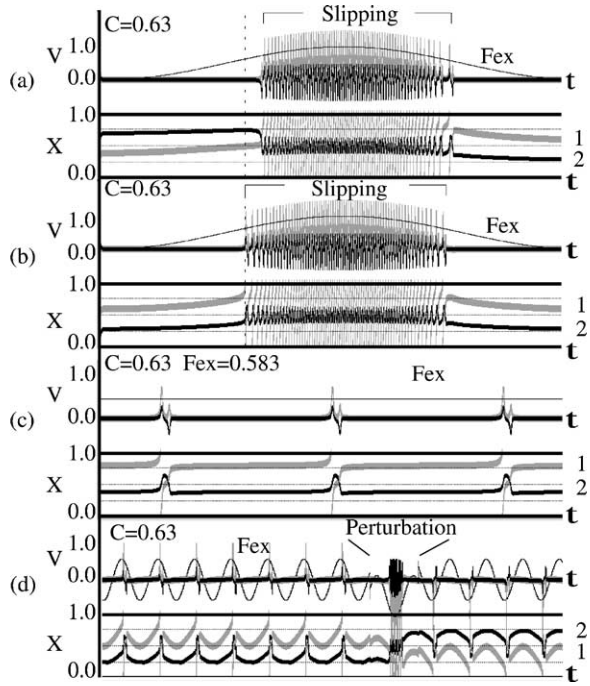

Fig. 4. Typical temporal evolutions of the velocity and phase of each oscillator for case II with a slowly changing F_{ex} in (a) and (b), a static F_{ex} with the value 0.583 which is just larger than $R_{smaller}$ in (c), and a slowly changing $F_{ex} = 0.61 \sin(t/T)$ and a perturbation in (d). In (a) $c = 0.63$ and $x_1^b < x_2^b$, and in (b), (c) and (d) $c = 0.63$ and $x_1^b > x_2^b$. Shades and widths of the curves, and the numbers at the right of the figures have the same meanings as in Fig. 3.

follows. Fig. 4 displays typical temporal evolutions of each oscillator's velocity and phase for case II. Here, $c = 0.63$ and in (a) $x_1^b < x_2^b$ and in (b) $x_1^b > x_2^b$. In contrast to cases I and III, the arrangement of oscillators is kept fixed as F_{ex} is increased until the first oscillator starts to slip. We also found that slippage begins at a later time for $x_1^b < x_2^b$ than for $x_1^b > x_2^b$. This means that R depends on the oscillator configuration: R for $x_1^b < x_2^b$ is larger than that for $x_1^b > x_2^b$. Moreover, the state with $x_1 > x_2$ is realized after slippage of the first oscillator whether $x_1^b > x_2^b$ or $x_1^b < x_2^b$. Fig. 4(c) plots the temporal evolutions of each oscillator's velocity and phase for $c = 0.63$ with $F_{ex} = 0.583$, which is slightly larger than $R_{smaller}$. If F_{ex} is constant with $|F_{ex}| > R$, the first oscillator slips and the second oscillator oscillates periodically. As shown in Fig. 4(c), the time required to switch from $x_1 > x_2$ to $x_1 < x_2$ is much longer than that to switch from $x_1 < x_2$ to $x_1 > x_2$ when F_{ex} is slightly larger than $R_{smaller}$. This supports the fact that the state with $x_1 > x_2$ is realized after slippage. (Because of the symmetry of this system, $x_1 < x_2$ is necessarily realized after slippage of the first oscillator if $F_{ex} < 0$.)

The above fact means that, in case II, R in the same direction as the direction of previous slippage is smaller than that in the opposite direction. Because of this fact, this system exhibits rectification behavior in this case. Now, to demonstrate this behavior, we consider the case $c = 0.63$ and $F_{ex} = 0.61 \sin(t/T)$, with sufficiently large T. Thus, in this case, the external force acts symmetrically in both positive and negative directions. Here, we choose the value 0.61 because it is halfway between the two values of R for $c = 0.63$. For this system, as shown in the left-hand side of Fig. 4(d), the first oscillator can move only in one direction, even though $F_{\rm ex}$ alternates between positive and negative values. However, if a sufficiently strong force is applied instantaneously to the first oscillator in the direction opposite to its motion, its motion can reverse direction (see the right-hand side of Fig. 4(d)). This means we can control the slippage direction in the system.

3. Mechanism of the simulation results

In the following, we try to understand the mechanism for the occurrence of behaviors like I, II and III. We consider only the case that $F_{\rm ex} > 0$ and $F_{\rm ex}$ varies smoothly and very slowly.

By considering the balance equations obtained by setting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ in Eqs. (2.1) and (2.2) and choosing the oscillator configuration before the slippage, we can obtain c–R curves plotted in Fig. 2. First, we consider the case with $x_1 > x_2$. With this condition with $\dot{x}_2 = 0$, we obtain $x_1 = 2x_2$. The R curve is obtained as the maximum values of $F_{\rm ex}$ with $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ under the condition $x_1 > x_2$. By setting $X = 2\pi x_2$, the curve is given by $R = -[\sin(2X) + c\sin(X)]$ with $X = \arccos[(-c - (c^2 + 32)^{1/2})/8]$. We name this curve R_a. This curve is consistent with the numerical result for R for $c < \hat{c}_{\rm crit^2}$. Second, we consider the case $x_1 < x_2$. Then, the line $R = c$ is consistent with the numerical results of R for $c > \hat{c}_{\rm crit^1}$, which is obtained from $R = -[c\sin(2\pi(x_1 - x_2)) + \sin(2\pi x_1)]|_{x_1=0.5, x_2=0.75}$. Here $x_1 = 0.5$ and $x_2 = 0.75$ correspond to the values for which the force from the second to the first oscillator is maximal under the condition $x_1 < x_2$. We name this line R_b. These curves, R_a and R_b, give the thresholds that determine the stability of stationary states with $x_1 > x_2$ and $x_1 < x_2$, respectively, as $F_{\rm ex}$ is increased, where the stationary state means the state with no slippage. If $|F_{\rm ex}| > R_a$ and $|F_{\rm ex}| > R_b$ are satisfied, the first oscillator shows slippage without exception.

Now, we consider the dynamical aspects of the first and second oscillators in the case where c is so large that $R_a < R_b$ holds. The previous results obtained from the balance equation give the following two facts for this case: (i) the first oscillator moves when $|F_{\rm ex}| > R_b$. (ii) When $|F_{\rm ex}|$ decreases to R_b, the slippage of the first oscillator stops with the condition $x_1 < x_2$ if the situation with $x_1 = 0.5$ and $x_2 = 0.75$ can be realized. If the first oscillator moves quasi-statically, situation (ii) can be realized. However, if $|F_{\rm ex}|$ decreases from $|F_{\rm ex}| > R_b$, each oscillator has a finite velocity from the beginning at $|F_{\rm ex}| = R_b$. Thus in the following, we consider the effect of a finite velocity of each oscillator at $|F_{\rm ex}| = R_b$. Because $F_{\rm ex}$ works only on the first oscillator, the first oscillator moves faster than the second most of the time. Then, it is expected that the first oscillator tends to reach the position $x = 0.5$ before the second oscillator reaches the position $x = 0.75$. In such a case, the first oscillator continues to move with a finite velocity because this situation does not satisfy the previous balance equations. This means that the slippage of the first oscillator does not stop with $x_1 < x_2$ even when $|F_{\rm ex}| = R_b$. If $|F_{\rm ex}|$ becomes smaller, the motion of the first oscillator becomes slower. Then, x_2 can reach such a position where the force from the second to the first oscillator is large enough to balance $F_{\rm ex}$ at $x_1 < x_2$. From these facts, in this case, the first oscillator moves with a finite velocity, we need to decrease $|F_{\rm ex}|$ to $|F_{\rm ex}| = R_{\rm stop} < R_b$ in order to stop the first oscillator's movement with $x_1 < x_2$.

In the case with small c region ($c \to 0$), $R_a > R_b$ holds. In such a small c, the system is approximated by a system with one degree of freedom, which means the system's state is determined by only the relation between $|F_{\rm ex}|$ and R_a. Then, the curve R_a gives the critical force.

On the other hand, for the case with large c ($c \to 1$), R_a is smaller than $R_{\rm stop}$. Then, when $F_{\rm ex}$ increases from $F_{\rm ex} < R_a$ to $F_{\rm ex} > R_a$, the first oscillator cannot cross the second oscillator, while it can cross the potential barrier like in Fig. 3(c) and (d). This means that the line R_b gives the critical force.

In the region with smaller c, $R_{\rm stop}$ is expected to become smaller than a value on the curve R_a. Then, the situation with $R_{\mathrm{stop}} < R_a < R_b$ can be realized for intermediate value of c. This means that as soon as F_{ex} exceeds R_a, the first oscillator starts the slippage if the stationary state with $x_1 > x_2$ is realized. On the other hand, the stationary state with $x_1 < x_2$ remains until $|F_{\mathrm{ex}}|$ exceeds R_b. Thus, the critical force possesses two values depending on the relation between x_1 and x_2 at the stationary state, which is obtained as the case II in our simulation. Moreover, the relation $R_{\mathrm{stop}} < R_a$ means that the slippage of first oscillator cannot be stopped with $x_1 < x_2$ when $|F_{\mathrm{ex}}| > R_a$. On the other hand, with the decrease of $|F_{\mathrm{ex}}|$, the time when the first oscillator crosses the potential barrier becomes long, which means the motion of the first oscillator during the crossing of the potential barrier approaches the quasi-static motion. From these facts, the time required to switch from $x_1 > x_2$ to $x_1 < x_2$ becomes longer than that to switch from $x_1 < x_2$ to $x_1 > x_2$ with the decrease of F_{ex} like in Fig. 4(c). Thus, the slippage of the first oscillator is stopped at $F_{\mathrm{ex}} < R_a$ with $x_1 > x_2$ in the case II.

4. Three oscillators model

The phenomena discussed above can also be observed in systems with a larger number of degrees of freedom. As an example, we consider a system that consists of three oscillators in a spatially periodic field, with the motions of each oscillator's phase obeying the equation

$$\dot{x}_i = \sum_{i \neq j} c_i c_j \frac{1 + \cos(2\pi(x_i - x_j))}{2} \sin(2\pi(x_i - x_j)) + c_i c_p \frac{1 + \cos(2\pi x_i)}{2} \sin(2\pi x_i) + \delta_{i,1} F_{\mathrm{ex}} \tag{4.1}$$

($x_i \ldots \mathrm{mod}\ 1$). The characteristic length of the interactions between the oscillators in this system is shorter than that of the system described by Eqs. (2.1) and (2.2). In the case with $c_1 = c_2 = 1$ and $c_3 = c_p = c > 0$, behavior similar to that exhibited in case II for the two-oscillator system is observed over a wide

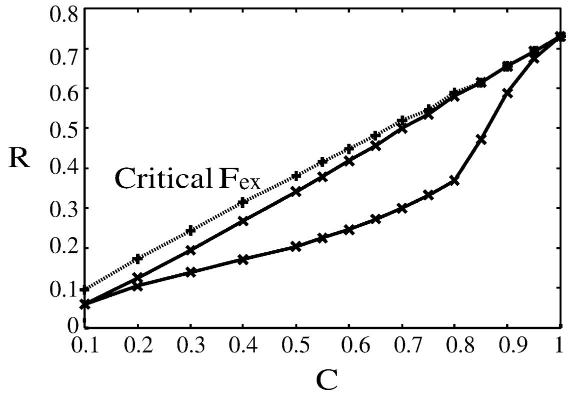

Fig. 5. The solid curves represent typical critical force R as the function of c in the system consisting of three oscillators for two configurations, the (1, 2, 3) configuration (lower curve) and the (1, 3, 2) configuration (upper curve) before slippage. The dotted curve represents the critical values of F_{ex}, for which the (1, 3, 2) is preserved under slippage.

range of values of c with $F_{\mathrm{ex}} = F(1 - \cos(t/T))/2$, for $F > 0$ and sufficiently large T.

The solid curves in Fig. 5 represent R as a function of c ($0.1 \leq c \leq 1$). As shown, R takes two values for each value of c, depending on the relationships among the x_i^b. Fig. 6(a)–(c) displays typical temporal evolutions of each oscillator's velocity and phase with $c = 0.5$ for two oscillator configurations before slippage: the (1, 2, 3) configuration in which the oscillators are arranged in the order 1, 2, 3 with respect to the direction of F_{ex} (the direction of increasing x in this case), as in (a), and the (1, 3, 2) configuration, as in (b) and (c). Here, $F = 0.5$ in (a) and (b), and $F = 0.36$ in (c). By comparing (a) and (b), we find that R for the (1, 3, 2) configuration is larger than that for the (1, 2, 3) configuration. Also it is seen that the (1, 2, 3) configuration is always realized after slippage in both cases considered in (a) and (b). We are thus led to the conclusion that, as in the system with two oscillators, under a certain condition, R for the direction of the previous slippage is always smaller than R for the direction opposite to the previous slippage. For the three-oscillator system, this condition is that F be larger than a particular value, which we discuss below.

As stated above, the (1, 2, 3) configuration is apparently always realized after slippage in the situations considered in Figs. 5(a) and (b). If F is not too large,

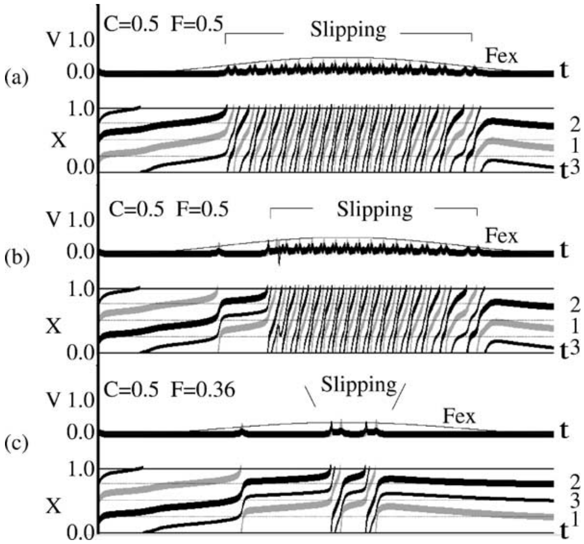

Fig. 6. Typical temporal evolutions of the velocity and phase of each oscillator in the three-oscillator system with $c = 0.5$. The shades and widths of the curves have the same meanings as in Fig. 3. The external forces F_{ex} used here are $F_{ex} = F(1 - \cos(t/T))/2$ in (a), (b), and (c). The oscillator configuration before slippage is $(1, 2, 3)$ in (a) and $(1, 3, 2)$ in (b) and (c).

however, the $(1, 3, 2)$ configuration can be preserved upon slippage. In fact, this is the case for the situation considered in Fig. 6(c). The dotted curve in Fig. 5 represents critical values of F_{ex}, below which the oscillator configuration is preserved upon slippage. Thus, in this system, the maximum strength of the external force at the slippage determines whether or not there is a direction dependence of R. We found that with a properly chosen F_{ex}, this system too can exhibit rectification behavior.

5. Summary and discussion

In this paper, we have investigated the history dependence of phase slippages using simple systems consisting of two and three oscillators with external forces. In these systems, we found that, in some cases, the magnitude of the minimal external force needed to start the phase slippage can depend on the direction of the slippage. In particular, the minimal external force needed for the phase slip is smaller when the previous slippage was in the same direction than when it was in the opposite direction. By this property, a oscillator in this system can continue to slip along a single direction even in the case that the direction of the external force change in time.

Recently, such a rectification behavior has been studied, for example, in some ratchet systems [8–10] that are simple models of molecular motors. In these models, with an external force that favors neither direction, particles can move in only one direction, which is completely determined by the shape of the potential. Our simulation results cannot be compared directly with a ratchet nor molecular motors because the external force is not noisy, unlike such systems. However, we expect that the obtained rectifier mechanism can give a hint to understand complex behaviors observed in some molecular motors [11].

Further analytical study of systems is necessary to clarify the behavior presented, while the study of systems with three or more oscillators will be reported. The present model can also be regarded as a simple extension of a model of the static and dynamic frictions investigated previously [12,13]. Thus, further extension of the present model that can realize some types of memory effects of friction [12–14] and charge density wave systems [3], is also an important future problem.

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