Versal S_5-varieties of dimension two

Shinzo Bannai
Tokyo Metropolitan University, shinzo.bannai@gmail.com

1 Introduction and Motivation.

X, Y: norm. proj. algebraic varieties over the complex numbers $\mathbf{C}$.
$\pi: X \to Y$: surjective finite morphism.
$\mathbf{C}(X)$ (resp. $\mathbf{C}(Y)$) : function fields of X (resp. Y).

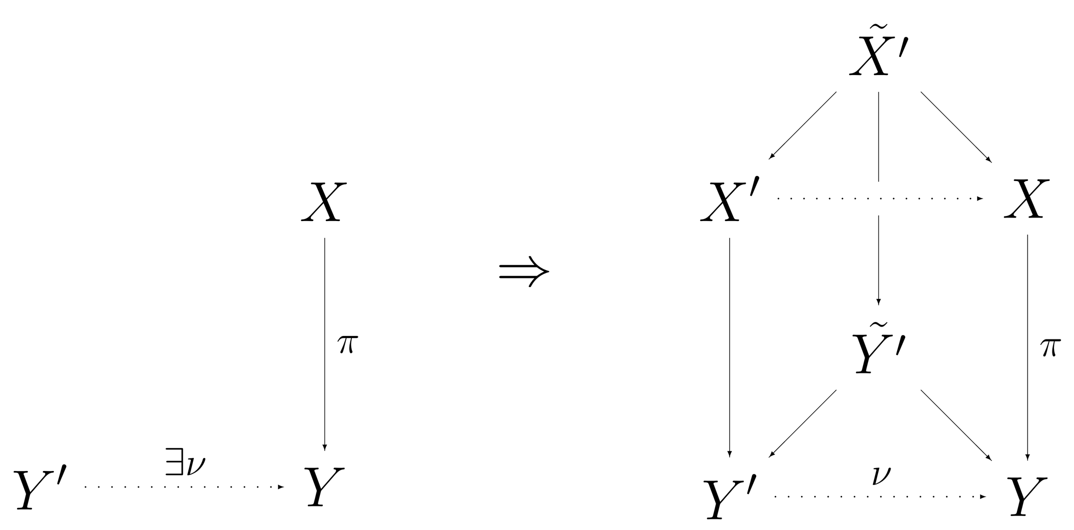
Definition 1.1
$\pi: X \to Y$ is a Galois cover. $\Leftrightarrow$ *$\mathbf{C}(X)/\mathbf{C}(Y)$ is an Galois extension.*

Note
$\pi: X \to Y$ is a Galois cover $\Rightarrow \exists G \curvearrowright X$ and $Y \cong X/G$.
If $\mathrm{Gal}(\mathbf{C}(X)/\mathbf{C}(Y)) \cong G$, we say that π is a G-cover.

The Main Problem in Galois covers

The Inverse Galois Problem
Given G and a projective variety Y', construct a G-cover over Y'. i.e. construct a projective variety X' and a surjective finite morphism $\pi': X' \to Y'$ such that π' is a G-cover.

Namba's approach [6]: (c.f. T.Yasumura's poster)
Given a finite group G, a projective variety Y' , and a G-cover $\pi: X \to Y$, and if a G-indecomposable rational map $\nu: Y' \dashrightarrow Y$ exists,
$\Rightarrow$ **We obtain a G-cover over Y' by "pulling-back" the G-cover structure of $\pi: X \to Y$.**



where $\tilde{Y'}$ is a resolution of indeterminacies of ν and $\tilde{X'} = Y' \times_Y X$.
This construction works only if ν exists. This leads to the following definition of a versal G-cover given in [11] and [9].

Versal G-cover and versal G-variety
Let $\varpi: X \to Y$ be a G-cover.

Definition 1.2
$\varpi: X \to Y$ is versal, if for any G-cover $\pi: Z \to W$,
- *$\exists \mu: Z \to X$ G-equivariant rational map such that*
- *$\mu(Z) \not\subset \mathrm{Fix}(X, G) := \{x \in X | G_x \neq \{1\}\}$*

If $\varpi: X \to Y$ is versal, we say that X is a versal G-variety.

Theorem 1.1 (Namba [6])
A versal G-cover exists for any finite group G.

Definition 1.3 (Buhler-Rehistein [2],Tokunaga [10])
The essential dimension $\mathrm{ed}_{\mathbf{C}}(G)$ of a finite group G is
$\mathrm{ed}_{\mathbf{C}}(G) := \min\{\dim X | \pi: X \to Y$ *is a versal G-cover* $\}$

Problems in versal G-varieties
- Find versal G-varieties X such that $\dim X = \mathrm{ed}_{\mathbf{C}}(G)$.
- Classify versal G-varieties.
- Study the relation between two G-varieties.

2 The case $G = S_5$: the symmetric group of degree 5.

Fact
- $\mathrm{ed}_{\mathbf{C}}(S_5) = 2$

Theorem 2.1 (Tokunaga, [10])
If $\mathrm{ed}_{\mathbf{C}}(G) = 2$ and if $\varpi: X \to Y$ is a versal G-cover such that $\dim(X) = 2$, then X is rational.

Hence the classification is reduced to the case of rational S_5-surfaces.

The three birational equivalence classes of rational S_5-surfaces
The classification of rational G-surfaces is given by Dolgachev and Iskovskikh in [3]. There are three distinct birational equivalence classes represented by:
- $\mathbf{P^1} \times \mathbf{P^1}$.
- Dp_5: Del Pezzo Surface of degree 5.
- Cb_3:Clebsch cubic surface.

$[x_0, x_1, x_2, x_3, x_4]$: hom. coord. of $\mathbf{P}^4$
$S_5 \curvearrowright \mathbf{P}^4$ by permutation of coordinates.

$$Cb_3 : \sum_{i=0}^{4} x_i = \sum_{i=0}^{4} x_i^3 = 0$$

Then Cb_3 is a del Pezzo surface of degree 3 and

$$S_5 \curvearrowright Cb_3$$

$\mathbf{P}^1 \times \mathbf{P}^1$ is not versal by elementary arguments. Dp_5 is versal. This is shown in [1]. Cb_3 is versal. This is essentially equivalent to the following equivalent theorems by Hermite.

Hermite's S_5-covariant, and Normal Form for quintics

Theorem 2.2 (Hermite [4])
Let L be any field. Let $S_5 \curvearrowright L^5$ by permutation of the coordinates. There exists a faithful S_5-covariant

$$\phi: L^5 \to L^5$$

such that

$$\phi(L^5) \subset \{(x_1, \cdots, x_5) \in L^5 | \sum_{i=1}^{5} x_i = \sum_{i=1}^{5} x_i^3 = 0\}$$

Theorem 2.3 (Hermite [4])
Let L/K be a separable field extension of degree 5. Then there exists $\theta \in L$ such that $L = K(\theta)$ and

$$\theta^5 + b\theta^3 + d\theta + d = 0 \quad (b, d \in K)$$

Conclusion

Main result

Theorem 2.4
There are exactly two distinct birational equivalence classes of versal S_5-surfaces represented by Dp_5 and Cb_3 with S_5-action.

3 Wish List.

- Want to find more methods in determining whether a variety is versal or not!! Is there some kind of good invariant?
- Want to actually apply the construction of Namba using versal G-varieties!!
- Want to understand Hermite's covariant in geometric terms!!
- Want to know the relation between Dp_5 and Cb_3.

References

[1] S. Bannai and H. Tokunaga, A note on embeddings of S_4 and A_5 into the two-dimensional Cremona group and versal Galois covers, Publ. RIMS, Kyoto Univ. **43** (2007), 1111–1123

[2] J. Buhler and Z. Reichstein, On the essential dimension of a finite group, Compositio Math. **106** (1997), no. 2, 159–179

[3] I. Dolgachev and V. Iskovskikh, Finite subgroups of the plane Cremona group, (available at http://www.math.lsa.umich.edu/ idolga/torinofinal.pdf)

[4] C. Hermite : Sur l 'invariant du 18^e ordre des formes du cinquième degré et sur le rôle qu ' il joue dans la résolution de l 'équation du cinquième degré, extrait de deux lettres de M. Hermite à l ' editeur. J. reine angew. Math. 59 (1861) 304–305.

[5] H. Kraft, A result of Hermite and equations of degree 5 and 6, J. Algebra vol. 297 (2006), 234-253

[6] M. Namba, On finite Galois Coverings of projective manifolds, *J. Math. Soc. Japan*, **41** (1989), 391-403.

[7] M. Namba, Finite branched coverings of complex manifold, *Sugaku Expositions* **5** (1992), 193-211.

[8] H. Tokunaga, Galois covers for $\mathfrak{S}_4$ and $\mathfrak{A}_4$ and their applications, *Osaka J. Math.* **39**(2002), 621-645.

[9] H. Tokunaga: *Note on a 2-dimensional Versal D_8-cover*, Osaka J. Math **41** (2004), 831–838.

[10] H. Tokunaga, 2-dimensional versal G-covers and Cremona embeddings of finite groups,*Kyushu J. of Math*, **60**(2006), 439-456.

[11] H. Tsuchihashi, Galois coverings of projective varieties for the dihedral groups and the symmetric groups, *Kyushu J. of Math.* **57**(2003), 411-427.