Branched coverings and three manifolds
Second lecture

José María Montesinos-Amilibia

Universidad Complutense

Hiroshima, March 2009
Second Lecture. Universal branching sets
Universal branching sets in surfaces

- Take an arbitrary unbounded, surface $\Sigma$ with an orientation $O$ and a triangulation $K$ of it.
Universal branching sets in surfaces

- Take an arbitrary unbounded, surface $\Sigma$ with an orientation $O$ and a triangulation $K$ of it.
- Subdivide barycentrically $K$ to obtain another triangulation $K'$. The vertexes of $K'$ fall naturally in three classes: barycenters of vertexes (resp. edges, faces) of $K$ called, respectively, bary-vertexes, bary-edges or bary-faces.
Any face of $K'$ has a natural orientation $o$, namely the one given by the following ordering of its vertexes: bary-vertex, bary-edge, bary-face.
Any face of $K'$ has a natural orientation $o$, namely the one given by the following ordering of its vertexes: bary-vertex, bary-edge, bary-face.

Color this face **white** iff $o = O$. Otherwise color it **black**. Then we have obtained a check-board coloration of the faces of $K'$ because two different faces sharing an edge get different colors.
Think of the sphere $S^2$ as the result of pasting linearly together two triangles (one white; the other black) along their edges. Call the resulting vertexes 0, 1, 2.
Think of the sphere $S^2$ as the result of pasting linearly together two triangles (one white; the other black) along their edges. Call the resulting vertexes $0$, $1$, $2$.

Map linearly white (black) triangles of the surface $\Sigma$ to the white (black) triangle of $S^2$ in such a way that bary-vertexes (resp. bary-edges, bary-faces) go to $0$ (resp. $1$, $2$).
Think of the sphere $S^2$ as the result of pasting linearly together two triangles (one white; the other black) along their edges. Call the resulting vertexes 0, 1, 2.

Map linearly white (black) triangles of the surface $\Sigma$ to the white (black) triangle of $S^2$ in such a way that bary-vertexes (resp. bary-edges, bary-faces) go to 0 (resp. 1, 2).

This map is a branched cover:
Theorem (Ramírez)

Every unbounded, orientable surface $\Sigma$ is a covering of the sphere $S^2$ branched over three points
The argument of Ramírez works in fact for every triangulated unbounded, oriented $n$-manifold. For $n = 3$, therefore, we have proved the following Theorem.
Theorem (Ramírez)

Every unbounded, orientable 3-manifold is a (combinatorial) branched covering of the sphere with branching set the set $G$ of edges of a tetrahedron embedded in the sphere:

The graph $G$
Corollary (M)

Every unbounded, orientable 3-manifold is a (combinatorial) branched covering of the sphere with branching set the set:

whose exterior has fundamental group of rang two.
Universal branching set

The graph \( G \) is a \textsc{universal branching set} in the sense that every 3-manifold branches over it. But note that, while in the case of surfaces, the branching set is a manifold, this is not the case if the dimension of the manifold is \( \geq 3 \).
Problem

Is there a universal branching set which is a manifold for every dimension?
History:
González-Acuña asked this question. Open for $n > 3$. 
W. Thurston found (in an unpublished paper) the first example of a (complicated) link in the 3-sphere $S^3$ that was a universal branching set.
1. W. Thurston found (in an unpublished paper) the first example of a (complicated) link in the 3-sphere $S^3$ that was a universal branching set.

2. Thurston also asked if some familiar knots and links, (like the figure eight knot, Whitehead link or the Borromean rings) were in fact universal branching sets.
This was answered positively by Hilden-Lozano-M (see also the work of Uchida).
1. This was answered positively by Hilden-Lozano-M (see also the work of Uchida).

2. It was also clear at the time that some knots and links could not be universal branching sets. (like the trefoil knot).
Theorem (Hilden-Lozano-Montesinos)

The figure-eight knot and the Whitehead and Borromean links are universal branching sets for all closed, orientable 3-manifolds.
The proof

- Start with a closed, orientable 3-manifold $M^3$. 
The proof

- Start with a closed, orientable 3-manifold $M^3$.
- Let $p : M^3 \to S^3$ be a simple 3-fold covering branched over the colored link $L$. 
- Assume $L$ is a closed braid.
• Assume $L$ is a closed braid.

• Applying Montesinos moves to $L$ we can assume every crossing of $L$ has 3 colors.
Assume $L$ is a closed braid.

Applying Montesinos moves to $L$ we can assume every crossing of $L$ has 3 colors.

Figure: Montesinos move
Figure: Every crossing of $L$ has 3 colors.
Using Montesinos moves we can assume all crossings are “positive”:

\[ r \]
\[ \theta \]

\textit{POSITIVE} \hspace{2cm} \textit{NEGATIVE}
In fact:

Figure: Montesinos move
Replace each crossing with a new small circle component:
After doing this our link \( L \) has two types of components; “braid" or "horizontal" components and “small circle" components:
Use the following the Montesinos transformation:
to replace each small circle component by three components as in the right hand side of:
Up to isotopy, each big circle component of $L$ extends over the top and bottom of all the horizontal components:
Now isotope the small circle components, **one at a time**, so that they become braid or horizontal components. As we do this to a particular small circle component “c" it becomes the topmost braid component:
Now our link $L$ has two types of components, horizontal components and vertical components. There are also two types of crossings:

$\theta$

$\theta$

VERTICAL

HORIZONTAL
Crucial observation: Every horizontal crossing is 3-colored:
We use the two following Montesinos transformations illustrated in what follows (both are useful) to replace each horizontal crossing by a vertical crossing.
First transformation (from right to left):
Second transformation (first and second step):
Second transformation (third step, from right to left):

Figure: Final step
We use these two Montesinos transformations to replace each horizontal crossing by a vertical crossing. In the course of doing this, new components, contained in the "peanut shaped" balls indicated by a "P" or "Q" are introduced:

**First transformation:**
Second transformation:
We will use for simplicity the $P$-peanut shape:

(We can use the $P$-peanut shape or the $Q$-peanut shape but never both in the same proof.)
Finally, after a slight isotopy our link $L$ has three types of components; horizontal, vertical and “special”. Each special component is contained in a “peanut shaped" topological ball:
Theorem

Let $M^3$ be a closed oriented 3–manifold. Then there is a 3–fold simple branched covering $p : M^3 \to S^3$ branched over a link $L$.

The link $L$ has three types of components.

a. **Horizontal.**
b. **Vertical.**
c. **Special.** These have local projections as in either the left or right hand side of the next figure:
A portion of the image of the link $L$ appears as follows:

Some of the “peanut shaped” balls contain two component links and arcs from a vertical and horizontal component, others contain only arcs from a vertical and horizontal component.
We will work out both together.
Define two rotations $T_1$ and $T_2$ of $S^3 = E^3 \cup \{\infty\}$. The rotation $T_1$ is simply the $m$–fold rotation about the $z$–axis; the rotation $T_1$ leaves invariant the set of horizontal and the set of vertical components of the link $L$. 
The rotation $T_2$ has as its axis a circle. It leaves the set of horizontal and the set of vertical components of $L$ invariant. It cyclically permutes the horizontal components and it sends each vertical component to itself. Its restriction to a vertical component is just the usual $n$–fold rotation of a circle.
The rotation $T_2$ has as its axis a circle. It leaves the set of horizontal and the set of vertical components of $L$ invariant. It cyclically permutes the horizontal components and it sends each vertical component to itself. Its restriction to a vertical component is just the usual $n$–fold rotation of a circle.

We can and shall assume that both rotations $T_1$ and $T_2$ leave the peanuts of the link $L$ invariant.
Consider the map \( f : S^3 \rightarrow S^3 / T_1 = S^3 \) which is an \( m \)-fold cyclic branched covering \( S^3 \rightarrow S^3 \) with branch set the trivial knot or z-axis, induced by rotation \( T_1 \).
Consider the map \( f : S^3 \to S^3 / T_1 = S^3 \) which is an \( m \)-fold cyclic branched covering \( S^3 \to S^3 \) with branch set the trivial knot or \( z \)-axis, induced by rotation \( T_1 \).

(Alternative: a branched cover \( f_1 : S^3 \to S^3 \) coinciding with \( f \) out of a tubular nbd of the branch set which is the double of the trivial knot trivial knot, and with branch indexes 1 and 2).
The branch set for the composite map $f \circ p : M^3 \to S^3$ consists of the branch set for $f$ (the $z$-axis) plus the image under $f$ of the branch set of $p$. 
The branch set for the composite map $f \circ p : M^3 \to S^3$ consists of the branch set for $f$ (the $z$-axis) plus the image under $f$ of the branch set of $p$.

(Alternative: using $f_1$ the branch set for the composite map $f_1 \circ p : M^3 \to S^3$ consists of the branch set for $f_1$ (the double of the $z$-axis) plus the image under $f$ of the branch set of $p$.)
The part $f(L)$ of the branch set of $f \circ p : M^3 \to S^3$ has one vertical component and $n$–horizontal components and $n$–“peanut" components (we depict also the image of the $z$-axis of rotation):

branch set of $f \circ p$  branch set of $f_1 \circ p$
Consider the map \( g : S^3 \to S^3 / T_2 = S^3 \) which is an \( n \)-fold cyclic branched covering \( S^3 \to S^3 \) with branch set the trivial knot, induced by rotation \( T_2 \).

(Alternative: a branched cover \( g_1 : S^3 \to S^3 \) coinciding with \( g \) outside a tubular nbhd of the branch set which is the double of the trivial knot, and with branch indexes 1 and 2).
The branch set for the composite map $g \circ f \circ p : M^3 \to S^3$ consists of the branch set for $g$ (a circle) plus the image under $g$ of the branch set of $f \circ p$. 
The branch set for the composite map $g \circ f \circ p : M^3 \rightarrow S^3$ consists of the branch set for $g$ (a circle) plus the image under $g$ of the branch set of $f \circ p$.

(Alternative: using $g_1$ the branch set for the composite map $g_1 \circ f \circ p : M^3 \rightarrow S^3$ consists of the branch set for $g_1$ (the double of a circle) plus the image under $g$ of the branch set of $f_1 \circ p$.)
The branching set of $g \circ f \circ p : M^3 \to S^3$ (or of $g_1 \circ f_1 \circ p$):

Branch set of $g \circ f \circ p : M^3 \to S^3$

Branch set of $g_1 \circ f_1 \circ p$
Can be isotoped to the link:

\[ g \circ f \circ p : M^3 \to S^3 \]

Branch set of \[ g \circ f \circ p : M^3 \to S^3 \]

Branch set of \[ g_1 \circ f_1 \circ p \]
The first link has a 3–fold symmetry. Let $T_3$ be this 3–fold rotation and let $h : S^3 \to S^3 = S^3 / T_3$ be the resulting branched covering.
The first link has a 3–fold symmetry. Let $T_3$ be this 3–fold rotation and let $h : S^3 \to S^3 = S^3 / T_3$ be the resulting branched covering.

Let $h_1 : S^3 \to S^3$ be equal to $h$ except in solid torus nbd of the axis of rotation of $h$ which it is replaced with two parallel axes and the branch indexes of $h_1$ are 1 and 2).
The map \( h \circ g \circ f \circ p \) is a 9\( mn \) to 1 branched covering of \( S^3 \) by \( M^3 \) with branch set \( h \) (branch set \( g \circ f \circ p \)) together with the image of the rotational axis:
We summarize this result in the form of a theorem:
Theorem (Hilden-Lozano-M)

Let $M^3$ be a closed orientable 3–manifold. Then $M^3$ is a branched covering of $S^3$ with branch set the Borromean rings. That is, the Borromean rings is a universal branching set.
Using the second link and the map $h_1$ instead of $h$ we have proved:
Theorem (Lozano-M)

Let $M^3$ be a closed orientable 3–manifold. Then $M^3$ is a branched covering of $S^3$ with branch set the double Borromean rings and the branching indexes of the covering are 1 and 2.
If in the above proofs we use the peanut $Q$

instead of $P$ we get
Theorem (Hilden-Lozano-M)

Let $M^3$ be a closed orientable 3–manifold. Then $M^3$ is a branched covering of $S^3$ with branch set the Whitehead link. That is, the Whitehead link is a universal branching set.
Theorem (Lozano-M)

Let $M^3$ be a closed orientable 3–manifold. Then $M^3$ is a branched covering of $S^3$ with branch set the double Whitehead link and the branching indexes of the covering are 1 and 2.

Figure: The double of the Whitehead link