Epimorphisms among 2-bridge knot groups and related problems

作間 誠 (広島)

- Knot group of a knot $K \subset S^3$

$$G(K) := \pi_1(S^3 - K)$$

<u>Fact</u> $G(K)$ is a complete invariants for prime knots

ie For prime knots K, K',

$K \cong K'$ equivalent

ie $(S^3, K) \cong (S^3, K')$

$\Leftrightarrow$ $G(K) \cong G(K')$

<u>Def</u> $K \geq K'$ $\Leftrightarrow$ $\exists$ epimorphism $G(K) \twoheadrightarrow G(K')$

<u>Fact</u> "$\geq$" is a partial order on {prime knots}.

ie $K \geq K'$ & $K' \geq K$ implies $K \cong K'$

<u>Problem</u> Study the partial order. (Simon)

Boileau - Wang
Boileau - Boyer - Reid - Wang
Kitano - Suzuki
Ohtsuki - Riley - S
Silver - Whitten
Gonzalez-Acuna, Raminez, ...

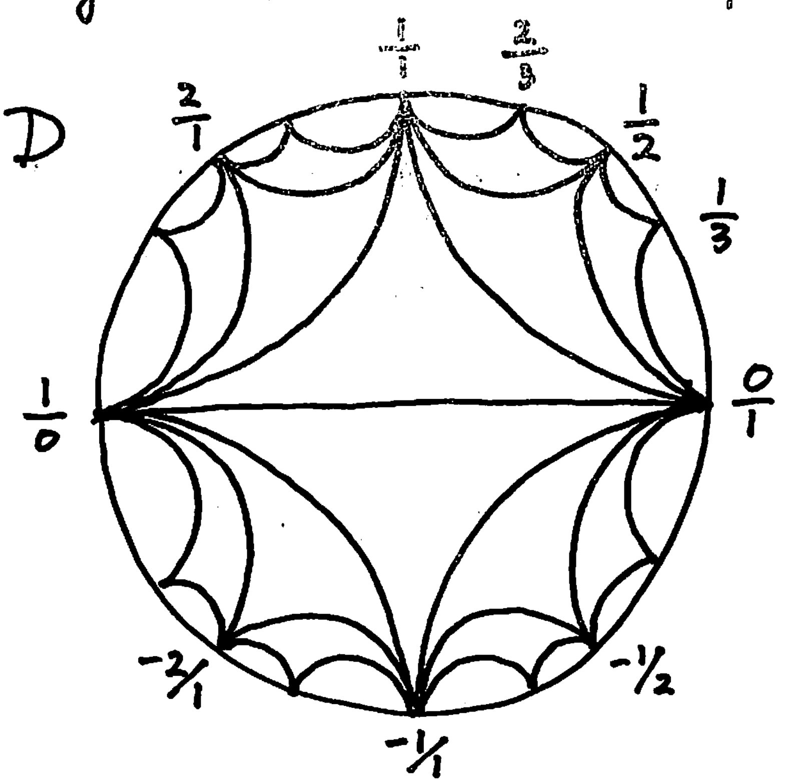
Plan of the talk

1. Systematic construction of epimorphisms among 2-bridge knot groups via branched fold maps.
   (Joint work with Ohtsuki and Riley)

2. Relation with:

   - End invariants of $SL(2.\mathbb{C})$-representation of once-punctured torus group
   - Geometric structure of the knot complement
   - $SL(2.\mathbb{C})$-character variety of knot group
   - McShane's identity
   - Degree one maps among 3-manifolds
   - Generalization

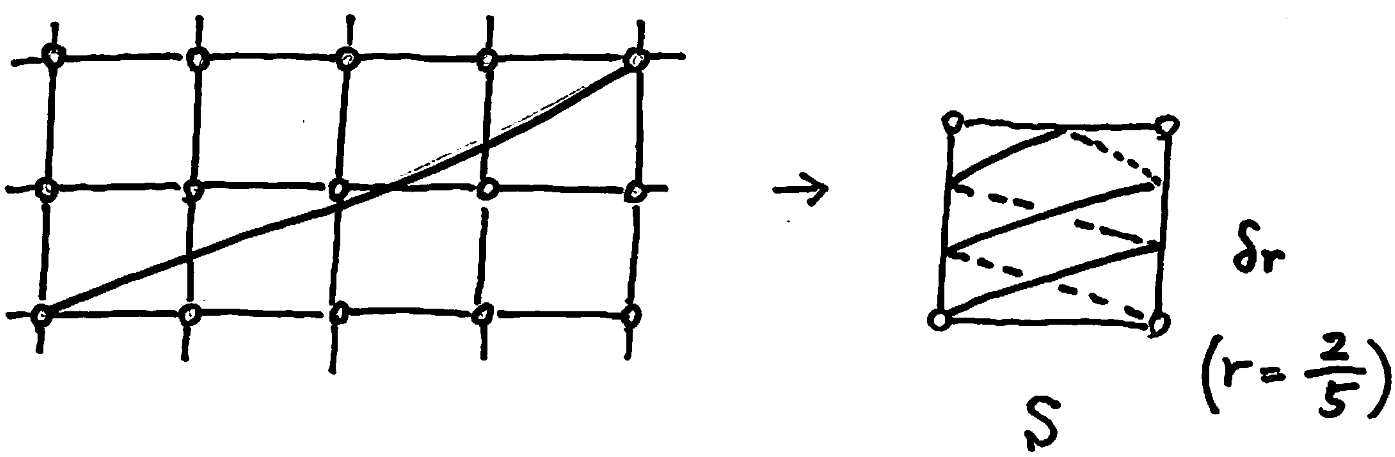
Farey tessellation and 4-punctured sphere S



Vertex set of D $=$ $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\frac{1}{0}\}$ $\qquad r$

$\underset{1:1}{\longleftrightarrow}$ {essential simple loops on S} $\qquad \alpha_r$

$\underset{1:2}{\longleftrightarrow}$ {essential simple arcs on S} $\qquad \delta_r$



- $\left\langle \frac{q_1}{p_1}, \frac{q_2}{p_2} \right\rangle$ is an edge of D

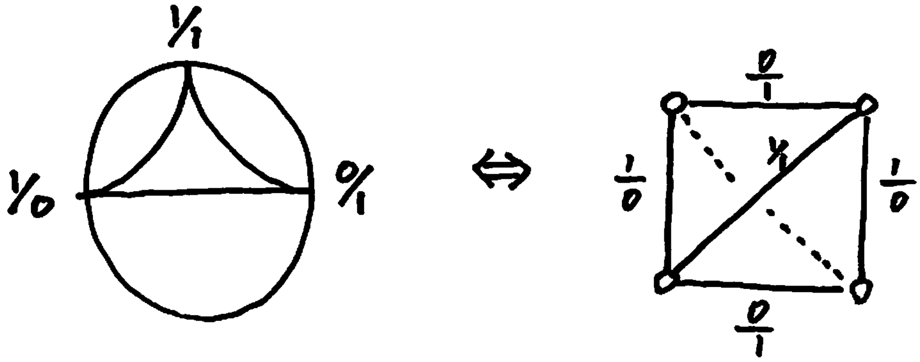
  $\Leftrightarrow \quad \begin{vmatrix} q_1 & q_2 \\ p_1 & p_2 \end{vmatrix} = \pm 1$

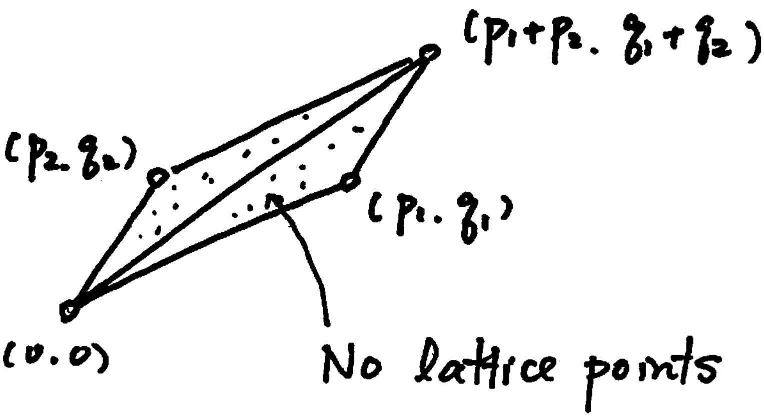
  $\Leftrightarrow \quad \delta_{q_1/p_1}$ and δ_{q_2/p_2} are disjoint

- 2-simplex $\left\langle \frac{q_1}{p_1}, \frac{q_1+q_2}{p_1+p_2}, \frac{q_2}{p_2} \right\rangle$

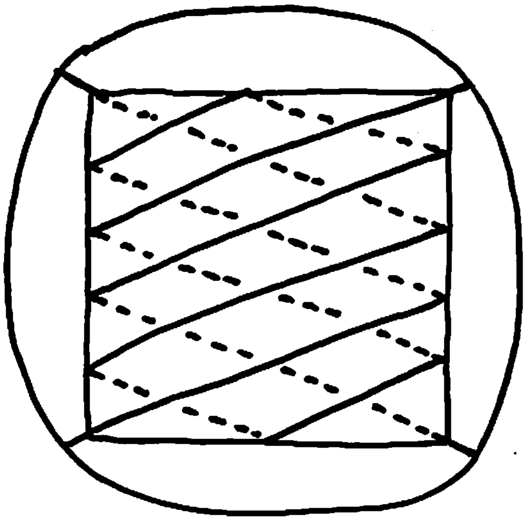
  $\Leftrightarrow$ ideal triangulation of S

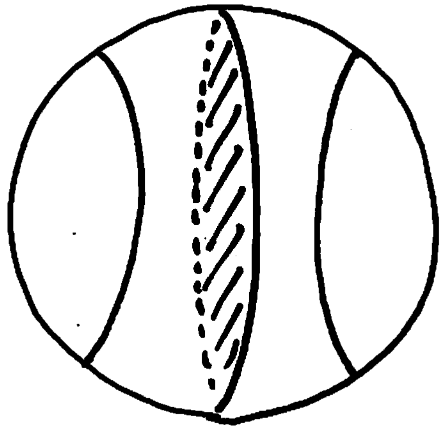
eg





- Rational tangle $(B^3, t(r))$ of slope r



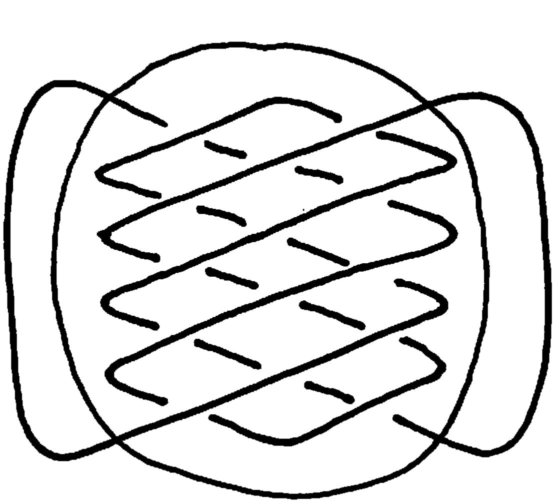


$(B^3, t(\frac{2}{5}))$ $\longleftrightarrow$ $(B^3, t(\infty))$

homeomorphic but not isotopic rel ∂

- 2-bridge knot (or link) $K(r)$ of slop r

$$(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$$



$\cong$

Figure-eight knot

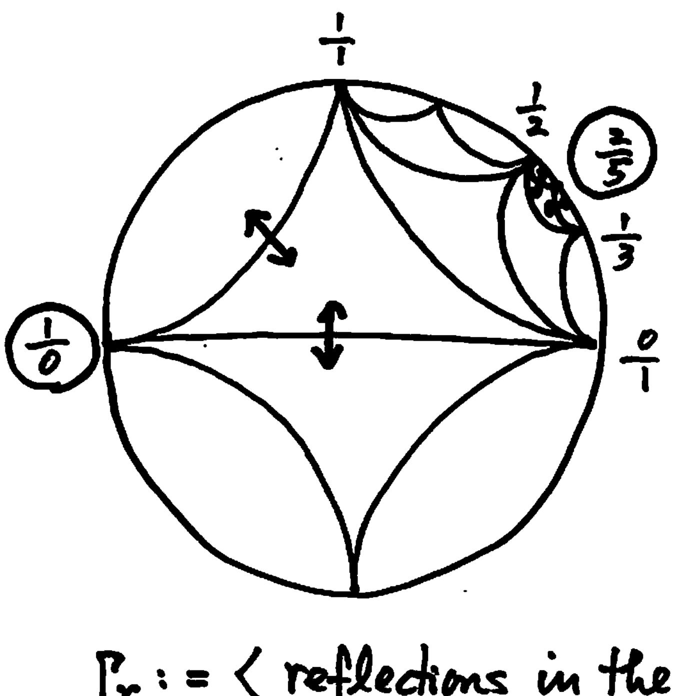
$K(\frac{2}{5})$

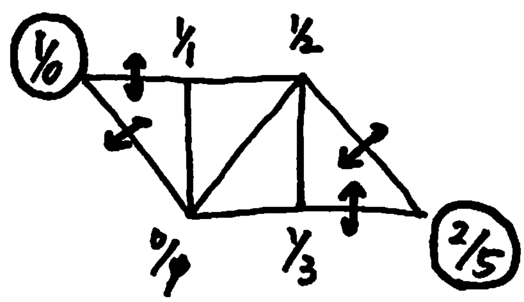
[Schubert]

Two 2-bridge knots $K(r)$ and $K(r')$ are equivalent iff the relative positions of $\{\infty, r\}$ and $\{\infty, r'\}$ in D are equivalent, i.e.

$$\exists\ \varphi : D \to D \quad \text{combinatorial isomorphism}$$
$$\text{st}\quad \varphi\{\infty, r\} = \{\infty, r'\}$$

- The group $\hat{\Gamma}_r$ associated with $K(r)$





$\Gamma_r := \langle$ reflections in the edges of D with an end point $r\rangle$

$\cong$ infinite dihedral group D_∞

$$\hat{\Gamma}_r := \langle \Gamma_\infty, \Gamma_r \rangle$$
$$\cong \Gamma_\infty * \Gamma_r \qquad \text{if} \quad d(\infty, r) \geq 2$$

Theorem

If $\tilde{r}$ belongs to the $\hat{\Gamma}_r$-orbit of r or ∞, then there is an epimorphism

$$G(K(\tilde{r})) \twoheadrightarrow G(K(r)) \quad (= \pi_1(S^3 - K(r))).$$

Moreover, the epimorphism is realized by an "almost level-preserving" "branched fold" proper map

$$f : (S^3, K(\tilde{r})) \to (S^3, K(r))$$

Namely, $K(\tilde{r}) = f^{-1}(K(r))$ (proper)

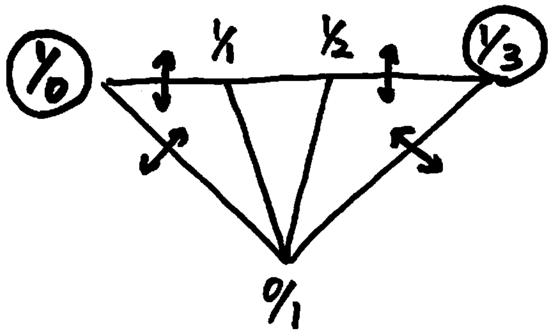
and hence f induces a map

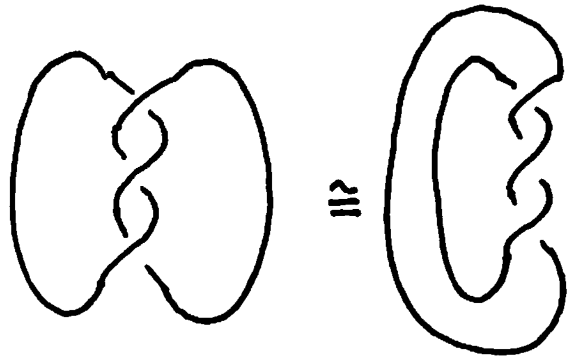
$$f : S^3 - K(\tilde{r}) \to S^3 - K(r),$$

and it induces an epimorphism

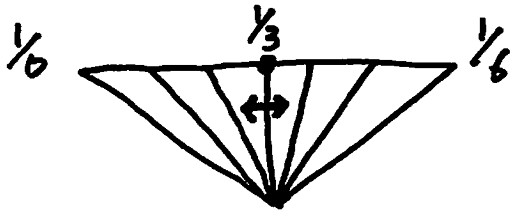
$$\begin{array}{ccccc} f_* : & \pi_1(S^3 - K(\tilde{r})) & \twoheadrightarrow & \pi_1(S^3 - K(r)) \\ & \parallel & & \parallel \\ & G(K(\tilde{r})) & & G(K(r)) \end{array}$$

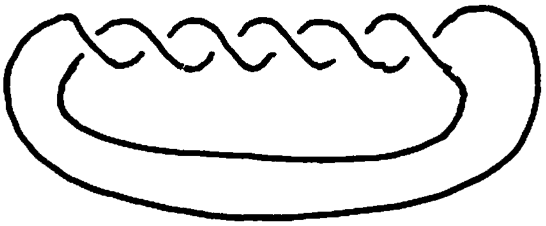
<u>Example</u> $r = \frac{1}{3}$.



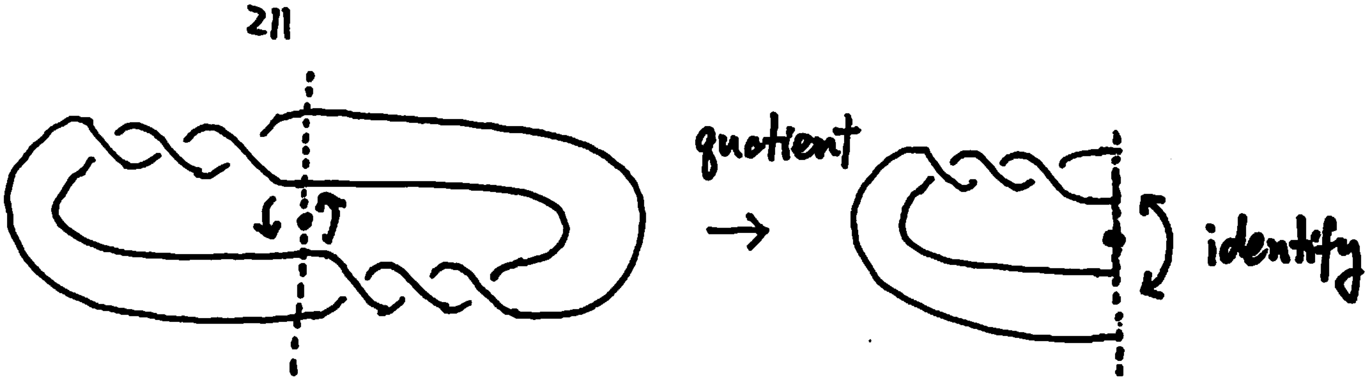


• $\tilde{r} = \frac{1}{6}$





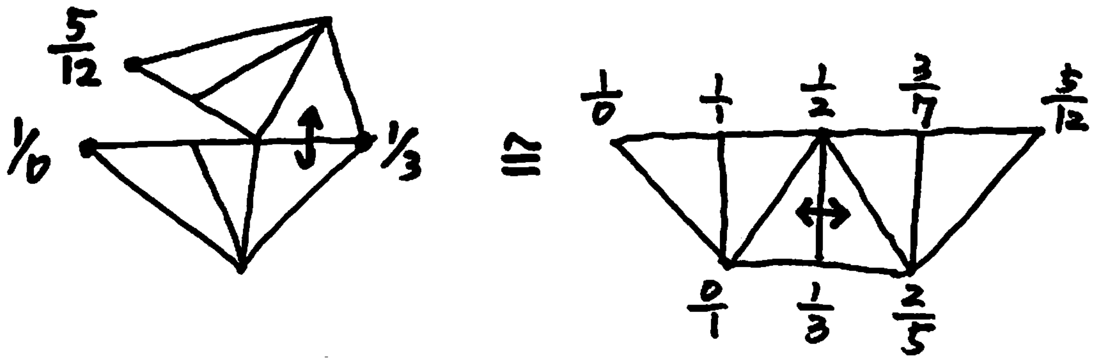
$K(\tilde{r})$

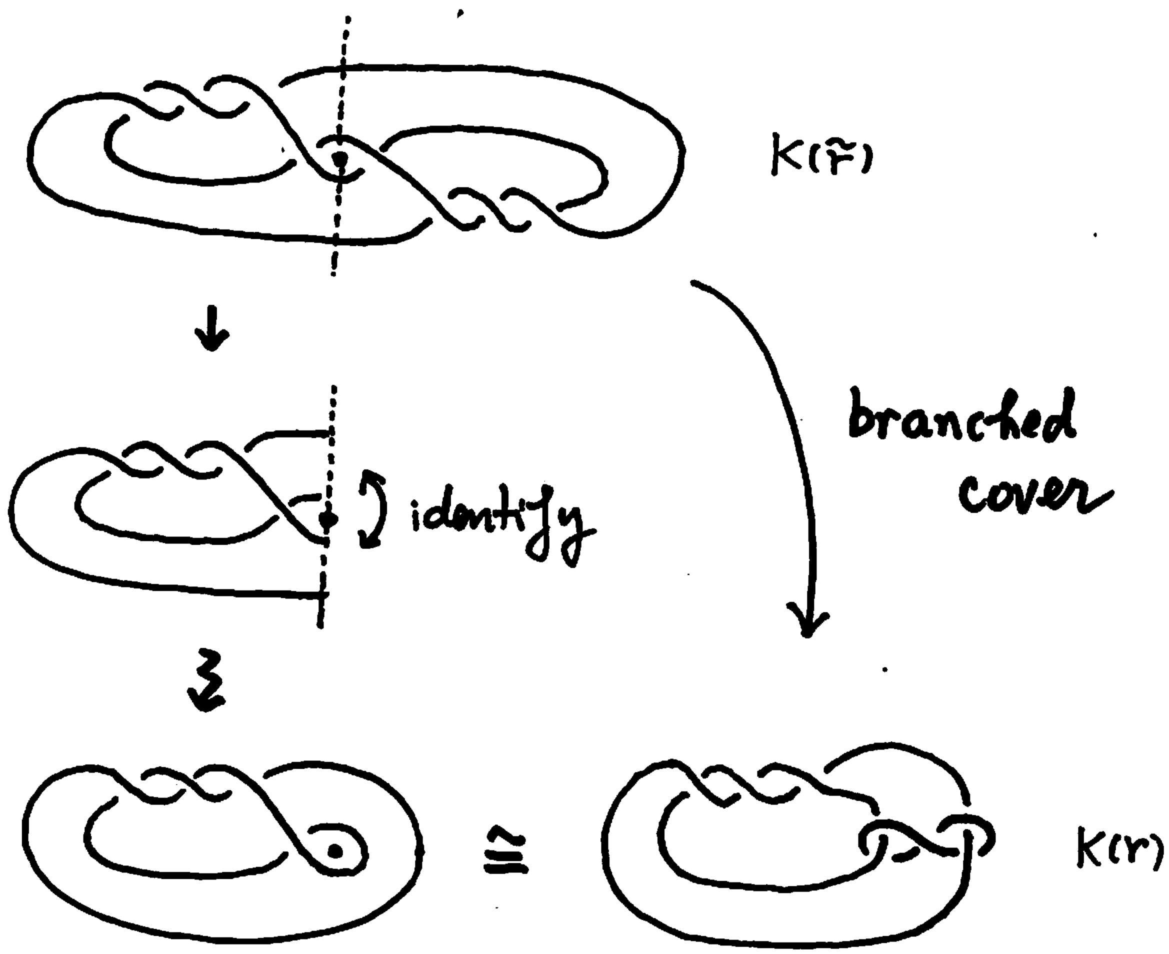


2-fold symmetry

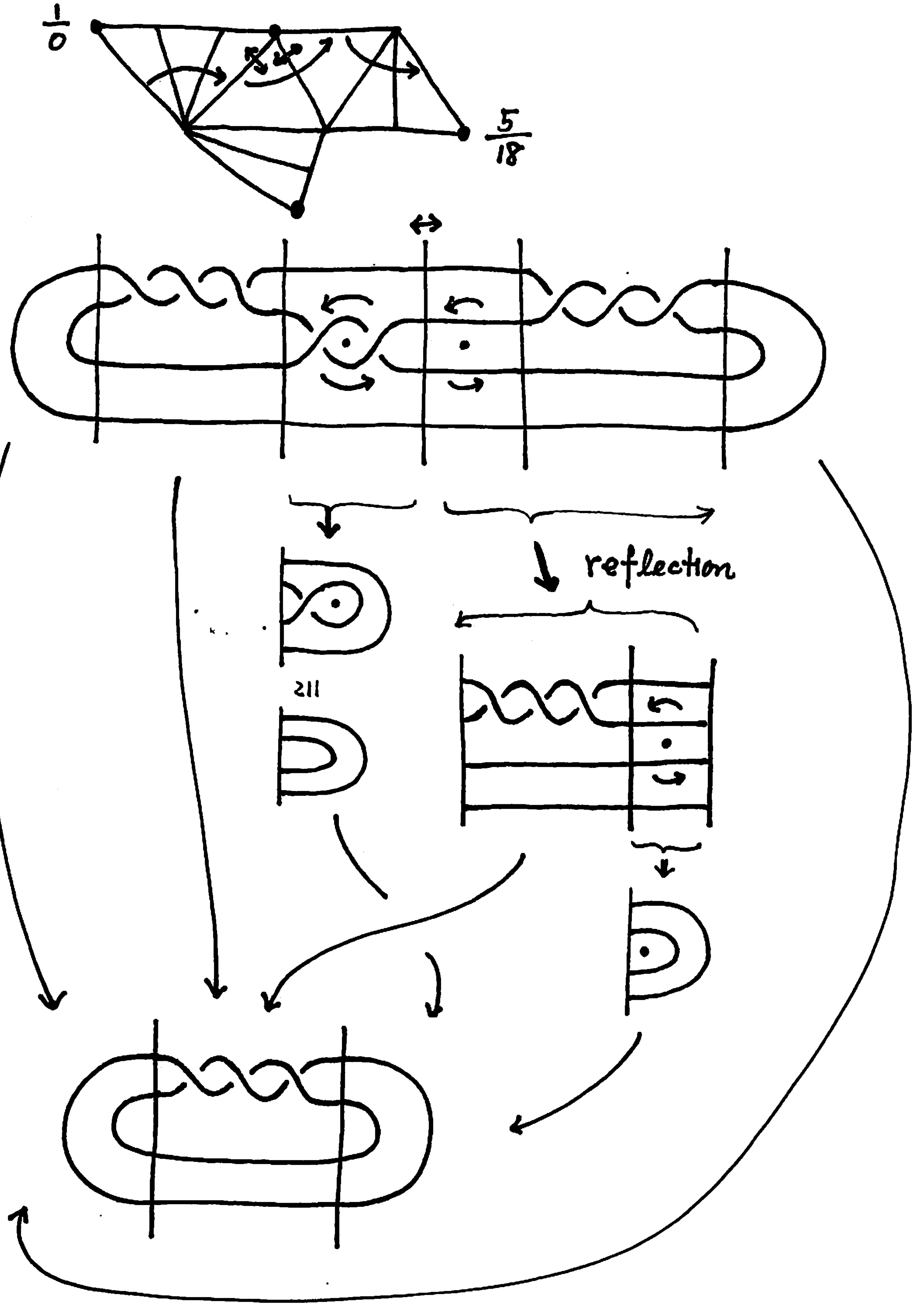
Branched cover
branched over
the trivial knot

$\cdot\ \tilde{r} = \frac{5}{12} = [2,2,2] = \cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}} = [3,-2,3]$

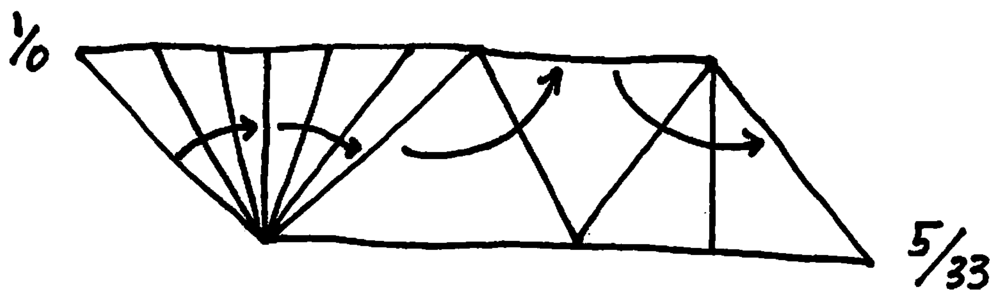


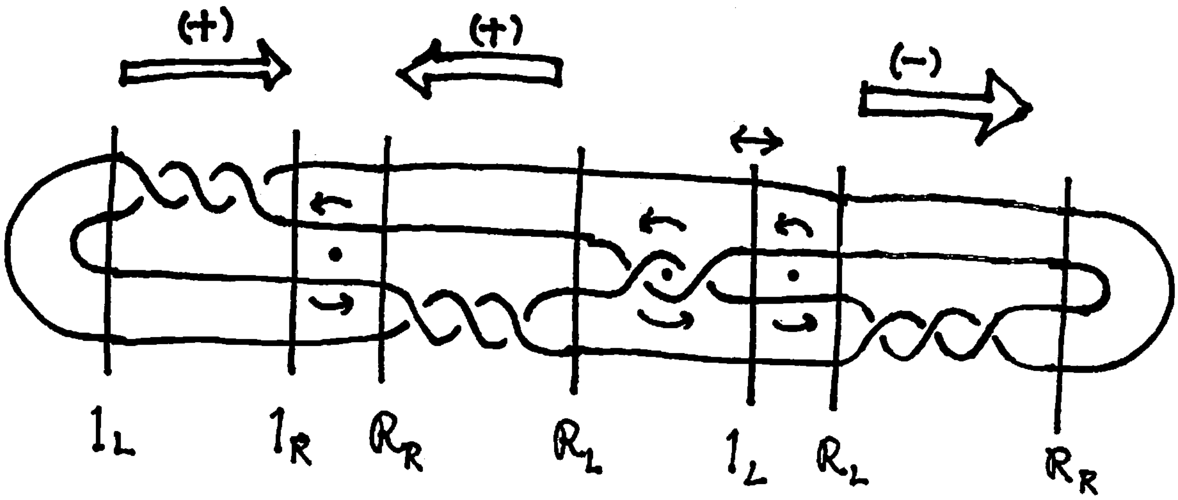


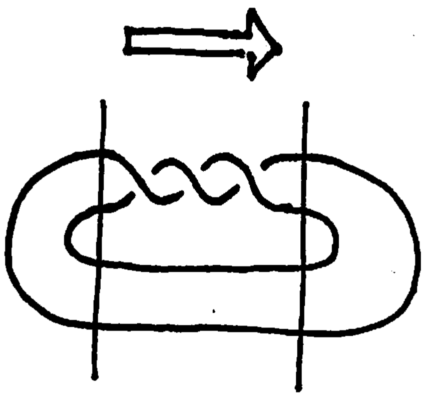
• $\tilde{r} = \frac{5}{18} = [3, 1, 1, 2] = [3, +2, -3]$



• $\tilde{r} = \dfrac{5}{33} = [6.1.1.2] = [3.0.3.2.-3]$







$\deg f = (+1) + (+1) + (-1) = +1$

(Proof of the first part of Theorem)

<u>Lemma</u> $G(K(r)) \cong \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$

$\odot$ $$\pi_1(S^3 - K(r)) \cong \pi_1(B^3 - t(\infty)) \underset{\pi_1(S)}{*} \pi_1(B^3 - t(r))$$

$$\cong \left(\frac{\pi_1(S)}{\langle\langle \alpha_\infty \rangle\rangle} \right) \underset{\pi_1(S)}{*} \left(\frac{\pi_1(S)}{\langle\langle \alpha_r \rangle\rangle} \right)$$

$$\cong \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$$

<u>Remark</u> Since $\pi_1(S) / \langle\langle \alpha_\infty \rangle\rangle \cong F_2$,

$G(K(r)) \cong \left(\pi_1(S) / \langle\langle \alpha_\infty \rangle\rangle \right) / \langle\langle \alpha_s \rangle\rangle$

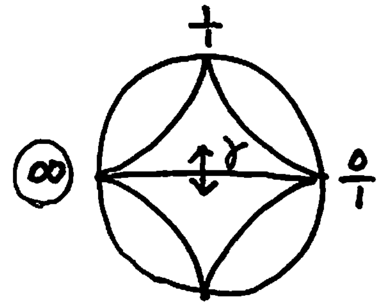
has a presentation with 2 generator and 1 relation.

e.g. $G(K(1/3)) = \langle x, y \mid x y x \bar{y} \bar{x} \bar{y} \rangle$

<u>Lemma</u> For $\gamma \in \hat{\Gamma}_r$ and $s \in \hat{\mathbb{Q}}$,

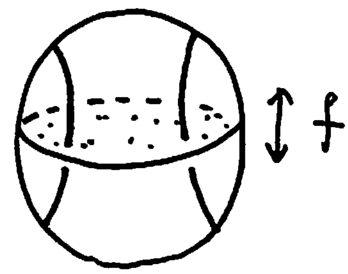
$$\alpha_{\gamma(s)} = \alpha_s \quad \text{in} \quad G(K(r))$$

$\because$ Suppose $\gamma(s) = -s$ ie



Then γ is "induced" by

$$f : (B^3, t(\infty)) \xrightarrow[\cong]{} (B^3, t(\infty))$$



i.e

$$\begin{tikzcd}
\{\text{essential loops on } S\} \arrow[r, "f_\#"] \arrow[d, equal] & \{\text{essential loops on } S\} \arrow[d, equal] \\
\mathcal{D}^{(0)} \arrow[r, "f_\# = \gamma"] & \mathcal{D}^{(0)}
\end{tikzcd}$$

ie $f(\alpha_s) = \alpha_{-s} = \alpha_{\gamma(s)}$

Moreover

$$\begin{tikzcd}
\pi_1(B^3 - t(\infty)) \arrow[r, "f_* = id"] \arrow[d, equal] & \pi_1(B^3 - t(\infty)) \arrow[d, equal] \\
\pi_1(S)/\langle\langle \alpha_\infty \rangle\rangle \arrow[d] & \pi_1(S)/\langle\langle \alpha_\infty \rangle\rangle \arrow[d] \\
\pi_1(S)/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle \arrow[r, dashed, "id", "\cong"'] & \pi_1(S)/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle
\end{tikzcd}$$

Hence $\alpha_s = f_*(\alpha_s) = \alpha_{\gamma(s)}$ in $G(K(r))$

The same argument works for every generator of $\hat{\Gamma}_r$ $\square$

Cor If $\tilde{r} \in \hat{\Gamma}_r\{\infty, r\}$, then

$$\alpha_{\tilde{r}} = 1 \quad \text{in } G(K(r))$$

$\because$ $\alpha_{\tilde{r}} = \alpha_\infty \ (\text{or } \alpha_r) = 1$

(Proof of Theorem)

Suppose $\tilde{r} \in \Gamma_r\{\infty, r\}$.

Then $\alpha_\infty = 1$, $\alpha_{\tilde{r}} = 1$ in $G(K(r))$

Hence the identity map $\pi_1(S) \to \pi_1(S)$

induces

$$\begin{array}{ccc} \dfrac{\pi_1(S)}{\langle\langle \alpha_\infty, \alpha_{\tilde{r}} \rangle\rangle} & \twoheadrightarrow & \dfrac{\pi_1(S)}{\langle\langle \alpha_\infty, \alpha_r \rangle\rangle} \\ \| & & \| \\ G(K(\tilde{r})) & & G(K(r)) \end{array}$$

$\square$

Question Does the coverse to Theorem hold?

i.e. if $G(K(\tilde{r})) \twoheadrightarrow G(K(r))$,

then $\tilde{r} \in \hat{\Gamma}_r\{\infty, r\}$?

[Gonzaléz-Acuna, Raminez]

Yes, if $r = \frac{1}{(\text{odd})}$.

[Boileau - Boyer - Reed - Wang]

$G(K(\tilde{r}))$ can dominate only 2-bridge knot groups. Moreover it can dominate only finitely many knot groups.

(Partial positive answer to Simon's conj.)

Conjecture A The converse to Cor holds.

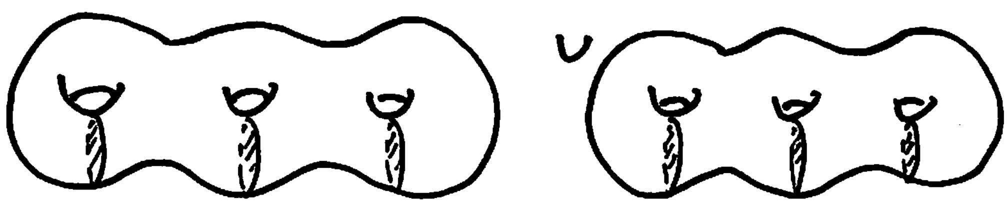
i.e. if $\alpha_s = 1$ in $G(K(r))$,

then $s \in \hat{\Gamma}_r \{\infty, r\}$

- Yes if $r = 1/n$ $(n \in \mathbb{N})$
- [Eguchi] Yes, if $r = \dfrac{a}{2a+1}$ $(1 \leq a \leq 10)$

[Minsky]

Generalization of the conjecture to simple loops on Heegaard surfaces.

$M =$ 

Which simple loop on a Heegaard surface is null-homotopic in the 3-manifold?

$PSL(2,\mathbb{C})$-representations of $G(K(r))$

Put $r = q/p$

(i) If $q \not\equiv \pm 1 \pmod p$, then

$S^3 - K(r)$ admits a complete hyperbolic structure.

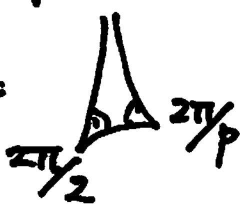
The holonomy reprentation

$$\rho : G(K(r)) \hookrightarrow PSL(2, \mathbb{C})$$

is faithful discrete.

(ii) If $q \equiv \pm 1 \pmod p$, then

$S^3 - K(r)$ is a Seifert fibered space

over the orbifold $D(2,p) =$ 

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \arrow[r] & G(K(r)) \arrow[r] \arrow[dr, dashed, "\rho"'] & \pi_1^{orb}(D(2,p)) \arrow[r] \arrow[d, "\rho"] & 1 \\
& & & PSL(2, \mathbb{R}) &
\end{tikzcd}$$

Refined Conjecture B

For the representation

$$\rho : \pi_1(S) \longrightarrow G(K(r)) \hookrightarrow PSL(2.\mathbb{C}),$$

we have $\mathcal{E}(\rho) = \Lambda(\hat{\Gamma}_r) \subset \hat{\mathbb{R}}$.

Here · $\mathcal{E}(\rho)$ is the set of the end invariants of ρ,

· $\Lambda(\hat{\Gamma}_r)$ is the limit set of $\Lambda(\hat{\Gamma}_r)$.

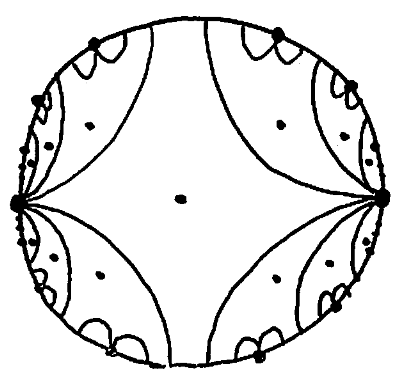
The limit set $\Lambda(\hat{\Gamma}_r)$

$= \{$accumulation points of $\hat{\Gamma}_r \cdot x$ in $\mathbb{H}^2\}$

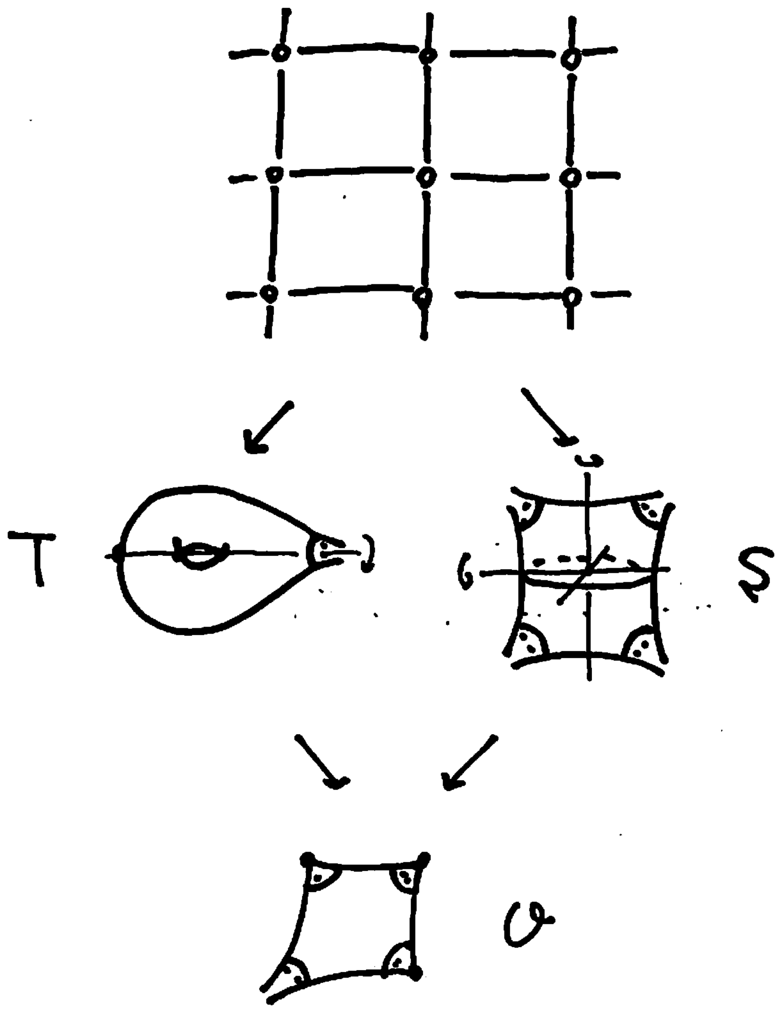
where $x \in \mathbb{H}^2$

$= \overline{\hat{\Gamma}_r \{0.1\}}$

$\cong$ Cantor set



- 4-punctured sphere S is commensurable with once-punctured torus T



$\pi_1(T)$ $\quad$ $\pi_1(S)$

$\triangleright$ $\quad$ $\triangleleft$

$\pi_1(\mathcal{O})$

- A line of slope $r \in \hat{\mathbb{Q}}$ projects to a simple loop $\alpha_r \subset S$

  $\qquad\qquad\qquad\qquad$: $\qquad$: $\qquad$ $\beta_r \subset T$

st $\quad \alpha_r = \beta_r^2 \in \pi_1(\mathcal{O})$

Definition

- $\rho : \pi_1(T) \to PSL(2,\mathbb{C})$ is type-preserving, if
  - (i) irreducible
  - (ii) ρ (peripheral loop) is parabolic
- $\rho : \pi_1(S) \to PSL(2,\mathbb{C})$ is type-preserving, if
  - (i) irreducible
  - (ii) ρ (peripheral loop) is parabolic.
  - (iii) "some condition"

Fact

$$\mathcal{X} := \{ \rho : \pi_1(T) \to PSL(2,\mathbb{C}) \overset{\text{type-pres.}}{} \} / \text{conj}$$

$$\updownarrow$$

$$\{ \rho : \pi_1(S) \to PSL(2,\mathbb{C}) \text{ type-pres} \} / \text{conj}$$

$$\tilde{\mathcal{X}} := \{ \rho : \pi_1(T) \to SL(2,\mathbb{C}) \text{ type-pres} \} / \text{conj}$$

$\downarrow$ 4-1 covering

$\mathcal{X}$

Markoff (trace) map

For $\rho \in \widetilde{\mathcal{X}}$, set

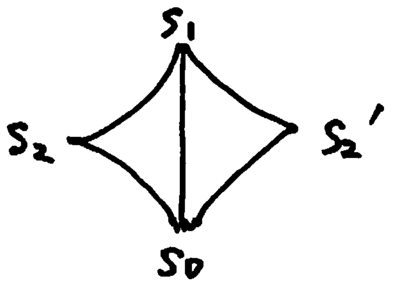
$$\phi = \phi_\rho : \hat{\mathbb{Q}} \longrightarrow \mathbb{C}$$
$$s \longmapsto \mathrm{tr}(\rho(\beta_s))$$

Then ϕ satisfies the following conditions:

(i) If $\langle s_0, s_1, s_2 \rangle$ is a Farey triangle and $(x, y, z) = (\phi(s_0), \phi(s_1), \phi(s_2))$ then

$$x^2 + y^2 + z^2 = xyz$$

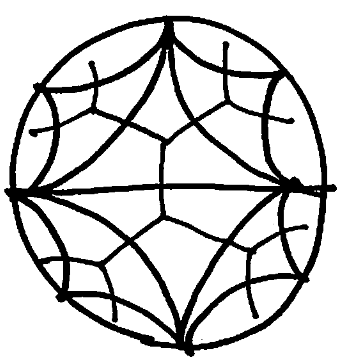
(ii) If

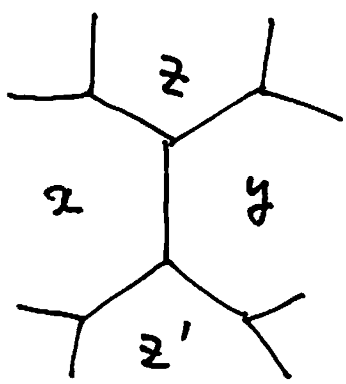


and $z' = \phi(s_2')$

then $z + z' = xy$

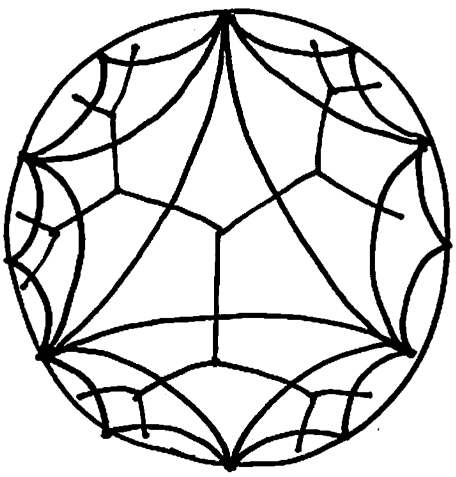
i.e.

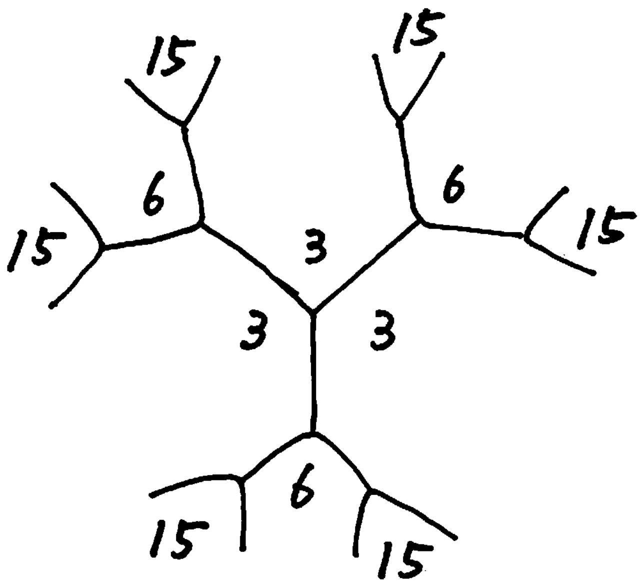




$$x^2 + y^2 + z^2 = xyz$$
$$z + z' = xy$$

Example The (unique) integral Markoff map.





Remark Any integral solution of the Markoff identity $x^2 + y^2 + z^2 = xyz$ arises in this way.

Example trivial Markoff map $\phi : \hat{\mathbb{Q}} \to \mathbb{C}$

$$\phi(s) = 0 \quad ({}^\forall s \in \hat{\mathbb{Q}})$$

Fact $\tilde{\mathcal{X}} = \{ \rho : \pi_1(T) \to SL(2,\mathbb{C}) \text{ type-pres} \} / \text{conj}$

$\cong$

$\bar{\Phi} := \{ \phi : \hat{\mathbb{Q}} \to \mathbb{C} \text{ nontrivial Markoff map} \}$

$\cong$

$\{ (x,y,z) \in \mathbb{C}^3 - \{0\} \mid x^2 + y^2 + z^2 = xyz \}$

Def. Let $\rho \in \tilde{\mathcal{X}}$ and $\phi = \phi_\rho \in \bar{\Phi}$.

Then $\lambda \in \hat{\mathbb{R}} = \mathcal{PML}(T)$ is an end invariant of ρ, if

$\exists \{S_n\} \subset \hat{\mathbb{Q}}$ mutually distinct elements

st (i) $|\phi(S_n)|$ is bounded from the above

(ii) $S_n \to \lambda$ in $\hat{\mathbb{R}}$

The set of the end invariants of ρ is denoted by $\mathcal{E}(\rho) \subset \hat{\mathbb{R}}$.

Bowditch ; Markoff triples and quasi-Fuchsian groups
Proc. London Math Soc. 77 (1998)

Tan, Wong, Zhang : End invariants for $SL(2,\mathbb{C})$ characters of the one-holed torus.
Amer. J. Math 130 (2008)

Examples

1. If Φ is the integral Markoff map then $\mathcal{E}(\rho) = \emptyset$ and

$$\rho : \pi_1(T) \longrightarrow PSL(2, \mathbb{R}) \text{ faithful discrete}$$

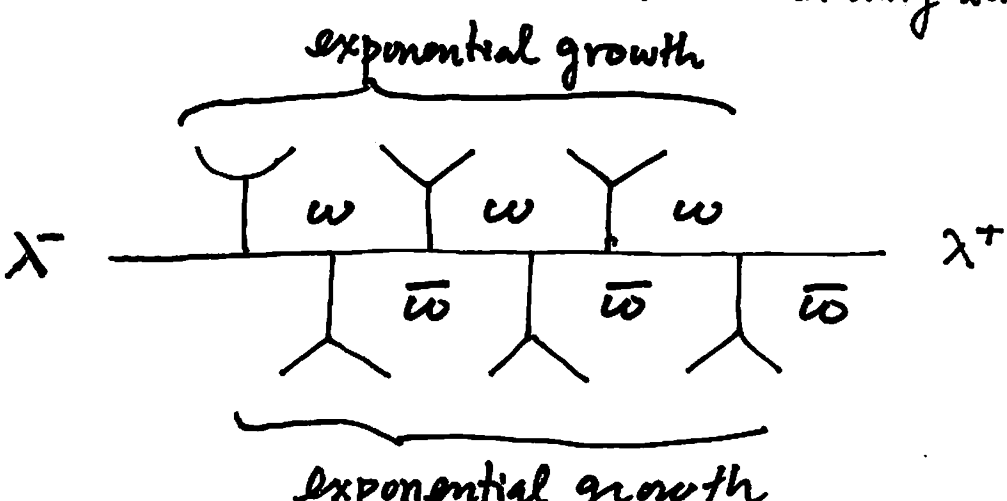
2. If ρ is quasi-fuchsian, then $\mathcal{E}(\rho) = \emptyset$

3. $\{ \rho \in \mathcal{X} \mid \mathcal{E}(\rho) = \emptyset \}$ is open

4. If ρ is faithful discrete, then

$$\mathcal{E}(\rho) = \begin{cases} \emptyset & \\ \{\lambda\} & \text{if } \rho : \text{singly degenerate} \\ \{\lambda^-, \lambda^+\} & \text{if } \rho : \text{doubly degenerate} \end{cases}$$

Here $\lambda^{\pm}$ are Thurston's ending laminations.



where $\omega = \frac{3+\sqrt{3}i}{2}$ $\lambda^{\pm} = \frac{-1 \pm \sqrt{5}}{2}$

[Tan-Wong-Zhang]

$\mathcal{E}(\rho)$ is a closed subset of $\hat{\mathbb{R}} = PML$ which has no interior points.

<u>Dendrite Conjecture</u> (Bowditch, Tan-Wong-Zhang)

If $\mathcal{E}(\rho)$ has more than 2 elements, then $\mathcal{E}(\rho)$ is a Cantor set.

[Tan-Wong-Zhang, S]

There exists $\rho \in \mathcal{X}$, st $\mathcal{E}(\rho)$ is a Cantor set.

[Tan-Wong-Zhang]

The dendrite conjecture is valid if ρ is a "discrete" representation or a pure-imaginary representation.

(ie $\phi(s) \in \mathbb{R} \cup i\mathbb{R}$ for ${}^{\forall}s \in \hat{\mathbb{Q}}$)

[Minsky's ending lamination theorem]

Two faithful discrete representations
ρ and ρ' are conjugate,
iff they have the same Thurston's
end invariants.

In particular, two doubly doubly
degenerate faithful discrete representations
ρ and ρ' are conjugate,
iff $\mathcal{E}(\rho) = \mathcal{E}(\rho')$.

<u>Question</u>

What can we say about representations
which are not necessarily discrete or
faithful ?

Analogy of the ending lamination conjecture

<u>Conjecture</u> (Tan-Wong-Zhang)

If $\mathcal{E}(\rho)$ has at least 2 elements, then ρ is "essentially" determined by $\mathcal{E}(\rho)$.

(To be precise:
Suppose $\mathcal{E}(\rho) = \mathcal{E}(\rho')$ has at least 2 elements. Then ρ' is conjugate to $\rho \circ \varphi_*$ for some

$$\varphi \in \mathrm{Aut}(\mathcal{D}, \mathcal{E}(\rho)) < \mathrm{Aut}(\mathcal{D}) \cong \pi_0 \mathrm{Diff}(T).$$

$$\begin{tikzcd} \pi_1(T) \arrow[d, "\varphi_*"] \arrow[dr, "\rho\circ\varphi_*"] & \\ \pi_1(T) \arrow[r, "\rho"'] & SL(2,\mathbb{C}) \end{tikzcd}$$
)

[Tan-Yamashita-S]

The above conjecture is not valid in a more general setting.

Lemma $\rho \in \tilde{\mathcal{X}}$ induces a $PSL(2.\mathbb{C})$-representation of $G(K(r))$, iff

$$\Phi(\infty) = \Phi(r) = 0$$

$\because$ Since $G(K(r)) = \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$,

we have :

ρ induces $G(K(r)) \longrightarrow PSL(2.\mathbb{C})$

$$\Leftrightarrow \quad \underset{\rho(\beta_\infty^2)}{\underset{\shortparallel}{\rho(\alpha_\infty)}} = \underset{\rho(\beta_r^2)}{\underset{\shortparallel}{\rho(\alpha_r)}} = 1 \in PSL(2.\mathbb{C})$$

$\Leftrightarrow$ $\rho(\beta_\infty)$, $\rho(\beta_r)$ are elliptic of order 2

$\Leftrightarrow$ $\Phi(\infty) = \Phi(r) = 0$

$\square$

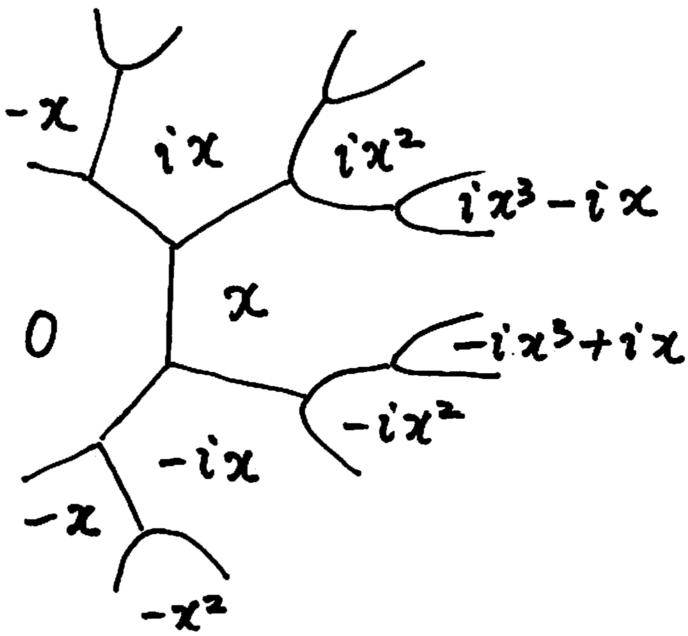
Observation If $\phi(\infty) = 0$,

then for any $\gamma \in \Gamma_\infty$,

$$\phi(\gamma(s)) = \pm \phi(s) \quad (\forall s \in \hat{\mathbb{Q}})$$

In particular $\infty \in \mathcal{E}(\rho)$

$\because$



Cor If $\phi(\infty) = \phi(r) = 0$

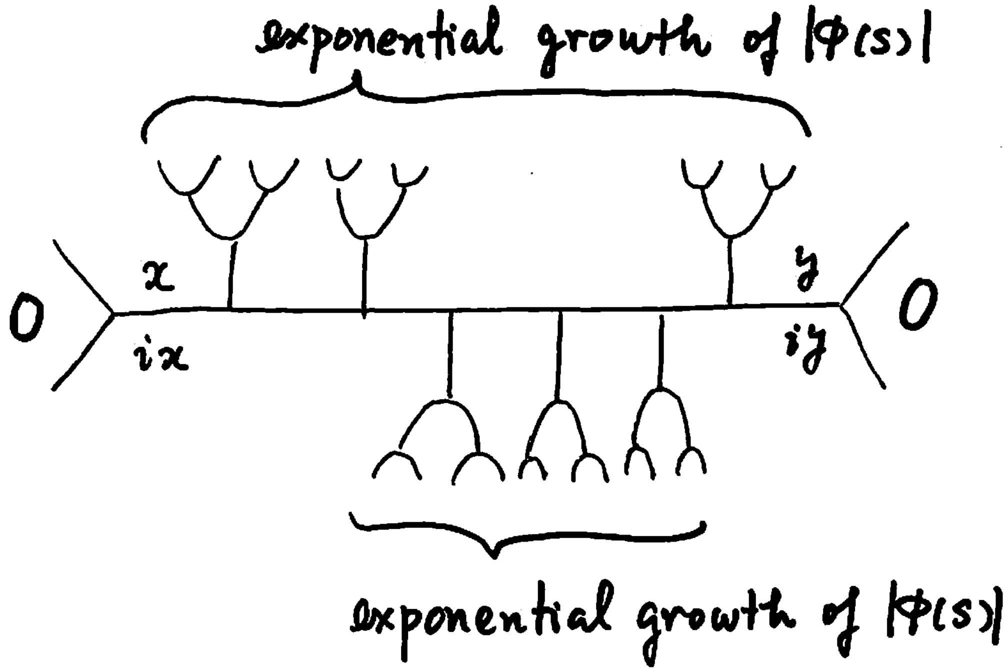
(ie if ρ induces a representation of $G(K(r))$)

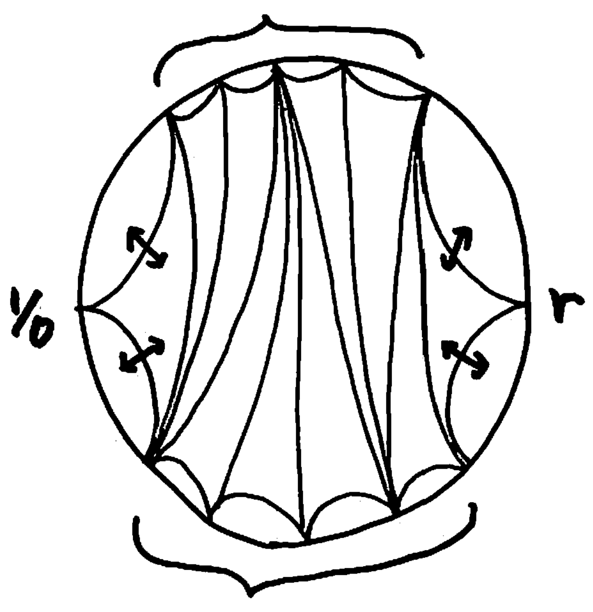
then $\mathcal{E}(\rho) \supset \Lambda(\hat{\Gamma}_r)$

Conjecture B

If ρ_r induces a faithful discrete representation of $G(K(r))$, then $\mathcal{E}(\rho_r) = \Lambda(\hat{\Gamma}_r)$.

In other words,





The space of Kleinian groups

- $\mathcal{X} \supset \mathcal{X}_{\text{discrete}} := \{ \operatorname{Im}(\rho) \text{ is discrete} \}$

$$\begin{array}{l} \quad\;\; \cup \\ \mathcal{D} := \{ \text{faithful \& discrete} \} \\ \quad\;\; \shortparallel \\ \overline{QF} \\ \quad\;\; \cup \\ QF := \{ \text{quasi fuchsian} \} \\ \quad\;\; \uparrow \\ \text{open set} \cong \mathbb{H}^2 \times \mathbb{H}^2 \end{array}$$

- For each (hyperbolic) 2-bridge knot $K(r)$ the element $\rho_r \in \mathcal{X}_{\text{discrete}} \subset \mathcal{X}$ inducing a faithful discrete representation of $G(K(r))$ is an isolated point of $\mathcal{X}_{\text{disc.}}$

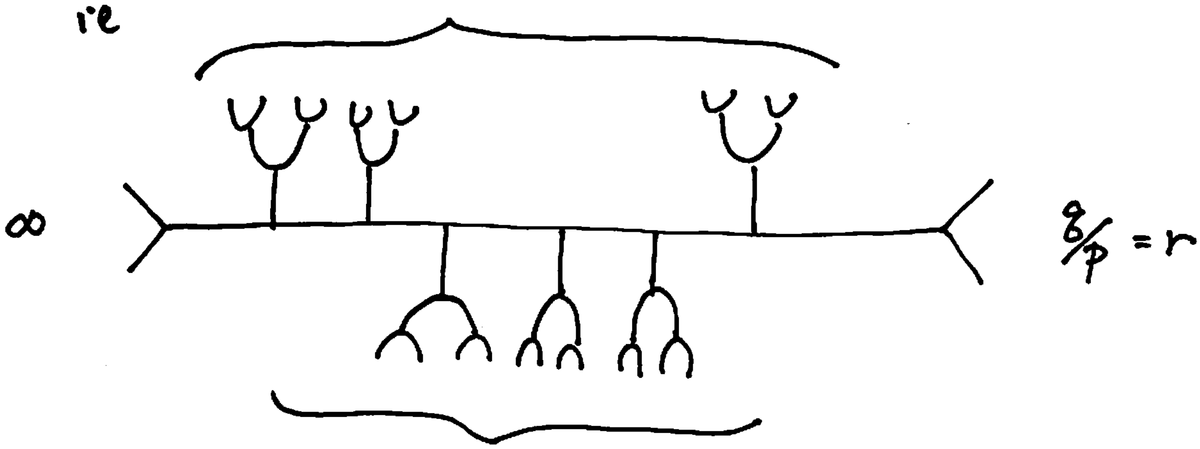
[Akiyoshi - S - Wada - Yamashita] LNM vol 1909

There is a natural path (actually a plane) in $\mathcal{X}$ joining the isolated point ρ_r with the open set QF.

Conjecture C

For every representation ρ in the natural path joining ρ_r with $Q\mathcal{F}$, the latter assertion in Conjecture B holds.

i.e.



exponential growth of $|\phi(s)|$

By [Tan-Wang-Zhang],
the subset of the natural path consisting those ρ satisfying Conj C is non-empty and open.

Question Is it closed?

Proposition

For each representation ρ in the natural path joining ρ_r with $Q7$, the Markoff map $\Phi = \Phi_\rho$ does not vanish at the following slopes.