Splitting curves on a rational ruled surface, the Mordell-Weil groups of hyperelliptic fibrations and Zariski pairs

Hiro-o TOKUNAGA

Abstract

Let Σ be a smooth projective surface, let $f' : S' \to \Sigma$ be a double cover of Σ and let $\mu : S \to S'$ be the canonical resolution. Put $f = f' \circ \mu$. An irreducible curve C on Σ is said to be a splitting curve with respect to f if f^*C is of the form $C^+ + C^- + E$, where $C^- = \sigma_f^* C^+$, σ_f being the covering transformation of f and all irreducible components of E are contained in the exceptional set of μ. In this article, we show that a kind of "reciprocity" of splitting curves holds for a certain pair of curves on rational ruled surfaces. As an application, we consider the topology of the complements of certain curves on rational ruled surfaces.

Introduction

Let Y be a smooth projective variety, and let X' be a double cover, i.e., a normal variety with a finite surjective morphism $f' : X' \to Y$ of $\deg f' = 2$. We denote its smooth model by X and its resolution by $\mu_{f'} : X \to X'$ and put $f = f' \circ \mu_{f'}$. Let $\sigma_{f'}$ be the covering transformation of f' and we assume that $\sigma_{f'}$ inducdes an involution σ_f on X.

Definition 0.1 Let $\mathcal{D}$ be an irreducible divisor on Y.

(i) We say that D is a splitting divisor with respect to $f : X \to Y$ if $f^*\mathcal{D}$ is of the form

$$f^*\mathcal{D} = \mathcal{D}^+ + \mathcal{D}^- + E,$$

where $\mathcal{D}^- = \sigma_f^*\mathcal{D}^+$, and irreducible components of E are all exceptional divisors of $\mu_{f'}$. In case of $\dim Y = 2$, we call $\mathcal{D}$ a splitting curve.

(ii) If f' is uniquely determined by the branch locus $\Delta_{f'}$ of f', then we say that D is a splitting divisor with respect to $\Delta_{f'}$ (see §1 for the terminologies for branched covers). Note that if Y is simply connected, then any double cover is uniquely determined by its branch locus.

Remark 0.1 In the case of $\dim Y = 2$, if $\mathcal{D}$ is a splitting curve on Y with respect to $f : X \to Y$, $\mathcal{D}$ does not meet $\Delta_{f'}$ transversely at smooth parts of $\mathcal{D}$ and $\Delta_{f'}$.

In this article, we study splitting curves on a smooth surface from two different viewpoint:

(I) The study of dihedral covers and their application.

(II) A problem motivated by elementary number theory, i.e., to formulate "reciprocity law" of double covers.

In order to illustrate how the notion of splitting divisors works in the study of dihedral covers, let us recall some results in [2] (For notations and terminology, see §1):

Let C be a smooth conic in $\mathbb{P}^2$ and let $f : Z \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ with branch locus $\Delta_f = C$. Note that $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$. We denote the class of a divisor in $\mathrm{Pic}(Z)$ by a pair of integers (a, b). Then we have:

Proposition 0.1 ([2, Proposition 2]) *For an irreducible curve D in $\mathbb{P}^2$, there exists a D_{2n}-cover $\pi : X \to \mathbb{P}^2$ such that*

- $\Delta_\pi = C + \mathcal{D}$, *and*
- *the ramification index along C (resp. $\mathcal{D}$) is 2 (resp. n) (we say that f is branched at $2C + n\mathcal{D}$ if these conditions on the ramification are satisfied),*

if and only if both of two condititons below are stasified:

(i) $\mathcal{D}$ is a splitting curve with respect to C.

(ii) If we put $f^*\mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-$ and denote the class of $\mathcal{D}^+$ by (a, b), then $a - b$ is divisible by n.

In [2], we construct nodal rational curves $\mathcal{D}_1, \ldots, \mathcal{D}_k$ of degree m in $\mathbb{P}^2$ as follows:

- k is an integer not exceeding $m/2$.

- For each i, $\mathcal{D}_i$ is tangent to C at m distinct smooth points of $\mathcal{D}_i$.
- $f^*\mathcal{D}_i = \mathcal{D}_i^+ + \mathcal{D}_i^-, \mathcal{D}_i^+ \sim (m-i, i), \mathcal{D}_i^- \sim (i, m-i)$.

By Propositin 0.1, we can show that the following statement:

Proposition 0.2 *For any $i, j (i \neq j)$, there exists no homeomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ such that $f(C \cup \mathcal{D}_i) = C \cup \mathcal{D}_j$ for any i and j, i.e., there is no homeomorphism between pairs $(\mathbb{P}^2, C \cup \mathcal{D}_i)$ and $(\mathbb{P}^2, C \cup \mathcal{D}_j)$. Namely $(C \cup \mathcal{D}_1, \ldots, C \cup \mathcal{D}_k)$ is a Zarski k-plet (See [1] for the definition of a Zariski k-plet or a Zariski pair).*

In Proposition 0.2, one of clues is that $\mathcal{D}_i$ is rational, and it is a splitting curve with respect to C. In this article, we consider the case that $\mathcal{D}_i$ is *non-rational*.

We also remark that in [10] Shimada intensively studied splitting curves of degree ≤ 2 on $\mathbb{P}^2$ with respect to sextic curves with simple singularities. Such splitting curves play essential role to classify so called *lattice Zariski pairs*.

As for the viewpoint (II), let us recall a fact from number theory:

Let m be a square free integers and put $K = \mathbb{Q}(\sqrt{m})$. We denote the ring of integers of K by O_K and the discriminant of K by δ_K. Let p be an odd prime with $p \not| \delta_K$ and let (p) be the ideal of O_K generated by p. Then the statements below hold (See [7, Proposition 13.1.3], p.190, for example) :

(i) If $x^2 \equiv m \bmod p$ is solvable in $\mathbb{Z}$, then $(p) = \mathfrak{p}_1\mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are distinct prime ideals in O_K.

(ii) If $x^2 \equiv m \bmod p$ is not solvable in $\mathbb{Z}$, then (p) is a prime ideal in O_K.

Hence whether (p) splits or not depends on the solvability of $x^2 \equiv m \bmod p$. Moreover law of quadratic reciprocity gives a relation between the solvability of $x^2 \equiv q \bmod p$ and that of $x^2 \equiv p \bmod q$ for odd primes.

These facts suggest us to formulate the following problem:

Problem 0.1 Let Σ be a smooth projective surface. Let D_1, D_2 and D_3 be reduced divisors on Σ, We denote the irreducible decompostion of D_i $(i = 1, 2)$ by $D_i = \sum_j D_{i,j}$ $(i = 1, 2)$, respectively. Suppose that there exist double covers $p_i : \mathcal{S}_i \to \Sigma$ with $\Delta_{p_i} = D_i + D_3$ for $i = 1, 2$. Is there any law to determine whether $p_1^* D_{2,j}$ splits or not in terms of some propeties of $\mathcal{S}_2$?

In this article, keeping the viewpoint (I), in particular, application to the study of Zariski pairs (§5), in mind, we consider Problem 0.1 under the following setting:

Let Σ_d be the Hirzebruch surface of degree d. Throughout Introduction, we assume that d **is even**. $\Delta_{0,d}$ denotes the negative section, i.e., the section whose self-intersection number is $-d$ and F_d denotes a fiber of $\Sigma_d \to \mathbb{P}^1$. Note that $\mathrm{Pic}(\Sigma_d) = \mathbb{Z}\Delta_{0,d} \oplus \mathbb{Z}F_d$.

Let T_d be an irreducible divisor on Σ_d such that

(i) $T_d \sim (2g+1)(\Delta_{0,d} + dF_d)$ $(g \geq 1)$ and

(ii) T_d has only nodes (resp. at worst simple singularities) if $g \geq 2$ (resp. $g = 1$).

Let Δ be a section of Σ_d such that

(i) $\Delta \sim \Delta_{0,d} + dF_d$ and,

(ii) for all $x \in T_d \cap \Delta$, x is a smooth point of T_d and the local intersection number $(\Delta \cap T_d)_x$ at x is even.

Let $q_d : W_d \to \Sigma_d$ be a double cover branched at $2(\Delta_{0,d} + \Delta)$. Let $p'_d : S'_d \to \Sigma_d$ be a double cover branched at $2(\Delta_{0,d} + T_d)$ and let $\mu_d : S_d \to S'_d$ be the canonical resolution in the sense of [6]. Put $p_d = p'_d \circ \mu_d$. We can easily check the following properties:

- W_d is the Hirzebruch surface of degree $d/2$.
- The composition $\varphi_d : S_d \to \Sigma_d \to \mathbb{P}^1$ gives a hyperelliptic fibration of genus g on S_d. The preimage of $\Delta_{0,d}$ in S_d gives a section O. We denote the Mordell-Weil group of the Jacobian of the geneiric fiber $(S_d)_\eta$ by $\mathrm{MW}(\mathcal{J}_{S_d})$, O being the zero element.
- $p_d^*\Delta$ is of the form
$$p_d^*\Delta = s^+ + s^-,$$
where $s^\pm$ are sections of φ_d and $s^- = -s^+$ in $\mathrm{MW}(\mathcal{J}_{S_d})$. In particular, Δ is a splitting curve with respect to p_d.

Now we are in position to state our main result in this article:

Theorem 0.1 *Under the notation as above, T_d is a splitting curve with respect to q_d, if and only if s^+ is 2-divisible in* $\mathrm{MW}(\mathcal{J}_{S_d})$.

Note that Theorem 0.1 can be regarded as an analogy of the "reciprocity." In fact, we may interpret it as a double cover version of

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

It may be interesting problem to consider the "reciprocity" of double covers under more general setting.

As an application of Theorem 0.1, we study branched covers with branch locus $\Delta_{0,d} + \Delta + T_d$ and the topology of the pair $(\Sigma_d, \Delta_{0,d} \cup \Delta \cup T_d)$. Here is our statement:

Theorem 0.2 *We keep the notation as before. There exists a D_{2n}-cover branched at $2(\Delta_{0,d} + \Delta) + nT_d$ for any odd number n, if and only if s^+ is 2-divisible in $\mathrm{MW}(\mathcal{J}_{S_d})$*

An interesting corollary to Theorem 0.2 is as follows:

Corollary 0.1 *Let $(\Delta_1 \cup T_{d,1}, \Delta_2 \cup T_{d,2})$ be a pair of reduced divisors on Σ_d such that both of $\Delta_i \cup T_{d,i}$ $(i = 1, 2)$ satisfy the conditions of Δ and T_d in Theorem 0.1. Let $p_{d,i} : S_{d,i} \to \Sigma_d$ $(i = 1, 2)$ be the canonical resolutions of the double covers with branch locus $\Delta_{0,d} + T_{d,i}$ $(i = 1, 2)$, respectively. Let s_i^+ $(i = 1, 2)$ be sections coming from Δ_i $(i = 1, 2)$, respectively. If (i) s_1^+ is 2-divisible in $\mathrm{MW}(\mathcal{J}_{S_{d,1}})$ and (ii) s_2^+ is not 2-divisible in $\mathrm{MW}(\mathcal{J}_{S_{d,2}})$, then there is no homeomorphism $h : \Sigma_d \to \Sigma_d$ such that $h(\Delta_{0,d}) = \Delta_{0,d}$, $h(\Delta_1) = \Delta_2$ and $h(T_{d,1}) = T_{d,2}$.*

This article consists of 4 sections. In §1, we summarize on Galois covers, especially, dihedral covers, and the Mordell-Weil group of the Jacobian of the generic fiber of a fibered surface. We prove Theorems 0.1 and 0.2 in §2 and §3, respectively. In §4, we give some explicit examples in the case when $d = 2, g = 1$. In the last section, we will see that the study of splitting curves gives some examples of Zariski pairs, which is related our viewpoint I.

1 Preliminaries

1.1 Summary on Galois covers

1. Generalities

Let G be a finite group. Let X and Y be a normal projective varieties. We call X a G-cover, if there exists a finite surjective morphism $\pi : X \to Y$ such that the finite field extension given by $\pi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$ is a Galois extension with $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$. We denote the branch locus of π by Δ_π. Δ_π is a reduced divisor if Y is smooth([17]). Let B be a reduced divisor on Y and we denote its irreducible decomposition by $B = \sum_{i=1}^r B_i$. We say that a G-cover $\pi : X \to Y$ is branched at $\sum_{i=1}^r e_i B_i$ if

(i) $\Delta_\pi = \mathrm{Supp}(\sum_{i=1}^r e_i B_i)$ and

(ii) the ramification index along B_i is e_i for $1 \le i \le r$.

2. Cyclic covers and double covers

Let $\mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order n. We call a $\mathbb{Z}/n\mathbb{Z}$- (resp. a $\mathbb{Z}/2\mathbb{Z}$-) cover by an n-cyclic (resp. a double) cover. We here summarize some facts on cyclic and double covers. We first remark the following fact on cyclic covers.

Fact: Let Y be a smooth projective variety and B a reduced divisor on Y. If there exists a line bundle $\mathcal{L}$ on Y such that $B \sim n\mathcal{L}$, then we can construct a hypersurface X in the total space, L, of $\mathcal{L}$ such that

- X is irreducible and normal, and
- $\pi := \mathrm{pr}|_X$ gives rise to an n-cyclic cover, pr being the canonical projection $\mathrm{pr} : L \to Y$.

(See [3] for the above fact.)

As we see in [14], cyclic covers are not always realized as a hypersurface of the total space of a certain line bundle. As for double covers, however, the following lemma holds.

Lemma 1.1 *Let $f : X \to Y$ be a double cover of a smooth projective variety with $\Delta_f = B$, then there exists a line bundle $\mathcal{L}$ such that $B \sim 2\mathcal{L}$ and X is obtained as a hypersurface of the total space, L, of $\mathcal{L}$ as above.*

Proof. Let φ be a rational function in $\mathbb{C}(Y)$ such that $\mathbb{C}(X) = \mathbb{C}(Y)(\sqrt{\varphi})$. By our assumption, the divisor of φ is of the form

$$(\varphi) = B + 2D,$$

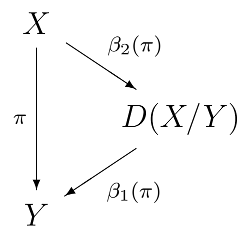
where D is a divisor on Y. Choose $\mathcal{L}$ as the line bundle determined by $-D$. This implies our statement. $\square$

By Lemma 1.1, note that any double cover X over Y is determined by the pair $(B, \mathcal{L})$ as above.

3. Dihedral covers

We next explain dihedral covers briefly. Let D_{2n} be a dihedral group of order $2n$ given by $\langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1 \rangle$. In [16], we developed a method of dealing with D_{2n}-covers. We need to introduce some notation in order to describe it.

Let $\pi : X \to Y$ be a D_{2n}-cover. By its definition, $\mathbb{C}(X)$ is a D_{2n}-extension of $\mathbb{C}(Y)$. Let $\mathbb{C}(X)^\tau$ be the fixed field by τ. We denote the $\mathbb{C}(X)^\tau$- normalization by $D(X/Y)$. We denote the induced morphisms by $\beta_1(\pi) : D(X/Y) \to Y$ and $\beta_2(\pi) : X \to D(X/Y)$. Note that X is a $\mathbb{Z}/n\mathbb{Z}$-cover of $D(X/Y)$ and $D(X/Y)$ is a double cover of Y such that $\pi = \beta_1(\pi) \circ \beta_2(\pi)$:



In [16], we have the following results for D_{2n}-covers (n:odd).

Proposition 1.1 *Let n be an odd integer with $n \geq 3$. Let $f : Z \to Y$ be a double cover of a smooth projective variety Y, and assume that Z is smooth. Let σ_f be the covering transformation of f. Suppose that there exists a pair $(D, \mathcal{L})$ of an effective divisor and a line bundle on Y such that*

(i) D and $\sigma_f^* D$ have no common components,

(ii) if $D = \sum_i a_i D_i$ denotes the irreducible decomposition of D, then $0 < a_i \leq (n-1)/2$ for every i;and the greatest common divisor of the a_i's and n is 1, and

(iii) $D - \sigma_f^* D$ is linearly equivalent to $n\mathcal{L}$.

Then there exists a D_{2n}-cover, X, of Y such that (a) $D(X/Y) = Z$, (b) $\Delta(X/Y) = \Delta_f \cup f(\operatorname{Supp} D)$ and (c) the ramification index along D_i is $n/\gcd(a_i, n)$ for $\forall i$.

For a proof, see [16]. A corollary related to a splitting divisor on Y, we have the following:

Corollary 1.1 *Let $\mathcal{D}$ be a splitting divisor on Y with respect to $f : Z \to Y$. Put $f^*\mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-$. If there exists a line bundle $\mathcal{L}$ on Z such that $\mathcal{D}^+ - \mathcal{D}^- \sim n\mathcal{L}$ for an odd number n, then there exists a D_{2n}-cover $\pi : X \to Y$ branched at $2\Delta_f + n\mathcal{D}$.*

Conversely we have the following:

Proposition 1.2 *Let $\pi : X \to Y$ be a D_{2n}-cover ($n \geq 3$, n: odd) of Y and let $\sigma_{\beta_1(\pi)}$ be the involution on $D(X/Y)$ determined by the covering transformation of $\beta_1(\pi)$. Suppose that $D(X/Y)$ is smooth. Then there exists a pair of an effective divisor and a line bundle $(D, \mathcal{L})$ on $D(X/Y)$ such that*

(i) *D and $\sigma^*_{\beta_1(\pi)}D$ have no common component,*

(ii) *if $D = \sum_i a_i D_i$ denotes the decomposition into irreducible components, then $0 \leq a_i \leq (n-1)/2$ for every i,*

(iii) *$D - \sigma^*_{\beta_1(\pi)}D \sim n\mathcal{L}$, and*

(iv) *$\Delta_{\beta_2(\pi)} = \mathrm{Supp}(D + \sigma^*_{\beta_1(\pi)}D)$.*

For a proof, see [16].

Corollary 1.2 *Let D_i be an arbitrary irreducible component of D in Proposition 1.2. The image $\beta_1(\pi)(D_i)$ is a splitting divisor with respect to $\beta_1(\pi) : D(X/Y) \to Y$*

1.2 A review on the Mordell-Weil groups for fibrations over curves

In this section, we review the results on the Mordell-Weil group and the Mordell-Weil lattices studied by Shioda in [11, 12].

Let S be a smooth algebraic surface with fibration $\varphi : S \to C$ of genus $g(\geq 1)$ curves over a smooth curve C. Throughout this section, we assume that

- φ has a section O and
- φ is relatively minimal, i.e., no (-1) curve is contained in any fiber.

Let S_η be the generic fiber of φ and let $K = \mathbb{C}(C)$ be the rational function field of C. S_η is regarded as a curve of genus g over K.

Let $\mathcal{J}_S := J(S_\eta)$ be the Jacobian variety of S_η. We denote the set of rational points over K by $\mathrm{MW}(\mathcal{J}_S)$. By our assumption, $\mathrm{MW}(\mathcal{J}_S) \neq \emptyset$ and it is well-known that $\mathrm{MW}(\mathcal{J}_S)$ has the structure of an abelian group.

Let $\mathrm{NS}(S)$ be the Néron-Severi group of S and let $Tr(\varphi)$ be the subgroup of $\mathrm{NS}(S)$ generated by O and irreducible components of fibers of φ. Under these notation, we have:

Theorem 1.1 *If the irregularity of S is equal to C, then* $\mathrm{MW}(\mathcal{J}_S)$ *is a finitely generated abelian group such that*

$$\mathrm{MW}(\mathcal{J}_S) \cong \mathrm{NS}(S)/Tr(\varphi).$$

See [11, 12] for a proof.

Let $p_d : S_d \to \Sigma_d$ be the double cover of Σ_d with branch locus $\Delta_{0,d} + T_d$ as in Introduction. Then we have

Lemma 1.2 *Let $p_d : S_d \to \Sigma_d$ be the double cover as before. There exists no unramified cover of S_d. In particular,* $\mathrm{Pic}(S_d)$ *has no torsion element.*

Proof. By Brieskorn's results on the simultaneous resolution of rational double points, we may assume that T_d is smooth. Since the linear system $|T_d|$ is base point free, it is enough to prove our statement for one special case. Chose an affine open set U_d of Σ_d isomorphic to $\mathbb{C}^2$ with a coordinate (x, t) so that a curve $x = 0$ gives rise to a section linear equivalent to Δ_∞. Choose T_d whose defining equation in U_d is

$$T_d : F_{T_d} = x^{2g+1} - \Pi_{i=1}^{(2g+1)d}(t - \alpha_i) = 0,$$

where α_i $(i = 1, \ldots, (2g+1)d)$ are distinct complex numbers. Note that

- T_d is smooth,
- singulair fibers of φ are over α_i $(i = 1, \ldots, (2g+1)d)$, and
- all the singular fibers are irreducible rational curves with unique singularity isomorphic to $y^2 - x^{2g+1} = 0$.

Suppose that $\widehat{S}_d \to S_d$ be any unramified cover, and let $\hat{g} : \widehat{S}_d \to \mathbb{P}^1$ be the induced fiberation. We claim that $\hat{g}$ has a connected fiber. Let $\widehat{S}_d \xrightarrow{\rho_1} C \xrightarrow{\rho_2} \mathbb{P}^1$ be the Stein factorization and let $\widehat{O}$ be a section coming from O. Then $\deg(\rho_2 \circ \rho_1)|_{\hat{g}} = \deg \hat{g}|_{\widehat{O}} = 1$, and $\hat{g}$ has a connected fiber.

On the other hand, since all the singular fibers of g are simply connected, all fibers over α_i ($i = 1, \ldots, (2g+1)d$) are disconnected. This leads us to a contradiction. □

Corollary 1.3 *The irreguarity $h^1(S_d, \mathcal{O}_{S_d})$ of S_d is 0. In particular,*

$$\mathrm{MW}(\mathcal{J}_{S_d}) \cong \mathrm{NS}(S_d)/Tr(\varphi),$$

where $Tr(\varphi)$ denotes the subgroup of $\mathrm{NS}(S_d)$ introduced as above.

Proof. By Lemma 1.2, we infer that $H^1(S_d, \mathbb{Z}) = \{0\}$. Hence the irregularity of S_d is 0. □

2 Proof of Theorem 0.1

Let us start with the following lemma:

Lemma 2.1 *$f : X \to Y$ be the double cover of Y determined by $(B, \mathcal{L})$ as in Lemma 1.1. Let Z be a smooth subvariety of Y such that (i) $\dim Z > 0$ and (ii) $Z \not\subset B$. We denote the inclusion morphism $Z \hookrightarrow Y$ by ι. If there exists a divisor B_1 on Z such that*

- *$\iota^* B = 2B_1$ and*
- *$\iota^* \mathcal{L} \sim B_1$,*

then the preimage $f^{-1}Z$ splits into two irreducible components Z^+ and Z^-.

Proof. Let $f|_{f^{-1}(Z)} : f^{-1}(Z) \to Z$ be the induced morphism. $f^{-1}(Z)$ is realized as a hypersurface in the total space of $\iota^* L$ as in usual manner (see [3, Chapter I, §17], for example). Our condition implies that $f^{-1}(Z)$ is reducible. Since $\deg f = 2$, our statement holds. □

Lemma 2.2 *Let Y be a smooth projective variety, let $\sigma : Y \to Y$ be an involution on Y, let R be a smooth irreducible divisor on Y such that $\sigma|_R$ is the identity, and let B be a reduced divisor on Y such that $\sigma^* B$ and B have no common component.*

If there exists a σ-invariant divisor D on Y (i.e., $\sigma^* D = D$) such that

- *$B+D$ is 2-divisible in* $\mathrm{Pic}(Y)$*, and*

- *R is not contained in* $\mathrm{Supp}(D)$*,*

then there exists a double cover $f : X \to Y$ branched at $2(B+\sigma^*B)$ such that R is a splitting divisor with respect to f.

Moreover, if there is no 2-torsion in $\mathrm{Pic}(Y)$*, then R is a splitting divisor with respect to $B+\sigma^*B$.*

Proof. By our assumption and Y is projective, there exists a divisor D_o on Y such that

1. R is not contained in $\mathrm{Supp}(D_o)$, and

2. $B+D \sim 2D_o$.

Hence $B+\sigma^*B \sim 2(D_o+\sigma^*D_o-D)$ Let $f : X \to Y$ be a double cover determined by $(Y, B+\sigma^*B, D_o+\sigma^*D_o-D)$ and let $\iota : R \hookrightarrow Y$ denote the inclusion morphism. Since $\sigma|_R = \mathrm{id}_R$,

$$\iota^*B = \iota^*\sigma^*B, \qquad \iota^*(D_o-D) = \iota^*(\sigma^*D_o-D),$$

we have

$$\begin{aligned} \iota^*B &\sim \iota^*(2D_o-D) \\ &= \iota^*D_o + \iota^*(\sigma^*D_o-D) \\ &= \iota^*(D_o+\sigma^*D_o-D). \end{aligned}$$

Hence, by Lemma 2.1, R is a splitting divisor with respect to f. Moreover, if there is no 2-torsion in $\mathrm{Pic}(Y)$, f is determined by $B+\sigma^*B$. Hence R is a splitting divisor with respect to $B+\sigma^*B$. □

Proposition 2.1 *Let $p_d : S_d \to \Sigma_d$ and $q_d : W_d \to \Sigma_d$ be the double covers as in Introduction. If there exists a σ_{p_d}-invariant divisor D on S_d such that $s^+ + D$ is 2-divisible in* $\mathrm{Pic}(S_d)$*, σ_{p_d} being the covering transformation of p_d, then T_d is a splitting divisor with respect to $\Delta_{0,d}+\Delta$.*

Proof. Let ψ_1 and ψ_2 be rational function on Σ_d such that $\mathbb{C}(W_d) = \mathbb{C}(\Sigma_d)(\sqrt{\psi_1})$ and $\mathbb{C}(S'_d)(=\mathbb{C}(S_d)) = \mathbb{C}(\Sigma_d)(\sqrt{\psi_2})$, respectively. Note that $(\psi_1) = \Delta_{0,d} + \Delta + 2D_1$ and $(\psi_2) = \Delta_{0,d} + T_d + 2D_2$ for some divisors D_1 and D_2 on Σ_d. Let X'_d be the $\mathbb{C}(\Sigma_d)(\sqrt{\psi_1}, \sqrt{\psi_2})$-normalization of Σ_d and let $\tilde{p}_d : X_d \to S_d$ be the induced double cover of S_d by the quadratic extension $\mathbb{C}(\Sigma_d)(\sqrt{\psi_1}, \sqrt{\psi_2})/\mathbb{C}(\Sigma_d)(\sqrt{\psi_2})$ and let $\tilde{\mu} : X_d \to X'_d$

be the induced morphsim. X'_d is a bi-double cover of Σ_d as well as a double cover of both W_d and S'_d. We denote the induced covering morphisms by $\tilde{q}_d : X'_d \to W_d$.

$$\begin{array}{ccccc} W_d & \xleftarrow{\tilde{q}_d} & X'_d & \xleftarrow{\tilde{\mu}} & X_d \\ {\scriptstyle q_d}\downarrow & & {\scriptstyle \tilde{p}'_d}\downarrow & & \downarrow{\scriptstyle \tilde{p}_d} \\ \Sigma_d & \xleftarrow{p'_d} & S_d & \xleftarrow{\mu} & S_d \end{array}$$

Since

$$(p_d^*\psi_1) = 2O + s^+ + s^- + 2p_d^*D_1, p_d = p'_d \circ \mu$$

and

$$(q_d^*\psi_2) = 2\Delta_{0,d/2} + q_d^*T_d + 2q_d^*D_2,$$

the branch loci of $\tilde{p}_d$ and $\tilde{q}_d$ are $p_d^*\Delta = s^+ + s^-$ and $q_d^*T_d$, respectively. Put

$$R := (p_d^*T_d)_{red} \setminus (\text{the exceptinonal set of } S_d \to S'_d).$$

Let $(T_d)_{sm}$ be the smooth part of T_d. Since $(q_d \circ \tilde{q}_d \circ \tilde{\mu})^*(T_d)_{sm} = (p_d \circ \tilde{p}_d)^*(T_d)_{sm}$, one can check that T_d is a splitting curve with respect to $\Delta_{0,d} + \Delta$ if and only if R is a splitting curve with respect to $s^+ + s^-$. Now by Lemma 2.2, our statement follows. □

We are now in position to prove Theorem 0.1

Proof of Theorem 0.1 We first note that the algebraic equivalence $\approx$ and the linear equivalence $\sim$ coincides on S_d by Lemma 1.2.

The case of $g \geq 2$. Let s_0 be an element in $\mathrm{MW}(\mathcal{J}_{S_d})$ such that $2s_0 = s^+$ on $\mathrm{MW}(\mathcal{J}_{S_d})$. By [12], there exists a divisor D on S_d which gives s_0. By [12], D satisfies a relation

$$2D \sim s^+ + (2DF - 1)O + \alpha\mathfrak{f} + \Xi,$$

where Ξ is a divisor whose irreducible components consist of those of singular fibers not meeting O. By our assumption on the singularity of T_d, we can infer that any irreducible component of Ξ is σ_{p_d}-invariant. As $\sigma_{p_d}^*O = O$, $\sigma_{p_d}^*\mathfrak{f} = \mathfrak{f}$, by Proposition 2.1, our statement follows.

The case of $g = 1$. Let s_0 be an element in $\mathrm{MW}(\mathcal{J}_{S_d})$ such that $2s_0 = s^+$. By Theorem 1.1 and Corollary 1.3, we have

$$2s_0 - s^+ \in Tr(\varphi).$$

Let $\phi : \mathrm{MW}(\mathcal{J}_{S_d}) \to \mathrm{NS}_{\mathbb{Q}}(:= \mathrm{NS}(S_d) \otimes \mathbb{Q})$ be the homomorphism given in [11, Lemmas 8.1 and 8.2]. Note that there will be no harm in considering $\mathrm{NS}_{\mathbb{Q}}$ since $\mathrm{NS}(S_d)$ is torsion free. By [11, Lemmas 8.1 and 8.2], $\phi(s)$ satisfies the following properties:

(i) $\phi(s) \equiv s \bmod Tr(\varphi)_{\mathbb{Q}}(:= Tr(\varphi) \otimes \mathbb{Q})$.

(ii) $\phi(s)$ is orthogonal to $Tr(\varphi)$.

Explicitly $\phi(s)$ is given by

$$\phi(s) = s - O - (sO + \chi(\mathcal{O}_{S_d}))\mathfrak{f} + \text{the correction terms},$$

where $\mathfrak{f}$ denote the fiber of φ. The correction terms is a $\mathbb{Q}$-divisor arising from reducible singular fiber in the following way:

Let $\mathfrak{f}_v$ be a singular fiber over $v \in \mathbb{P}^1$ and let $\Theta_{v,0}$ be the irreducible component with $O\Theta_0 = 1$.

- If s meets $\Theta_{v,0}$, then there is no correction term from $\mathfrak{f}_v$.
- If s does not meet $\Theta_{v,0}$, the correction term from $\mathfrak{f}_v$ is as follows:

  Let $\Theta_{v,1}, \dots, \Theta_{v,r_v-1}$ denote irreducible components of $\mathfrak{f}_v$ other than $\Theta_{v,0}$ and let $A := ((\Theta_{v,i}\Theta_{v,j}))$ be the intersection matrix of $\Theta_{v,1}, \dots, \Theta_{r_v-1}$. With these notation, the correction term is

$$\sum_i (\Theta_{v,1}, \dots, \Theta_{v,r_v-1})(-A^{-1}) \begin{pmatrix} s\Theta_{v,1} \\ \cdot \\ s\Theta_{v,r_v-1} \end{pmatrix}.$$

By our assumption,

$$\phi(s^+) = s^+ - O - \chi(\mathcal{O}_{S_d})\mathfrak{f}.$$

Put

$$\phi(s_0) = s_0 - O - (s_oO + \chi(\mathcal{O}_{S_d}))\mathfrak{f} + \sum_{v \in \mathrm{Red}} \mathrm{Corr}_v,$$

where $\mathrm{Red} = \{v \in \mathbb{P}^1 | \varphi^{-1}(v) \text{ is reducible.}\}$ and Corr_v denotes the correction term arising from the singular fiber $\mathfrak{f}_v$. Since $2s_0 - s^+ \in Tr(\varphi)$, $\phi(2s_0) - \phi(s^+) = 0$. Hence

$$(*) \quad 2s_0 - s^+ \sim_{\mathbb{Q}} O + (2s_0O + \chi(\mathcal{O}_{S_d}))\mathfrak{f} + 2\sum_{v \in \mathrm{Red}} \mathrm{Corr}_v.$$

Thus

$$2\sum_{v \in \mathrm{Red}} \mathrm{Corr}_v \sim_{\mathbb{Q}} E,$$

for some element $E \in Tr(\varphi)$.

Claim. $2\sum_{v\in \mathrm{Red}} \mathrm{Corr}_v \in Tr(\varphi)$.

Proof of Claim. We first note that $2\sum_{v\in \mathrm{Red}} \mathrm{Corr}_v = E$ in $Tr(\varphi)_{\mathbb{Q}}$. Since $O, \mathfrak{f}$ and all the irreducible components of reducible singular fibers which do not meet O form a basis of the free $\mathbb{Z}$-module $Tr(\varphi)$ as well as the $\mathbb{Q}$-vector space $Tr(\varphi)_{\mathbb{Q}}$, E is expressed as a $\mathbb{Z}$-linear combination of these divisors. As Corr_v is a $\mathbb{Q}$-linear combination of the irreducible components of reducible singular fibers which do not meet O, if $2\sum_{v\in \mathrm{Red}} \mathrm{Corr}_v \notin Tr(\varphi)$, then we have a nontrivial relation among $O, \mathfrak{f}$ and all the irreducible components of reducible singular fibers which do not meet O. This leads us to a contradiction. □

By Claim, we have

(i) $\mathrm{Corr}_v = 0$ if the singular fiber over v is of type either I_n (n: odd), IV or IV^* and

(ii) if $\mathrm{Corr} \neq 0$, one can write Corr_v in such a way that

$$\mathrm{Corr}_v = \frac{1}{2}D_{1,v} + D_{2,v},$$

where $D_{1,v}, D_{2,v} \in Tr(\varphi)$ and $D_{1,v}$ is reduced.

Since $s_0 + \sigma_{p_d}^* s_0 \in Tr(\varphi)$, we have

$$\frac{1}{2}(D_1 + \sigma_{p_d}^* D_1) \in Tr(\varphi).$$

Therefore we infer that we can rewrite D_1 in such a way that

$$D_1 = D_1' + \sigma_{p_d}^* D_1' + D_1'',$$

where

- $D_1' \neq D_1'$ and there is no common component between D_1' and $\sigma_{p_d}^* D_1'$, and
- each irreducible component of D_1'' is σ_{p_d}-invariant.

In particular, D_1 is σ_{p_d}-invariant. Now put

$$\begin{aligned} D &:= O + (2s_0O + \chi(\mathcal{O}_{S_d}) - 2\,[(2s_0O + \chi(\mathcal{O}_{S_d}))/2])\mathfrak{f} + D_1 \\ D_o &:= s_0 - [(2s_0O + \chi(\mathcal{O}_{S_d}))/2]\,\mathfrak{f} - D_2. \end{aligned}$$

Then the relation $(*)$ becomes

$$s^+ + D \sim 2D_o.$$

As $\sigma_{p_d}^* O = O$, $\sigma_{p_d}^* \mathfrak{f} = \mathfrak{f}$, by Proposition 2.1, our statement follows.

We now go on to prove the converse. Choose affine open subsets $V \subset W_d (= \Sigma_{d/2})$, and $U \subset \Sigma_d$ as follows:

(i) Both U and V are $\mathbb{C}^2$.

(ii) Let (t, x) and $(\tilde{t}, \zeta)$ be affine coordinates of U and V, respectively. Then q_d is given by

$$q_d : (\tilde{t}, \zeta) \mapsto (t, x) = (\tilde{t}, \zeta^2 + f(t)),$$

where $f(t)$ is a polynomial of degree $\leq d$. With respect to the coordinate (t, x), Δ_{q_d} is given by $\{x = \infty\} \cup \{x - f(t) = 0\}$

Since T_d is a splitting curve with respect to $q_d : W_d \to \Sigma_d$, we have $q_d^* T_d = T^+ + T^-$. As $T^\pm \sim (2g+1)(\Delta_{0,d/2} + dF_{d/2})$, we may assume that $T^\pm$ are given by the following equations on V:

$$\begin{aligned} T^+ : F(x,t) + \zeta G(x,t) &= 0 \\ T^- : F(x,t) - \zeta G(x,t) &= 0, \end{aligned}$$

where

$$F(x,t) = \sum_{l=0}^{g} a_{2l+1}(t)(x - f(t))^{g-l}, G(x,t) = \sum_{l=0}^{g} a_{2l}(t)(x - f(t))^{g-l}, a_0 = 1,$$

- $x = \zeta^2 + f(t)$, and
- $a_{2l}(t)$ $(l = 1, \ldots, g)$ and $a_{2l+1}(t)$ $(l = 0, \ldots g)$ are polynomials with $\deg a_{2l}(t) \leq dl$ and $\deg a_{2l+1}(t) \leq d(2l+1)/2$, respectively .

This implies that T_d is given by a defining equation of the form

$$F(x,t)^2 - (x - f(t))G(x,t)^2 = 0.$$

On the other hand , the generic fiber of $\varphi_d : S_d \to \mathbb{P}^1$ is given by

$$y^2 = F(x,t)^2 - (x - f(t))G(x,t)^2.$$

By considering the divisors of the rational functions on the generic fiber $(S_d)_\eta$ given by $y - F(x,t)$ an d$y + F(x,t)$ and the right hand side of the above equation, we infer that the sections given by $(f(t), \pm a_{2g+1}(t))$ is 2-divisible in $MW(\mathcal{J}_{S_d})$. AS $s^\pm$ are nothing but these sections, our statement follows.

3 Proof of Theorem 0.2

We first show that 2-divisibility of s^+ in $\mathrm{MW}(\mathcal{J}_{S_d})$ follows from the existence of a D_{2n}-cover for one odd number n.

Suppose that there exists a D_{2n}-cover $\pi_d : \mathcal{X}_d \to \Sigma_d$ branched at $2(\Delta_{0,d} + \Delta) + nT_d$ for some n. Let $\beta_1(\pi_d) : D(\mathcal{X}_d/\Sigma_d) \to \Sigma_d$ be the double cover canonically determined by $\pi_d : \mathcal{X}_d \to \Sigma_d$. As the branch locus of $\beta_1(\pi_d)$ is $\Delta_{0,d} + \Delta$, $D(\mathcal{X}_d/\Sigma_d) = W_d$ and $\beta_1(\pi_d) = q_d$. By Corollary 1.2, T_d is a splitting curve with respect to q_d. By Theorem 0.1, s^+ is 2-divisible in $\mathrm{MW}(\mathcal{J}_{S_d})$.

Conversely, suppose that s^+ is 2-divisible in $\mathrm{MW}(\mathcal{J}_{S_d})$. By Theorem 0.1, T_d is a splitting curve with respect to q_d. Hence we infer that $q_d^*T_d$ is of the form $T^+ + T^-$. Put

$$T^+ \sim a\Delta_{0,d/2} + bF_{d/2}.$$

Since

$$q_d^*T_d \sim (2g+1)(2\Delta_{0,d/2} + dF_{d/2}, \ \sigma_{q_d}^*T^+ = T^- \ \sigma_{q_d}^*\Delta_{0,d/2} = \Delta_{0,d/2} \text{ and } \sigma_{q_d}^*F = F,$$

we have

$$T^+ \sim T^- \sim (2g+1)\Delta_{0,d/2} + \frac{(2g+1)d}{2}F_{d/2}.$$

Hence by Corollary 1.1, There exists a D_{2n}-cover branched at $2\Delta + nT_d$ for any odd n.

4 Examples for the case of $g = 1$

In this section, we consider the case of $g = 1$. Namely, $\varphi : S_d \to \mathbb{P}^1$ is an elliptic fibration over $\mathbb{P}^1$ with section O. In this case, the involution induced by the covering transformation coincides with the one induced by the inversion morphism with respect to the group law on the generic fiber, O being the zero element.

Our main references are [8], [9] and [13]. As for the notation of singular fibers, we follow Kodaira's notation ([8]).

Let us start with the case when $d = 2$, i.e., $\varphi : S_2 \to \mathbb{P}^1$ is a rational elliptic surface.

Example 4.1 ([13, Example, p.198]) Let $\varphi : S_2 \to \mathbb{P}^1$ be the rational elliptic surface given by the following Weierstrass equation:

$$y^2 = x^3 + (271350 - 98t)x^2 + t(t - 5825)(t - 2025)x + 36t^2(t - 2025)^2,$$

t being an inhomogeneous coordinate of $\mathbb{P}^1$. $\varphi : S_2 \to \mathbb{P}^1$ satisfies the following properties:

(i) φ has 3 singular fibers over $t = 0, 2025, \infty$, of which types are of type I_2 over $t = 0, 2025$ and type III over $t = \infty$.

(ii) $\mathrm{MW}(\mathcal{J}_{S_2})$ has no torsion.

We infer that T_2 on Σ_2 given by the right hand side of the Weierstrass equation has 3 nodes (see [9, Table 6.2, p.551]) from (i) and is irreducible from (ii). In order to give an example, we use three sections given by [13] as follows:

$$s_0 : (0, 6t^2 - 12150t),\ s_1 : (-32t, 2t^2 - 6930t),\ s_2 : (-20t, 4t^2 - 4500t).$$

Let $\langle\ ,\ \rangle$ be the height pairing defined in [11]. Then we have

$$\langle s_0, s_0\rangle = \frac{1}{2},\ \langle s_i, s_i\rangle = 1\,(i = 1, 2),\ \langle s_1, s_2\rangle = 0,$$

and there is no other section s with $\langle s, s\rangle = 1/2$ other than $\pm s_0$.

The sections given by $2s_0$ and $s_1 + s_2$ are

$$2s_0 = \left(\frac{1}{144}t^2 + \frac{1231}{72}t - \frac{5143775}{144}, -\frac{1}{1728}t^3 - \frac{2335}{576}t^2 + \frac{13493375}{576}t - \frac{29962489375}{1728}\right)$$

$$s_1 + s_2 = \left(\frac{1}{36}t^2 + \frac{435}{2}t - \frac{921375}{4}, -\frac{1}{216}t^3 - \frac{1181}{24}t^2 - \frac{41625}{8}t + \frac{373156875}{8}\right)$$

Since $s_1 + s_2 \neq -2s_0$, we infer that $2s_0$ is 2-divisible, while $s_1 + s_2$ is not 2-divisible. Also, both $2s_0$ and $s_1 + s_2$ do not meet the zero section O. Let $\Delta^{(1)}$ and $\Delta^{(2)}$ be the sections which are the images of $2s_0$ and $s_1 + s_2$ in Σ_2, respectively. Put

$$B_1 = \Delta_{0,2} + \Delta^{(1)} + T_2, \quad B_2 = \Delta_{0,2} + \Delta^{(2)} + T_2.$$

Oen can check that, for each i, $\Delta^{(i)}$ and T_2 meet 3 distinct smooth points of T_2 in such a way that the intersection multiplicity at each point is 2. Let $q_2^{(i)} : W_2^{(i)} \to \Sigma_2$ $(i = 1, 2)$ be the double covers with branch locus $\Delta_{0,2} + \Delta^{(i)}$ $(i = 1, 2)$, respectively. Then T_2 is a splitting curve with respect to $q_2^{(1)}$, but not with respect to $q_2^{(1)}$. Hence by Corollary 0.1 there exists no homeomorphism $h : \Sigma_2 \to \Sigma_2$ such that $h(B_1) = B_2$.

Example 4.2 ([13, Example, p. 210]) Let $\varphi : S_2 \to \mathbb{P}^1$ be the rational elliptic surface given by the following Weierstrass equation:

$$y^2 = x^3 + (25t + 9)x^2 + (144t^2 + t^3)x + 16t^4,$$

t being an inhomogeneous coordinate of $\mathbb{P}^1$. (Note that the original Weierstrass equation in [13] is $y^2 - 6xy = x^3 + 25tx^2 + (144t^2 + t^3)x + 16t^4$. We change the equation slightly.) $\varphi : S_2 \to \mathbb{P}^1$ satisfies the following properties:

(i) φ has 2 singular fibers over $t = 0, \infty$, of which types are of type I_4 over $t = 0$ and type III over $t = \infty$.

(ii) $\mathrm{MW}(\mathcal{J}_{S_2})$ has no torsion.

Likewise Example 4.1, we infer that T_2 on Σ_2 has one a_3-singularity and one a_1-singularity and is irreducible. In order to give another example, we use three sections given by [13] as follows:

$$s_0 : (0, 4t^2),\ \ s_1 : (-16t, -48t),\ \ s_2 : (-15t, t^2 + 45t).$$

Let $\langle\ ,\ \rangle$ be the height pairing defined in [11]. Then we have

$$\langle s_0, s_0\rangle = \frac{1}{2},\ \ \langle s_i, s_i\rangle = \frac{3}{4}\,(i = 1, 2),\ \ \langle s_1, s_2\rangle = -\frac{1}{4},$$

and there is no other section s with $\langle s, s\rangle = 1/2$ other than $\pm s_0$. The sections given by $2s_0$ and $s_1 + s_2$ are

$$2s_0 = \left(\frac{1}{64}t^2 - \frac{41}{2}t + 315, -\frac{1}{512}t^3 - \frac{55}{32}t^2 + \frac{2637}{8} - 5670\right)$$
$$s_1 + s_2 = \left(t^2 + 192t + 8640, -t^3 - 301t^2 - 27936t - 803520\right)$$

Since $s_1 + s_2 \neq -2s_0$, we infer that $2s_0$ is 2-divisible, while $s_1 + s_2$ is not 2-divisible. Also, both $2s_0$ and $s_1 + s_2$ do not meet the zero section O. Let $\Delta^{(1)}$ and $\Delta^{(2)}$ be the sections which are the images of $2s_0$ and $s_1 + s_2$ in Σ_2, respectively. Put

$$B_1 = \Delta_{0,2} + \Delta^{(1)} + T_2, \quad B_2 = \Delta_{0,2} + \Delta^{(2)} + T_2.$$

For the pair (B_1, B_2), similar properties to those in Example 4.1 hold.

Remark 4.1 Note that the right hand sides of Weierstrass equations of Examples 4.1 and 4.2 can be rewritten as follows:

$$\begin{aligned} & x^3 + (271350 - 98t)x^2 + t(t - 5825)(t - 2025)x + 36t^2(t - 2025)^2 \\ = & \left\{\left(\frac{1}{12}t - \frac{5825}{12}\right)x + (6t^2 - 12150t)\right\}^2 + x^2\left(x - \frac{1}{144}t^2 - \frac{1231}{72}t + \frac{5143775}{144}\right), \\ & x^3 + (25t + 9)x^2 + (144t^2 + t^3)x + 16t^4 \\ = & \frac{1}{64}\left\{(t + 144)x + 32t^2\right\}^2 + x^2\left(x - \frac{1}{64}t^2 + \frac{41}{2}t - 315\right) \end{aligned}$$

Since the sections of Σ_2 given by $x - \frac{1}{144}t^2 - \frac{1231}{72}t + \frac{5143775}{144} = 0$ and $x - \frac{1}{64}t^2 + \frac{41}{2}t - 315 = 0$ are $\Delta^{(1)}$'s in Examples 4.1 and 4.2, respectively, we see that $\Delta^{(1)}$'s in both of examples are splitting curves with respect to $q_2^{(1)}$.

We show that an infinitely many examples of pairs (B_1, B_2) with similar properties to those in Examples 4.1 and 4.2. Let $\varphi : \mathcal{E} \to \mathbb{P}^1$ be an elliptic surface with section O given by

$$(*) \quad y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),$$

where $a_2(t), a_4(t), a_6(t) \in \mathbb{C}(t)$. We assume that $\varphi : \mathcal{E} \to \mathbb{P}^1$ satisfies the following two properties:

- $\mathrm{MW}(\mathcal{J}_{\mathcal{E}})$ has no torsion.
- There exists sections s_1 and s_2 in $\mathrm{MW}(\mathcal{J}_{\mathcal{E}})$ such that s_1 is 2-divisible $\mathrm{MW}(\mathcal{J}_{\mathcal{E}})$, while s_2 is not 2-divisible in $\mathrm{MW}(\mathcal{J}_{\mathcal{E}})$.

Let $\nu : C \to \mathbb{P}^1$ be a surjecticve morphism from a smooth curve C to $\mathbb{P}^1$ and let $\varphi_\nu : \mathcal{E}_\nu \to \mathbb{P}^1$ be the induced elliptic fibration obtained by the pull back by ν.

Lemma 4.1 *If we choose C and ν generally enough, the following properties are satisfied:*

1. $\mathrm{MW}(\mathcal{J}_{\mathcal{E}_\nu})$ *has no torsion of oder n.*
2. *Let $s_{\nu,1}$ and $s_{\nu,2}$ be the sections in $\mathrm{MW}(\mathcal{J}_{\mathcal{E}_\nu})$ induced by s_1 and s_2, respectively. $s_{\nu,1}$ is 2-divisible $\mathrm{MW}(\mathcal{J}_{\mathcal{E}_\nu})$, while $s_{\nu,2}$ is not 2-divisible in $\mathrm{MW}(\mathcal{J}_{\mathcal{E}_\nu})$.*

Proof. We first note that $\mathrm{MW}(\mathcal{J}_{\mathcal{E}})$ and $\mathrm{MW}(\mathcal{J}_{\mathcal{E}_\nu})$ are regarded as the set of $\mathbb{C}(t)$- and $\mathbb{C}(C)$- rational points of the elliptic curve $(*)$, respectively.

1. Let K_n be the field extension of $\mathbb{C}(t)$ obtained by addding all n-torsion points. If we choose C and ν generically enough so that $\mathbb{C}(C) \cap K_n = \mathbb{C}(t)$, then $\mathrm{MW}(\mathcal{J}_{\mathcal{E}_\nu})$ has no n-torsion.

2. The duplication formula for the group law of an elliptic curve shows that s_2 becomes 2-divisible over a certain field extension K of $\mathbb{C}(t)$ of degree at most 4. Hence if choose C and ν generically enough so that $\mathbb{C}(C) \cap K = \mathbb{C}(t)$, then $s_{\nu,2}$ is not 2-divisible in $\mathrm{MW}(\mathcal{J}_{\mathcal{E}_\nu})$. □

We now give examples of pairs (B_1, B_2) on Σ_{2n} as in Examples 4.1 and 4.2 for any n.

Example 4.3 Let $\varphi : S_2 \to \mathbb{P}^1$ be the rational elliptic surface in Example 4.1 or 4.2. Let $\nu_n : \mathbb{P}^1 \to \mathbb{P}^1$ be a generic rational map of degree n. Let $\varphi_{\nu_n} : \mathcal{E}_{\nu_n} \to \mathbb{P}^1$ be the induced relatively minimal elliptic fibration obtained via the pull-back of ν_n from φ. As we have seen at the beginning of this section, $\mathcal{E}_{\nu_n}$ is the canonical resolution of a double cover $p'_{2n} : \mathcal{E}'_{\nu_n} \to \Sigma_{2n}$ of Σ_{2n} such that

- the branch locus of p'_{2n} is of the form $\Delta_{0,2n} + T^{(\nu)}_{2n}$, $T^{(\nu)}_{2n} \sim 3(\Delta_{0,2n} + 2nF_{2n})$, and $T^{(\nu)}_{2n}$ has at worst simple singularities, and
- the involution induced by the covering transformation of p'_{2n} coincides with the one induced by the inversion morphsim of the group law on the generic fiber.

By Lemma 4.1, we may assume that $\mathrm{MW}(\mathcal{J}_{\mathcal{E}_{\nu_n}})$ has no 2-torsion. This implies that $T^{(\nu_n)}_{2n}$ is irreducible. Put

$$\begin{aligned} s_{1,\nu_n} &:= \text{the section arising from } 2s_o \\ s_{2,\nu_n} &:= \text{the section arising from } s_1 + s_2. \end{aligned}$$

and let $\Delta^{(1)}_{\nu_n}$ and $\Delta^{(1)}_{\nu_n}$ be the sections of Σ_{2n} induced by s_{1,ν_n} and s_{2,ν_n}, respectively. Put Put

$$B^{(\nu_n)}_1 = \Delta_{0,2} + \Delta^{(1)}_{\nu_n} + T^{(\nu_n)}_{2n}, \quad B^{(\nu_n)}_2 = \Delta_{0,2} + \Delta^{(2)}_{\nu_n} + T^{(\nu_n)}_{2n}.$$

For the pair $(B^{(\nu_n)}_1, B^{(\nu_n)}_2)$, similar properties to those in Example 4.1 hold.

5 Application to the study of Zariski pairs

In this section, we give two examples of Zariski pairs of sextic curves by using Corollary 0.1.

Let C be a smooth conic and let Q be an irreducible quartic such that

(i) Q has either $2a_1$ singularities or one a_3 singularity,

(ii) C is tangent to Q at 4 distinct smooth points of Q, and

(iii) one of the 4 intersection points of C and Q is an inflection point, x, of Q.

Likewise [16, §4], we consider a rational elliptic surface related to $C + Q$ as follows:

Let $\nu_1 : \mathbb{P}^2_x \to \mathbb{P}^2$ be a blowing-up at x. Let Q_x and E denote the proper transform of Q_x and the exceptional divisor of ν, respectively. Next let $\nu_2 : \widehat{\mathbb{P}}^2 \to \mathbb{P}^2_x$ be a blowing-up at $Q_x \cap E$, and let $\widehat{Q}, E_1$, and E_2 be the proper transforms of Q_x and E and the exceptional divisor of ν_2, respectively.

Let $\bar{l}_x$ be the proper transform of the tangent line at x. $\bar{l}_x$ is the exceptional curve of the first kind. By blowing down $\bar{l}_x$, we obtain Σ_2, and $\widehat{Q}$ and C give rise to divisors T_Q and Δ_C, respectively, such that $\Delta_C \sim \Delta_{0,2} + 2F_2$, $T_Q \sim 3(\Delta_{0,2} + 2F_2)$ and T_Q has $3a_1$ (resp. $a_3 + a_1$) singularities if Q had $2a_1$ (resp. one a_3).

Let $f' : \mathcal{E}' \to \Sigma_2$ be the double cover with branch locus $\Delta_{0,2} + T_Q$ and we denote its canonical resolution of $\mathcal{E}'$ by $\mathcal{E}$. $\mathcal{E}$ satisfies the following properties:

- $\mathcal{E}$ has 3 singular fibers whose configuration is $2I_2$ and III.
- The conic C gives rise to two sections s^+ and s^-.
- The covering transform induced by f' coincides with the one induced by the inversion morphism with respect to the group law.

Under these setting, one easily infers that

Proposition 5.1 *There exists a D_{2n}-cover branched at $2C + nQ$ if and only if there exists a D_{2n}-cover branched at $2(\Delta_{0,2} + \Delta_C) + nT_Q$.*

Example 5.1 Let $\mathcal{E}$ be the rational elliptic surface as in Example 4.1. By considering the reverse process observed as above starting from $\mathcal{E}$, we infer that

- two sections $2s_o$ and $s_1 + s_2$ in Examples 4.1 give rise to two conics C_1 and C_2, respectively and
- the corresponding T_2 gives rise to an irreducible quartic Q with $2a_1$ singularities.

By Proposition 5.1, we infer that $(C_1 \cup Q, C_2 \cup Q)$ is a Zariski pair.

We similarly obtain another Zariski pair starting from the rational elliptic surface in Example 4.2. It is a pair of sextic curves $(C_1 \cup Q', C_2 \cup Q')$ such that

- Q' is an irreducible quartic with one a_3 singularity.
- both of C_i $(i = 1, 2)$ are tangent to Q' at four distinct smooth points of Q'.

Remark 5.1 Both of Zariski pairs in Example 5.1 are in Shimada's list in [10]. Our reasoning why they are Zariski pairs gives another point of view different from Shimada's one.

References

[1] E. Artal Bartolo, J.-I. Codgolludo and H. Tokunaga: *A survey on Zariski pairs*, Adv.Stud.Pure Math., **50**(2008), 1-100.

[2] E. Artal Bartolo and H. Tokunaga: *Zariski k-plets of rational curve arrangements and dihedral covers*, Topology Appl. **142** (2004), 227-233.

[3] W. Barth, K. Hulek, C.A.M. Peters and A. Van de Ven: Compact complex surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete **4** 2nd Enlarged Edition, Springer-Verlag (2004).

[4] E. Brieskorn: *Über die Auflösung gewisser Singlaritäten von holomorpher Abbildungen*, Math. Ann. **166**(1966), 76-102.

[5] E. Brieskorn: *Die Auflösung der rationalen Singularitäten holomorpher Abbildungne*, Math. Ann. **178**(1968), 255-270.

[6] E. Horikawa: *On deformation of quintic surfaces*, Invent. Math. **31** (1975), 43−85.

[7] K. Ireland and M. Rosen: *A Classical Introduction to Modern Number Theory, Second Edition*, Graduate Text in Mathematics **84** Springer-Verlag (1990).

[8] K. Kodaira: *On compact analytic surfaces II*, Ann. of Math. **77** (1963), 563-626.

[9] R. Miranda and U. Persson: *On extremal rational elliptic surfaces*, Math. Z., **193**(1986), 537-558.

[10] I. Shimada: *Lattice Zariski k-ples of plane sextic curves and Z-splitting curves for double plane sextics*, arXive:0903.3308

[11] T. Shioda: *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Pauli **39** (1990), 211-240.

[12] T. Shioda: *Mordell-Weil lattices for higher genus fibration over a curve*, New trends in algebraic geometry (Warwick, 1996), 359–373, London Math. Soc. Lecture Note Ser., **264** (1999), 359-373.

[13] T. Shioda and H. Usui: *Fundamental invariants of Weyl groups and excellent families of elliptic curves*, Comment. Math. Univ. St. Pauli **41**(1992), 169-217.

[14] H. Tokunaga: *On a cyclic covering of a projective manifold*, J. of Math. Kyoto Univ. **30** (1990), 109 - 121

[15] H. Tokunaga: *Two remarks on non-Galois triple coverings*, Mem. Fac.Sci. Kochi Univ. **13**(1992), 19-33.

[16] H. Tokunaga: *On dihedral Galois coverings*, Canadian J. of Math. **46** (1994),1299 - 1317.

[17] O. Zariski: *On the purity of the branch locus of algebraic functions*, Proc. Nat. Acad. USA **44** (1958), 791-796.

Hiro-o TOKUNAGA
Department of Mathematics and Information Sciences
Graduate School of Science and Engineering
Tokyo Metropolitan University
1-1 Minami-Ohsawa, Hachiohji 192-0397 JAPAN
tokunaga@tmu.ac.jp