Hurwitz orbits of braid systems of "standard" type, and its application to braided surfaces

YOSHIRO YAGUCHI.

(Hiroshima University)

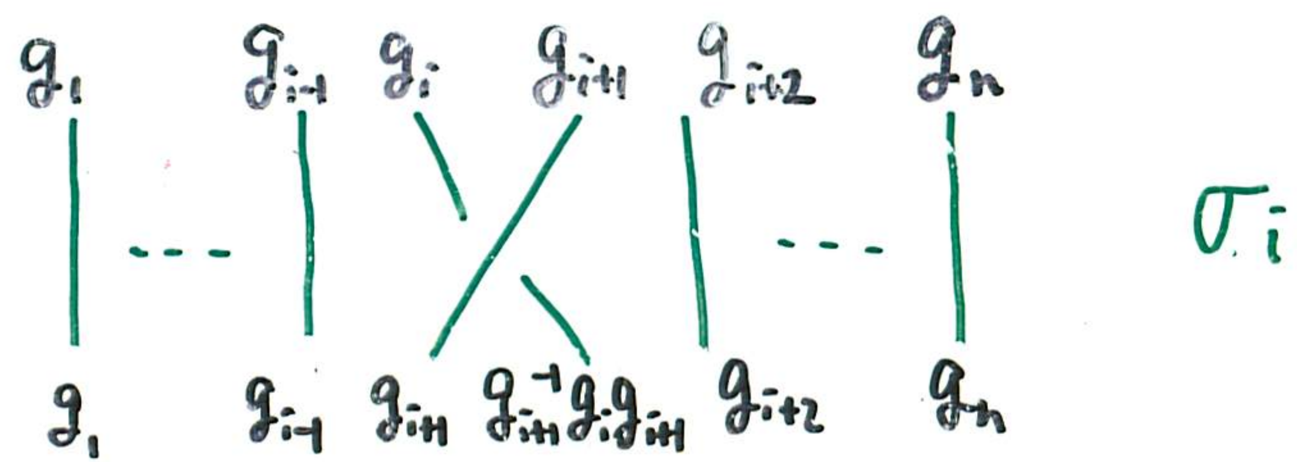
G : $\boxed{\text{a group}}$, B_n : $\boxed{\text{braid group of degree } n}$ ①

The right action $G^n \curvearrowleft B_n$ defined by

$$(g_1, \cdots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \cdots, g_n) \cdot \sigma_i$$

$$:= (g_1, \cdots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \cdots, g_n)$$

is called <u>Hurwitz action</u> of B_n on G^n



$(g_1, \cdots, g_n) \cdot B_n$: $\boxed{\text{Hurwitz orbit}}$

<u>Thm 1</u> [P. Kluitmann] (1988)

$\tau_1, \cdots, \tau_n \in S_{n+1}$: transpositions $\tau_i \neq \tau_j \ (i \neq j)$

$$\Rightarrow \#\left((\tau_1, \cdots, \tau_n) \cdot B_n\right) = (n+1)^{n-1}$$

$S_{n+1}^{\ n} \curvearrowleft B_n$

<u>Thm 2</u> [S.P. Humphries] (2004)

$\sigma_1, \cdots, \sigma_n \in B_{n+1}$: the standard generators

$$\Rightarrow \#\left((\sigma_1, \cdots, \sigma_n) \cdot B_n\right) = (n+1)^{n-1}$$

$B_{n+1}^{\ n} \curvearrowleft B_n$

<u>Thm 3</u> [Y]

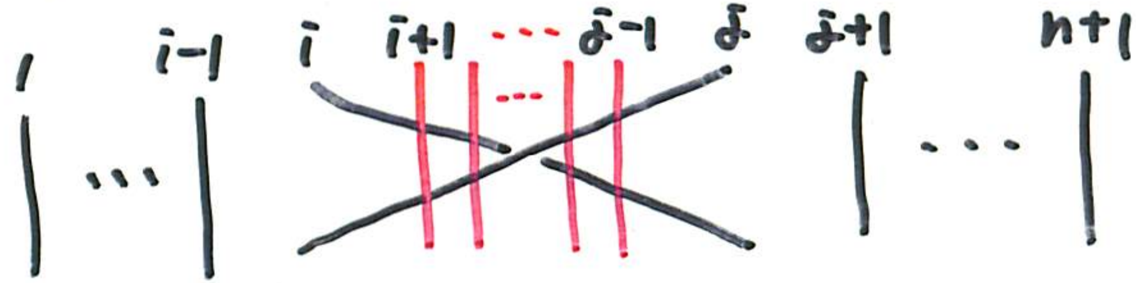
$\varphi \in \mathrm{Sym}\{1, \cdots, n\}$

$$\Rightarrow \#\left((\sigma_{\varphi(1)}, \cdots, \sigma_{\varphi(n)}) \cdot B_n\right) = (n+1)^{n-1}$$

$B_{n+1}^{\ n} \curvearrowleft B_n$

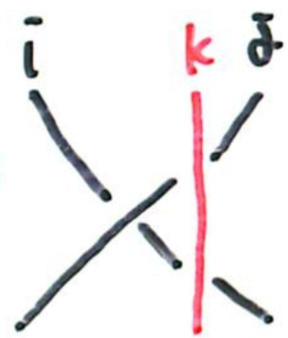
$\varphi \in \mathrm{Sym}\{1, \cdots, n\}$: fix

- $A := \{k \in \{2, \cdots, n\} \mid \varphi^{-1}(k-1) < \varphi^{-1}(k)\}$, $B := \{k \in \{2, \cdots, n\} \mid \varphi^{-1}(k-1) > \varphi^{-1}(k)\}$
- $S_{ij} \in B_{n+1}$ $(1 \leqq i < j \leqq n+1)$: following braid :

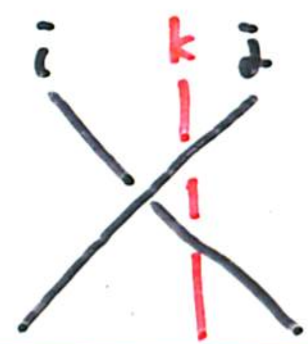


where

- $k \in A \Rightarrow$



- $k \in B \Rightarrow$



- $\Sigma := \{S_{ij} \in B_{n+1} \mid 1 \leqq i < j \leqq n+1\}$

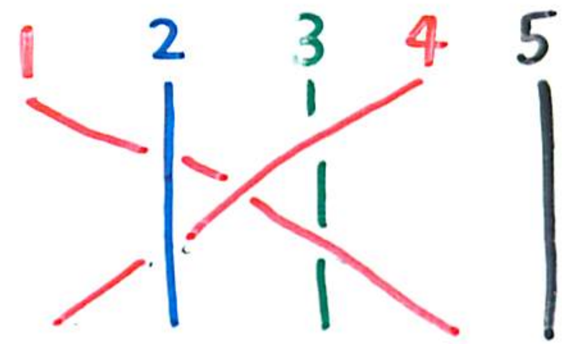
※ $\underset{\in B_{n+1}}{\sigma_i} = S_{i\,i+1}$

$\boxed{\text{Ex } (n=4)}$

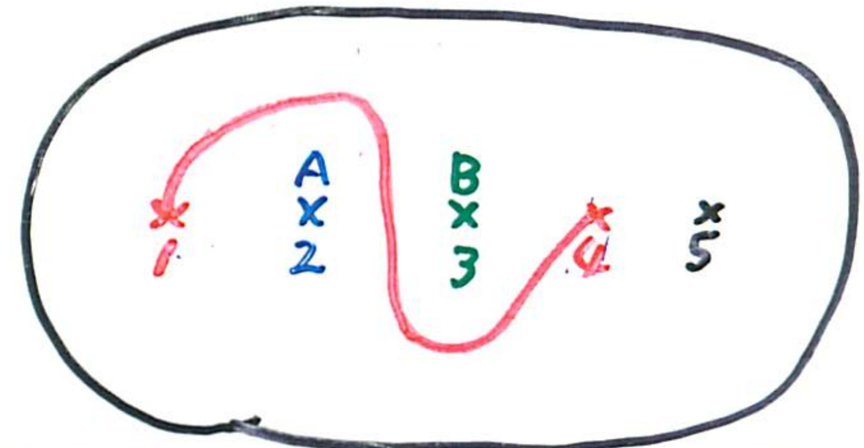
$\varphi \in \mathrm{Sym.}\{1,2,3,4\}$, $(\varphi(1), \varphi(2), \varphi(3), \varphi(4)) = (4,3,1,2)$

$\Rightarrow (\varphi^{-1}(1), \varphi^{-1}(2), \varphi^{-1}(3), \varphi^{-1}(4)) = (3,4,2,1)$

$\Rightarrow A = \{2\}$, $B = \{3,4\}$

$\Rightarrow$ 

S_{14}

$\mapsto$ 

arc corresponding to S_{14}

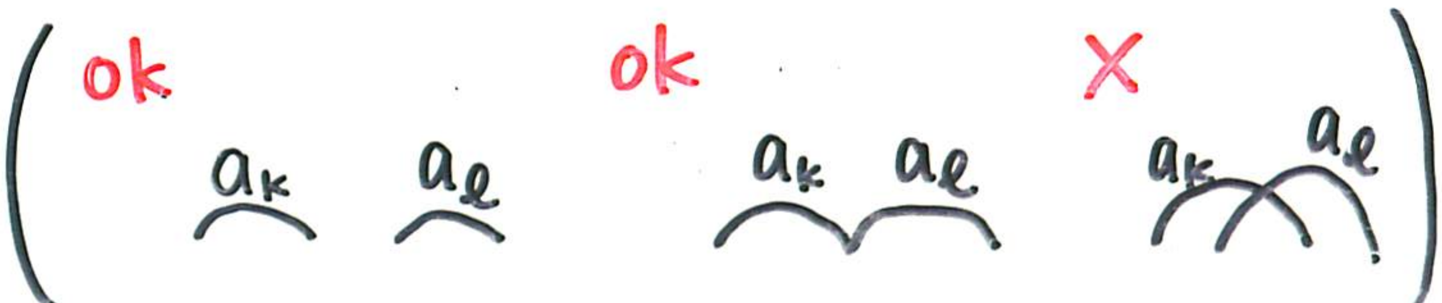
☆ $\Sigma \ni {}^{\forall} S_{ij} \mapsto \exists!$ arc corresponding to S_{ij}

- $\varphi \in \mathrm{Sym}\{1, \cdots, n\}$: fix
- $\vec{g} = (g_1, \cdots, g_n) \in \Sigma^n$: fix
- $a_1, \cdots, a_n$: the arcs corresponding to $g_1, \cdots, g_n$

$\vec{g}$ is φ-good $:\overset{\text{def}}{\Longleftrightarrow}$ For $\forall k, \ell \in \{1, \cdots, n\}$,

$\boxed{\text{(i)}}$ $k \neq \ell \Rightarrow a_k$ and a_ℓ are disjoint except possibly at their end points.

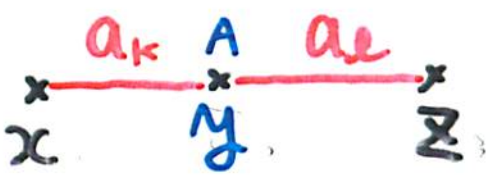
ex.



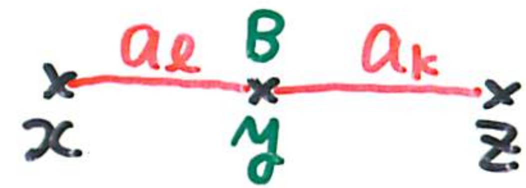
ii If $k < l$ and $a_k \cap a_l \neq \phi$, then

$(x < y < z)$

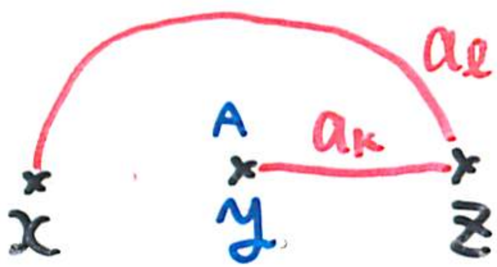
I



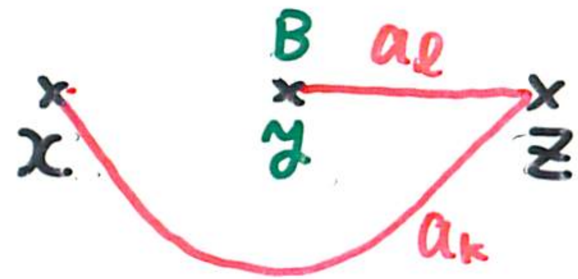
I'



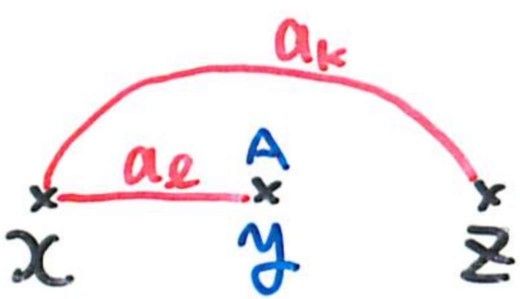
II



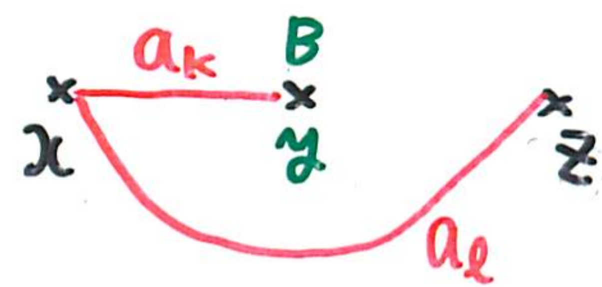
II'



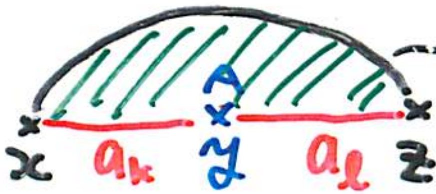
III



III'



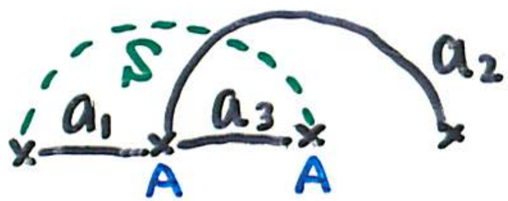
At $\boxed{\text{(ii)}}$, we define S by the triangle determined by a_k and a_ℓ.

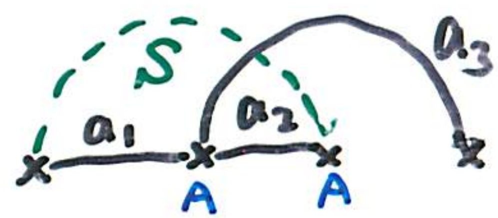
ex. (Ⅰ 

)

$\boxed{\text{(iii)}}$ If $(\mathrm{Int}\, S) \cap a_m \neq \phi$,

then $k < m < \ell$

ex. (OK 

× 

)

$\boxed{\text{(iv)}}$ $\bigcup_{k=1}^{m} a_k \ (\subset D^2)$ is a tree.

Lemma 1 ($\varphi \in \mathrm{Sym}\{1,\cdots,n\}$: fix)

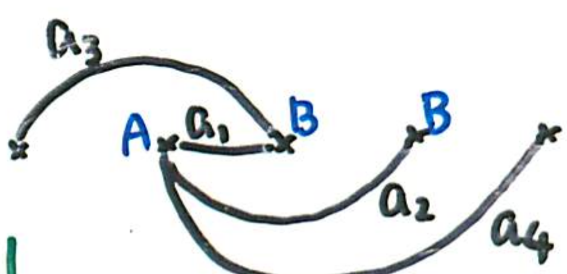
$\vec{g} = (g_1,\cdots,g_n) \in \Sigma^n$ is φ-good

$\Rightarrow \forall \beta \in B_n$, $\vec{g}\cdot\beta \in \Sigma^n$ and $\vec{g}\cdot\beta$ is φ-good

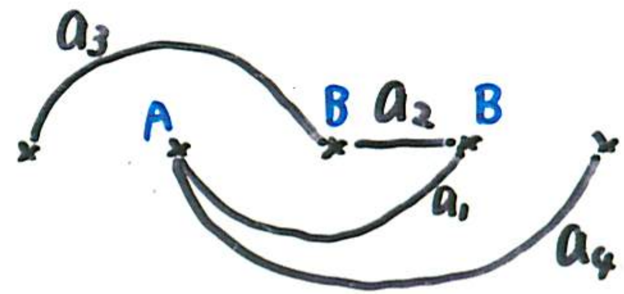
Ex $n=4$ ($B_5^4 \curvearrowright B_4$) $(\varphi(1),\varphi(2),\varphi(3),\varphi(4)) = (4,3,1,2)$

$$(g_1,g_2,g_3,g_4) \xmapsto{\cdot\sigma_1} (g_2, g_2^{-1}g_1g_2, g_3, g_4)$$

$$\Sigma^4 \ni (s_{23}, s_{24}, s_{13}, s_{25}) \mapsto (s_{24}, s_{34}, s_{13}, s_{25}) \in \Sigma^4$$



$\mapsto$



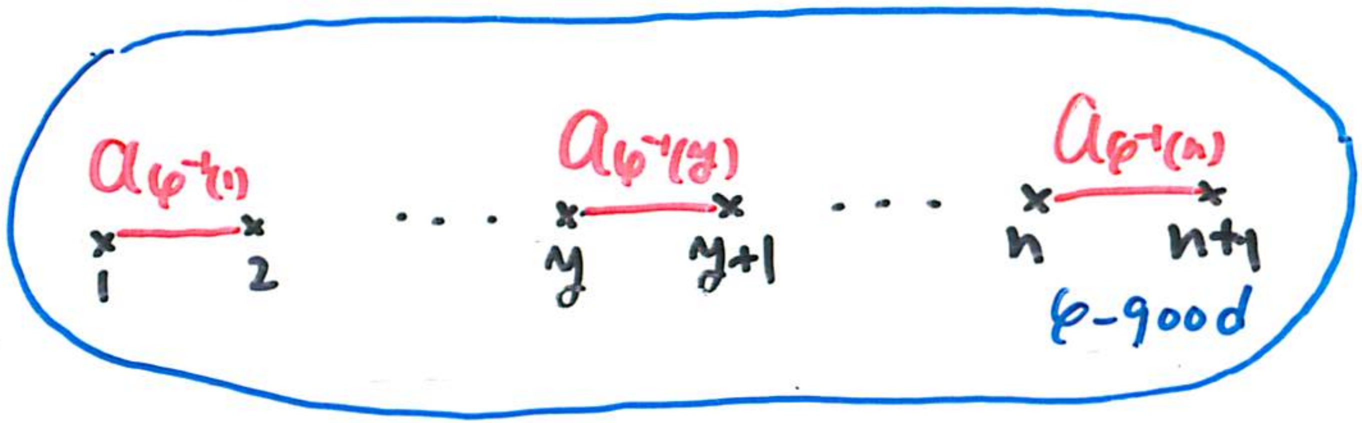
φ-good $\qquad$ φ-good

Thm 3

$$\forall \varphi \in \mathrm{Sym}\{1,\cdots,n\},\ \#\left((\sigma_{\varphi(1)},\cdots,\sigma_{\varphi(n)})\cdot B_n\right)=(n+1)^{n-1}$$

∵ $(\sigma_{\varphi(1)},\cdots,\sigma_{\varphi(n)}) \in \Sigma^n$ and

$(\sigma_{\varphi(1)},\cdots,\sigma_{\varphi(n)})$ is φ-good



By Lemma 1, $(\sigma_{\varphi(1)},\cdots,\sigma_{\varphi(n)})\cdot B_n \subset \Sigma^n$.

$p : B_{n+1} \to S_{n+1}$: the canonical projection

$$\Rightarrow \#\big((\underset{\Sigma^n}{\underbrace{\sigma_{\varphi(1)}, \cdots, \sigma_{\varphi(n)}}}) \cdot B_n\big) \overset{\uparrow}{=} \#\big(p^n((\sigma_{\varphi(1)}, \cdots, \sigma_{\varphi(n)}) \cdot B_n)\big)$$

(∵) $p(\Sigma) = \{\text{transpositions}\}$ and $p|_\Sigma$ is injective

$$\overset{\uparrow}{=} \#\big((p^n(\sigma_{\varphi(1)}, \cdots, \sigma_{\varphi(n)})) \cdot B_n\big)$$

(∵) Generally, the right diagram is commutative:

$$\begin{array}{ccc} G^n & \xrightarrow{f^n} & H^n \\ \cdot\beta \downarrow & & \downarrow \cdot\beta \\ G^n & \xrightarrow[f^n]{} & H^n \end{array}$$

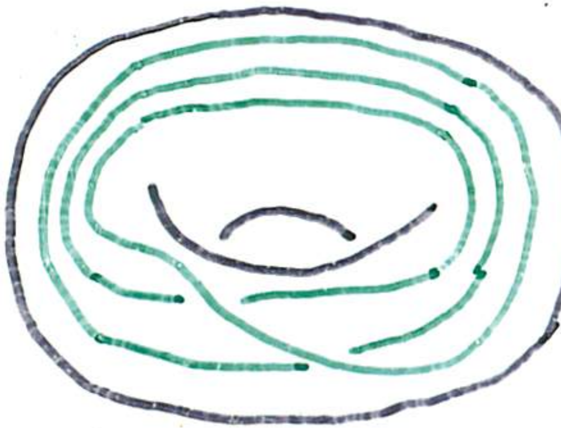
(G, H : group
$f : G \to H$: homomorphism
$\cdot\beta$: Hurwitz action by $\beta \in B_n$)

$$\overset{\uparrow}{=} (n+1)^{n-1}$$

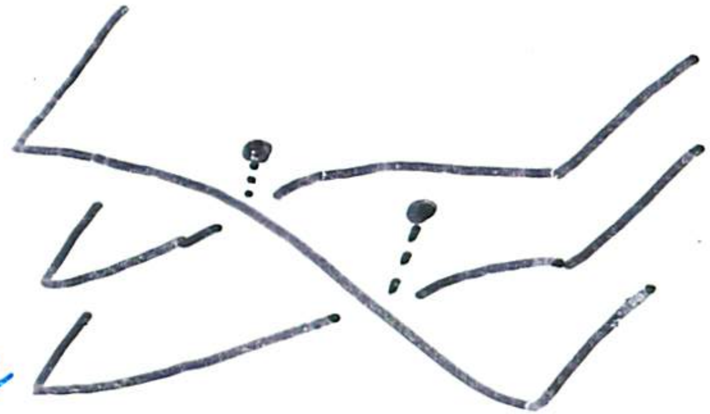
By P. Kluitmann (Thm 1)

Braided surface of degree m

The closure of a m-braid $\partial S \subset D_1^2 \times \partial D_2^2$

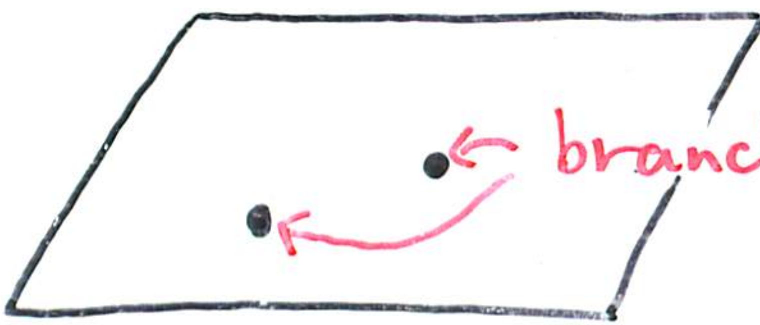


$D_1^2 \times D_2^2 \supset S$ ← an orientable surface



$p \downarrow$ $\qquad$ $\downarrow p|_S$: Branched covering projection. of degree m

D_2^2



branch points

S, S' : Braided surface of degree m

$\boxed{\text{Def}}$ S and S' are equivalent / isomorphic

$:\Leftrightarrow \exists \{h_u\}_{u \in [0,1]}$: ambient isotopy of $D_1^2 \times D_2^2$

s.t.

(1) $h_0 = \mathrm{id}$, $h_1(S) = S'$

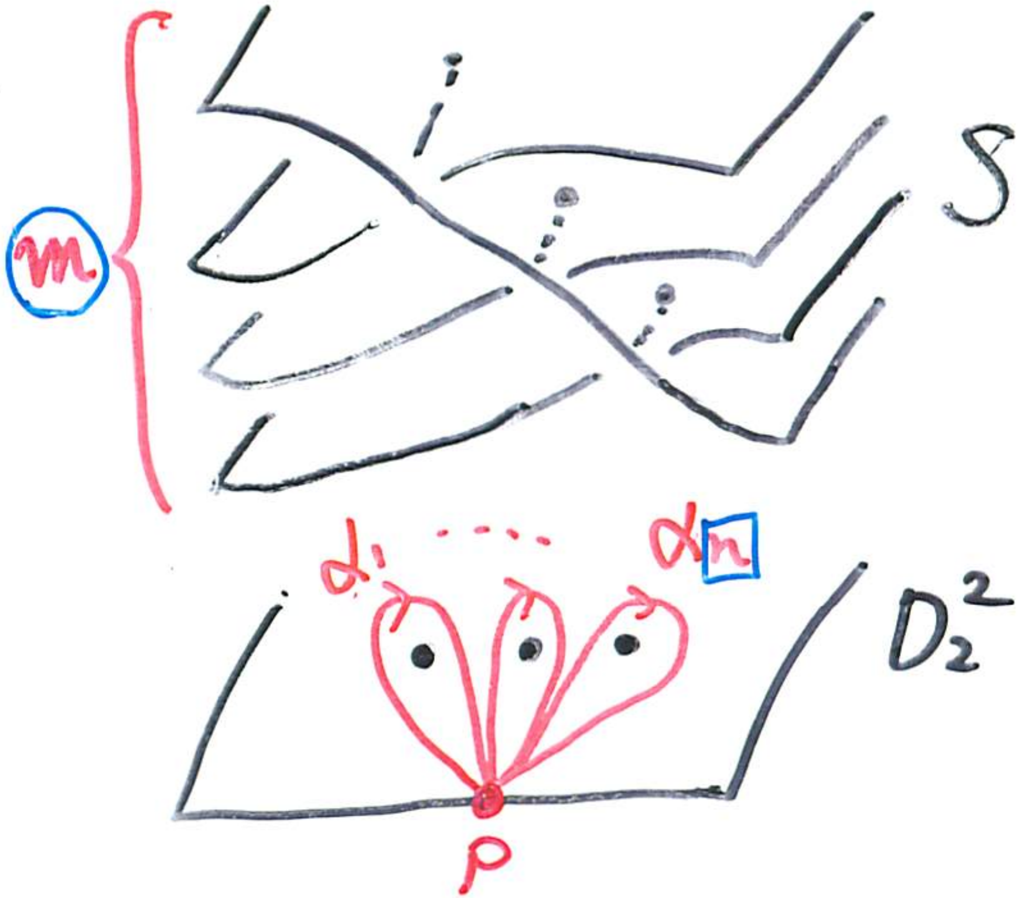
(2) $h_u |_{D_1^2 \times \partial D_2^2} = \mathrm{id}$ $(\forall u)$

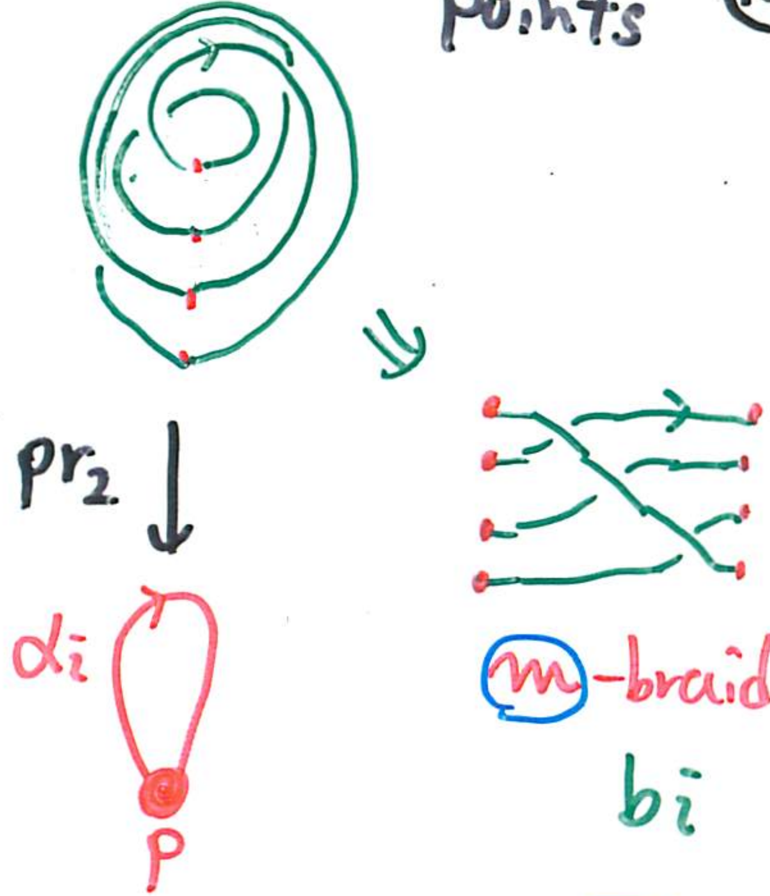
(3) $\exists \underline{h_u} : D_2^2 \to D_2^2$: homeo

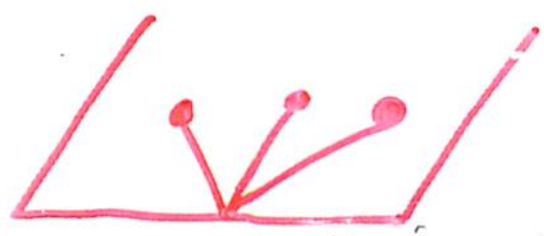
$$\begin{tikzcd} D_1^2 \times D_2^2 \supset S \arrow[r, "h_u|_S"] \arrow[d, "pr|_S"'] & h_u(S) \subset D_1^2 \times D_2^2 \arrow[d, "pr|_{h_u(S)}"] \\ D_2^2 \arrow[r, "\underline{h_u}", "\mathrm{id}"'] & D_2^2 \end{tikzcd} \quad (\forall u)$$

S : braided surface of degree m which has $\boxed{n}$ branch points

⑬







$$\underbrace{(b_1, \cdots, b_n)}_{\text{called "braid system of } S\text{"}} \in B_m^{\boxed{n}}$$

$\boxed{\text{Prop}}$

Let S be a braided surface whose braid system is $(b_1, \cdots, b_n) \in B_m^n$

Then, $(b_1, \cdots, b_n) . B_n \overset{1:1}{\longleftrightarrow} X_S$,

where X_S is the following set:

$$\left\{ \begin{array}{l|l} \text{isomorphism} & ① \ \Sigma(S') = \Sigma(S) \\ \text{class } [S'] & ② \ S' \text{ and } S \text{ are equivalent} \end{array} \right\}$$

the set of branch points

$\boxed{\text{Corollary of Thm 3}}$

Let S be a braided surface of degree $n+1$ whose braid system is $(\sigma_{\varphi(1)}, \cdots, \sigma_{\varphi(n)})$ for a $\varphi \in \mathrm{Sym}\{1, \cdots, n\}$.

Then, we have :

$$\#(X_S) = (n+1)^{n-1}$$