Construction of non-Galois triple coverings over $\mathbf{P^2}$ branched along quintic curves by pull-back construction

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Definition

Let X and Y be normal projective varieties. We denote the function fields of X and Y by $\mathbf{C(X)}$ and $\mathbf{C(Y)}$, respectively. Assume that there exists a finite surjective morphism $\pi : X \to Y$. Note that π induces the field extension $\mathbf{C(X)}/\mathbf{C(Y)}$. Then

- $\pi : X \to Y$: non-Galois triple covering $\overset{\text{def}}{\Longleftrightarrow}$ $\mathbf{C(X)}/\mathbf{C(Y)}$: non-Galois cubic extension.
- $\Delta_\pi := \{y \in Y \mid \sharp(\pi^{-1}(y)) < 3\}$ is called the branch locus of π.
- $p \in \Delta_\pi$: totally ramified point $\overset{\text{def}}{\Longleftrightarrow}$ $\sharp\pi^{-1}(p) = 1$.
- $p \in \Delta_\pi$: simply ramified point $\overset{\text{def}}{\Longleftrightarrow}$ $\sharp\pi^{-1}(p) = 2$.

Fact

1. For all non-Galois triple covering $\pi : X \to Y$, there exists an algebraic element ξ over $\mathbf{C(Y)}$ with minimal equation $\xi^3 + 3\alpha\xi + 2\beta = 0$ $(\alpha, \beta \in \mathbf{C(Y)})$ such that $\mathbf{C(X)} = \mathbf{C(Y)}(\xi)$
2. (H.Tokunaga) Let $\pi : \Sigma \to \mathbf{P^2}$ be a non-Galois triple covering with $\deg(\Delta_\pi) = 5$. Then Δ_π has one of the following two forms:

Ⅰ. $\Delta_\pi = C_2 + C_3$ (C_2: a conic, C_3: a cubic).

$$\pi \text{ is } \left\{\begin{matrix}\text{simply}\\ \text{totally}\end{matrix}\right\} \text{ ramified along } \left\{\begin{matrix}C_2\\ C_3\end{matrix}\right\}.$$

Ⅱ. $\Delta_\pi = Q + L$ (Q: a quartic, L: a line).

$$\pi \text{ is } \left\{\begin{matrix}\text{simply}\\ \text{totally}\end{matrix}\right\} \text{ ramified along } \left\{\begin{matrix}Q\\ L\end{matrix}\right\}.$$

In terms of the singularities of Q and relative position between Q and L, there exist 18 types for Type Ⅱ.

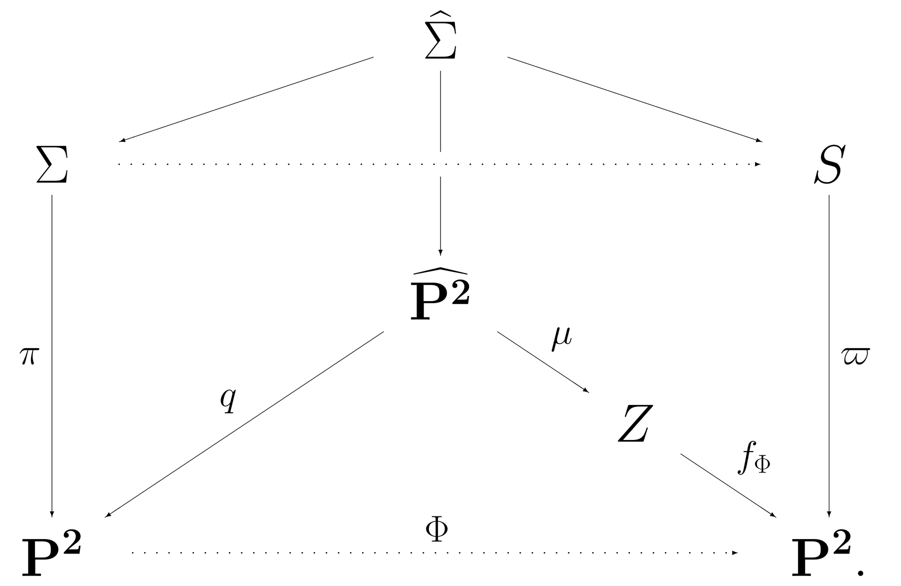
Pull-back construction

Let $\varpi : S \to \mathbf{P^2}$ be a non-Galois triple covering given by the cubic equation $\zeta^3 + 3u\zeta + 2v$, where (u, v) is inhomogeneous coordinate of $\mathbf{P^2}$. We consider the rational map

$$\Phi : \mathbf{P^2} \dashrightarrow \mathbf{P^2}$$
$$(x, y) \mapsto (u, v) = (\alpha, \beta),$$

where (x, y) is inhomogeneous coordinate of $\mathbf{P^2}$, and $\alpha, \beta \in \mathbf{C(P^2)}$.

Then, from Fact 1, we obtain a commutative diagram as follows:



Here, q is a resolution of indeterminacy of Φ. $\nu : Z \to \mathbf{P^2}$ is a stein factorization of the induced morphism by q. Then, $\widehat{\Sigma}$ is birationally equivalent to $S \times_{\mathbf{P^2}} \widehat{\mathbf{P^2}}$ over $\mathbf{P^2}$. In other words, Σ is obtained as a "rational" pull-back of $\varpi : S \to \mathbf{P^2}$.

Result

We apply "rational" pull-back construction to non-Galois triple coverings of Type Ⅱ in Fact 2 and obtain the rational maps explicitly as follows:

Φ_i	Q's singularities	$Q \cap L$	$\Phi_\pi^* u$	$\Phi_i^* v$	f_Φ
Φ_1	$2a_2$	(i)	x	$(y-1)(x-y)$	a double covering branched along a irreducible conic
Φ_2	$2a_2 + a_1$		$4x$	$3x^2 - 6x - 1 - y^2$	
Φ_3	$3a_2$		$-2x$	$6x^2 + 2x + 1 - y^2$	
Φ_4	a_5		x	$(x-y)(y-2) - 1$	
Φ_5	e_6		x	$y(x-y)$	
Φ_6	$a_5 + a_1$		x	$x^2 - y^2 - x/4$	
Φ_7	$2a_2$	(ii)	x	$y^2 - y$	a double covering branched along distinct two lines
Φ_8	$2a_2 + a_1$		x	$y^2 - 3x - 4$	
Φ_9	a_5		x	$y^2 - x$	
Φ_{10}	e_6		x	y^2	
Φ_{11}	$a_3 + a_2$	(iii),a_3	x	$y - xy$	an isomorphism
Φ_{12}	$a_3 + a_2 + a_1$		$x - 1$	$1 - xy$	
Φ_{13}	a_6	(iii),a_6	x	$1 - xy$	
Φ_{14}	$a_4 + a_2$	(v),a_4	x	$y - x^2$	
Φ_{15}	$2a_3$	(iv),$2a_3$	-1	xy	the image of f_Φ is a curve
Φ_{16}	$2a_3 + a_1$		-1	$1 - xy$	
Φ_{17}	a_7	(v),a_7	-1	$x - y^2$	
Φ_{18}	ordinary 4-ple point	(v), ordinary 4-ple point	-1	$y(y-3)$	

(i) L is a bitangent line to Q at two distinct smooth points.

(ii) L is a tangent line to Q at a smooth point with multiplicity 4.

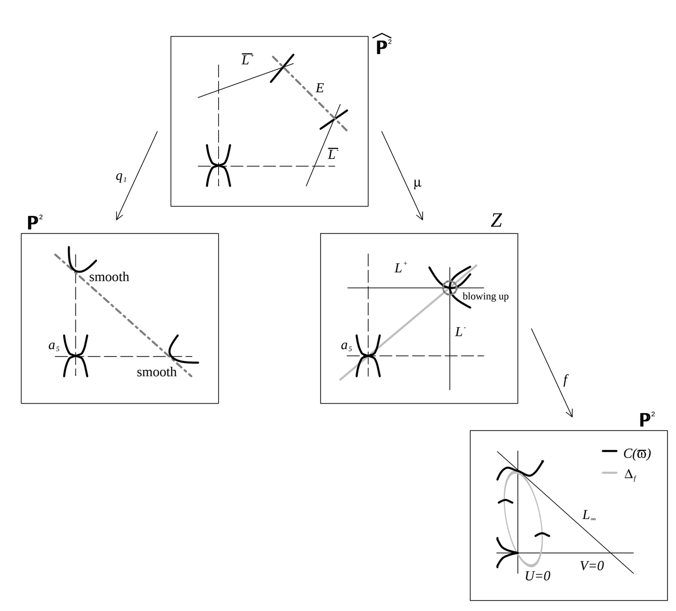
(iii) L is tangent to Q at one smooth point and passes through one singular point of Q.

(iv) L passes through two distinct singular points of Q.

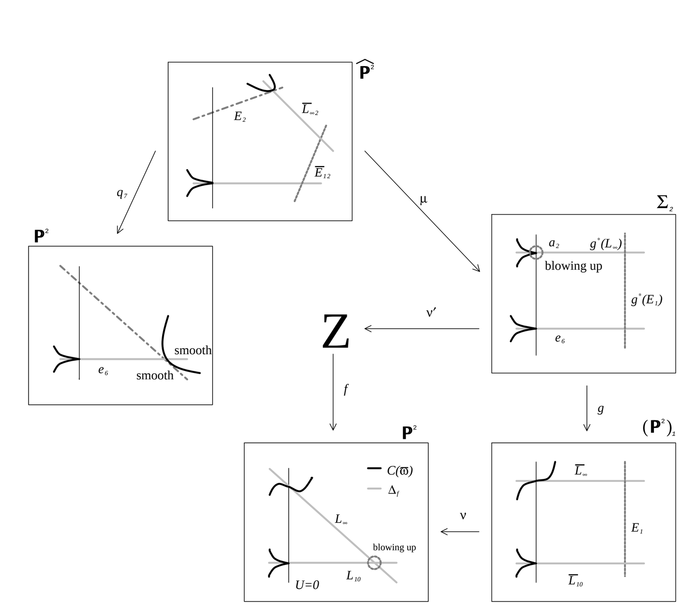
(v) L intersects Q at just one singular point.

The rational maps

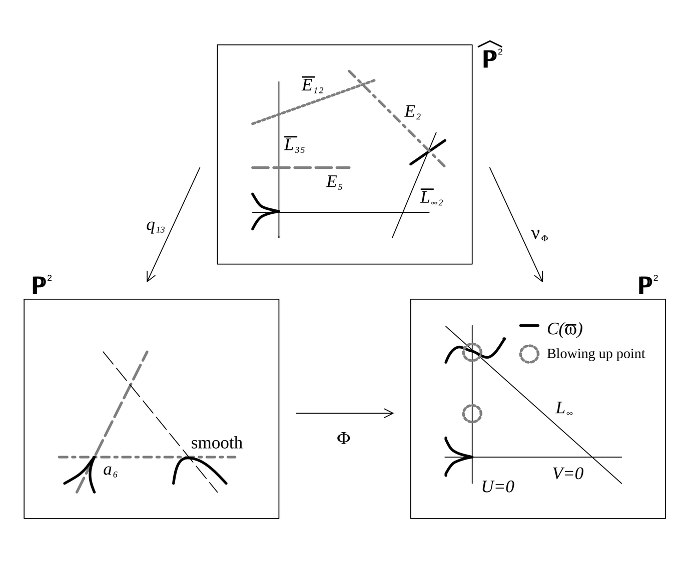
Following four pictures are the rational maps Φ_4, Φ_{10}, Φ_{13} and Φ_{16} in above table.



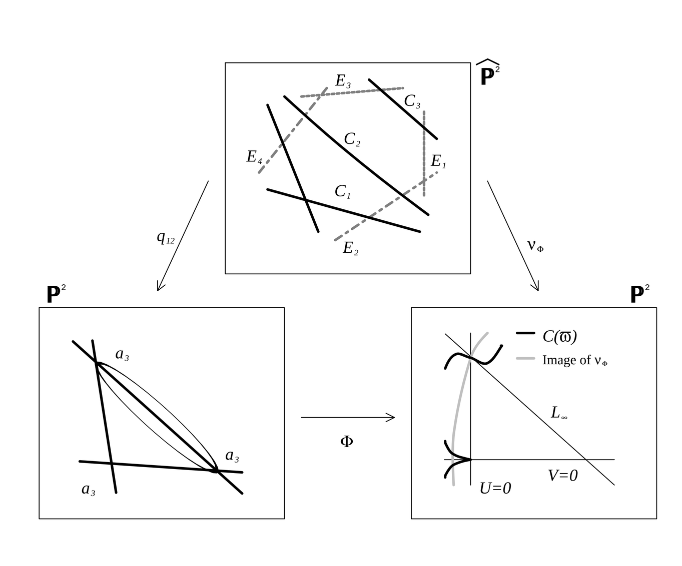
Φ_4



Φ_{10}



Φ_{13}



Φ_{16}