On the quasitorus decompositions of reduced non-generic affine quartics

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Introduction

$C \subset \mathbf{P}^2$: an irreducible projective curve of degree d.
C is a *generic curve* or *a non-generic curve* if C intersects L_∞ transversely or not.
$\Delta_C(t)$: the Alexander polynomial of C.
C^a: an affine part of a generic curve C.
Vik. S. Kulikov introduced *the Albanese dimension* $a(C^a)$ of C^a and showed that the defining polynomial $f(x,y)$ is *the quasitorus type* if $a(C^a) > 0$.

The quasitorus type

$f(x,y)$: a defining polynomial of C^a.
$f(x,y)$ is (p,q)-*quasitorus type* $\Leftrightarrow$ $\exists$ polynomials $g(x,y)$, $h(x,y)$ and $r(x,y)$ such that
(i) $\deg g > 0$, $\deg h > 0$ and $\deg r > 0$,
(ii) g, h and r are pairwise coprime,
(iii) g, h and r are coprime with f, and
(iv) coprime integers $p > 1$ and $q > 1$ such that

$$r(x,y)^{pq} f(x,y) = g(x,y)^p + h(x,y)^q.$$

Consider a k-cyclic extension $\mathbf{K}_k$ of the rational function field $\mathbf{C}(\mathbf{P}^2) = \mathbf{C}(x,y)$ of $\mathbf{P}^2$ given by

$$\zeta^k = f(x,y).$$

X'_k: the $\mathbf{K}_k$-normalization of $\mathbf{P}^2$.
X_k: its smooth model, *a cyclic multiple plane*.
$A(X_k)$: the Albanese variety of X_k.
$\alpha_k : X_k \longrightarrow A(X_k)$: the Albanese map.

The Albanese dimension

The number

$$a(C^a) = \max_{k \in \mathbf{N}} \left(\dim(\alpha_k(X_k)) \right)$$

is called the Albanese dimension of C^a.

General facts (Randell, Kulikov)

- $a(C^a) > 0 \Leftrightarrow \deg \Delta_C(t) > 0$ (Randell).
- $a(C^a) = 1 \Rightarrow f(x,y)$ possesses a unique quasitorus decomposition.
- If $f(x,y)$ possesses different quasitorus decompositions, then $a(C^a) = 2$ (Kulikov).

Our purpose

In [Y], for the non-generic case, we studied and found that the topology of $\mathbf{P}^2 - (C \cup L_\infty)$ becomes rather complicated when C is a non-generic quartic. We call such a quartic and line configuration *a QL-configuration* ([Y]). In this poster, we study the quasitorus decomposition of QL-configurations.

Results

The irreducible defining polynomials of QL-configurations which can be the branch loci of D_6-covers have a unique quasitorus decomposition except to 3-cuspidal quartic.

- 3-cuspidal case.

(i) $-32\left(x + \frac{3}{2}\right)^3 + \left(x^2 + y^2 + 12x + 9\right)^2$

(ii) $\frac{1}{256}\left(37\,x^2 - 240\,x + 18\sqrt{3}\,xy + 7\,y^2 + 144 - 48\sqrt{3}y\right)^2 + \frac{1}{2}\left(5\,x + \sqrt{3}y - 6\right)^3$

(iii) $\frac{1}{256}\left(37\,x^2 - 240\,x - 18\,x\sqrt{3}y + 7\,y^2 + 144 + 48\sqrt{3}y\right)^2 + \frac{1}{2}\left(5\,x - \sqrt{3}y - 6\right)^3$

(iv) $4\,i\left(-\frac{1}{3}iy^2 + \left(\frac{2}{3}x + 1\right)y + \frac{1}{3}ix^2 - ix\right)^3 - \frac{1}{27}\left(-2\,x^3 + 9\,x^2 + 6\,iyx^2 + 6\,y^2x - 2\,iy^3 - 27 + 9\,y^2\right)^2$

(v) $-4\,i\left(\frac{1}{3}iy^2 + \left(\frac{2}{3}x + 1\right)y - \frac{1}{3}ix^2 + ix\right)^3 - \frac{1}{27}\left(2\,x^3 - 9\,x^2 + 6\,iyx^2 - 6\,y^2x + 27 - 9\,y^2 - 2\,iy^3\right)^2$

Oka's line degeneration theory

$\{C_s \mid \|s\| \le 1\}$: an analytic family of reduced curves (C_s intersects L transversely) for $s \neq 0$ such that

$$\lim_{s \to 0} C_s = C_0 = D_0 + jL,$$

where L is a line and we call L, D_0 *a limit line*,*a limit curve* respectively. For C_0 and C_s (for $s \neq 0$), Oka proved the divisibility of Alexander polynomials:

$$\Delta_{C_s}(t) \mid \Delta_{D_0}(t, L) \ (s \neq 0).$$

Note that $\Delta_{C_s}(t)$ is the Alexander polynomial and $\Delta_{D_0}(t,L)$ is the tangential Alexander polynomial. Some of our QL-configurations are (limit curve) +(limit line) configurations of certain $(2,3)$-torus sextics given by the following form

$$F_6(X,Y,Z,s) = F_2(X,Y,Z,s)^3 + F_3(X,Y,Z,s)^2 \ (s \neq 0).$$

When $s \to 0$,

$$F_6 \to Z^2 F_4(X,Y,Z) = Z^3 F_1(X,Y,Z)^3 + Z^2 F_2(X,Y,Z)^2.$$

References

[Y] K. Yoshizaki, *On the topology of the complements of quartic and line configurations*, SUT Journal of Mathematics, **44** (2008), No.1, 125–152.