Characteristic Varieties of Quasiprojective Varieties

Enrique ARTAL BARTOLO

Departmento de Matemáticas
Facultad de Ciencias
Instituto Universitario de Matemáticas y sus Aplicaciones
Universidad de Zaragoza

Branched Coverings, Degenerations
and Related Topics 2010
Hiroshima March 8th 2010

Joint work with J.I. Cogolludo and D. Matei

Settings

- A *quasiprojective* group is the fundamental group of a quasiprojective smooth variety.

Settings

- A *quasiprojective* group is the fundamental group of a quasiprojective smooth variety.
- Goal 1: Find obstructions to determine if a group is quasiprojective.

Settings

- A *quasiprojective* group is the fundamental group of a quasiprojective smooth variety.
- Goal 1: Find obstructions to determine if a group is quasiprojective.
- Goal 2: Relate geometrical properties of a quasiprojective variety in terms of algebraic properties of the fundamental group.

Settings

- A *quasiprojective* group is the fundamental group of a quasiprojective smooth variety.
- Goal 1: Find obstructions to determine if a group is quasiprojective.
- Goal 2: Relate geometrical properties of a quasiprojective variety in terms of algebraic properties of the fundamental group.
- Zariski-Lefschetz: it is enough to consider curves and surfaces.

Settings

- A *quasiprojective* group is the fundamental group of a quasiprojective smooth variety.
- Goal 1: Find obstructions to determine if a group is quasiprojective.
- Goal 2: Relate geometrical properties of a quasiprojective variety in terms of algebraic properties of the fundamental group.

- Any finitely presented group is the fundamental group of compact orientable 4-manifold.

Settings

- A *quasiprojective* group is the fundamental group of a quasiprojective smooth variety.
- Goal 1: Find obstructions to determine if a group is quasiprojective.
- Goal 2: Relate geometrical properties of a quasiprojective variety in terms of algebraic properties of the fundamental group.

- Any finitely presented group is the fundamental group of compact orientable 4-manifold.
- Kähler groups are fundamental groups of compact Kähler 4-manifolds: there are obstructions for a group to be Kähler.

Character Torus

- G (finitely presented) group, X space such that $G = \pi_1(X)$.

Character Torus

- G (finitely presented) group, X space such that $G = \pi_1(X)$.
- $H := G/G' = H_1(X;\mathbb{Z})$, $k := \operatorname{rk} H$, $\operatorname{Tors} H$ the torsion of H.

Character Torus

- G (finitely presented) group, X space such that $G = \pi_1(X)$.
- $H := G/G' = H_1(X;\mathbb{Z})$, $k := \operatorname{rk} H$, $\operatorname{Tors} H$ the torsion of H.
- $\mathbb{T}_G := H^1(X;\mathbb{C}^*) = H^1(G;\mathbb{C}^*) = \operatorname{Hom}(G,\mathbb{C}^*) = \operatorname{Hom}(H,\mathbb{Z})$.

Character Torus

- G (finitely presented) group, X space such that $G = \pi_1(X)$.
- $H := G/G' = H_1(X;\mathbb{Z})$, $k := \operatorname{rk} H$, $\operatorname{Tors} H$ the torsion of H.
- $\mathbb{T}_G := H^1(X;\mathbb{C}^*) = H^1(G;\mathbb{C}^*) = \operatorname{Hom}(G,\mathbb{C}^*) = \operatorname{Hom}(H,\mathbb{Z})$.
- $\Lambda := \Lambda_{\mathbb{C}} = \mathbb{C}[H]$, $\mathbb{T}_G = \operatorname{Spec}\Lambda$.

Character Torus

- G (finitely presented) group, X space such that $G = \pi_1(X)$.
- $H := G/G' = H_1(X;\mathbb{Z})$, $k := \operatorname{rk} H$, $\operatorname{Tors} H$ the torsion of H.
- $\mathbb{T}_G := H^1(X;\mathbb{C}^*) = H^1(G;\mathbb{C}^*) = \operatorname{Hom}(G,\mathbb{C}^*) = \operatorname{Hom}(H,\mathbb{Z})$.
- $\Lambda := \Lambda_{\mathbb{C}} = \mathbb{C}[H]$, $\mathbb{T}_G = \operatorname{Spec} \Lambda$.
- $\mathbb{T}_G$ is an abelian complex Lie group, $\mathbb{T}_G^{\mathbf{1}}$ the component through the origin $\mathbf{1}$ is isomorphic to $(\mathbb{C}^*)^k$.

Character Torus

- G (finitely presented) group, X space such that $G = \pi_1(X)$.
- $H := G/G' = H_1(X;\mathbb{Z})$, $k := \operatorname{rk} H$, $\operatorname{Tors} H$ the torsion of H.
- $\mathbb{T}_G := H^1(X;\mathbb{C}^*) = H^1(G;\mathbb{C}^*) = \operatorname{Hom}(G,\mathbb{C}^*) = \operatorname{Hom}(H,\mathbb{Z})$.
- $\Lambda := \Lambda_{\mathbb{C}} = \mathbb{C}[H]$, $\mathbb{T}_G = \operatorname{Spec}\Lambda$.
- $\mathbb{T}_G$ is an abelian complex Lie group, $\mathbb{T}_G^{\mathbf{1}}$ the component through the origin $\mathbf{1}$ is isomorphic to $(\mathbb{C}^*)^k$.
- $1 \to \mathbb{T}_G^{\mathbf{1}} \to \mathbb{T}_G \to \operatorname{Hom}(\operatorname{Tors} H, \mathbb{C}^*) \to 1$.

Construction of the Alexander Invariant

X

Construction of the Alexander Invariant

$$\begin{array}{l} \tilde{X} \text{ maximal abelian covering of } X \\ \Big\downarrow \pi \\ X \end{array}$$

Construction of the Alexander Invariant

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"] & \circlearrowleft H \\ X \end{tikzcd}$$

Construction of the Alexander Invariant

Alexander Invariant

$H_1(\tilde{X};\mathbb{C}) = (G'/G'') \otimes_{\mathbb{Z}} \mathbb{C}$ is a Λ-module, $\Lambda = \mathbb{C}[H]$.

Construction of the Alexander Invariant

Alexander Invariant

$H_1(\tilde{X};\mathbb{C}) = (G'/G'') \otimes_{\mathbb{Z}} \mathbb{C}$ is a Λ-module, $\Lambda = \mathbb{C}[H]$.

Alexander presentation

- Choose X to be a CW-complex and consider the chain complex $C_*(X;\mathbb{C})$.

Construction of the Alexander Invariant

Alexander Invariant

$H_1(\tilde{X};\mathbb{C}) = (G'/G'') \otimes_{\mathbb{Z}} \mathbb{C}$ is a Λ-module, $\Lambda = \mathbb{C}[H]$.

Alexander presentation

- Choose X to be a CW-complex and consider the chain complex $C_*(X;\mathbb{C})$.
- $\tilde{X}$ inherits a CW-complex structure

Construction of the Alexander Invariant

Alexander Invariant

$H_1(\tilde{X};\mathbb{C}) = (G'/G'') \otimes_{\mathbb{Z}} \mathbb{C}$ is a Λ-module, $\Lambda = \mathbb{C}[H]$.

Alexander presentation

- Choose X to be a CW-complex and consider the chain complex $C_*(X;\mathbb{C})$.
- $\tilde{X}$ inherits a CW-complex structure
- Given σ an r-cell of X, fix $\tilde{\sigma}$ an r-cell in $\tilde{X}$ such that $\pi(\tilde{\sigma}) = \sigma$.

Construction of the Alexander Invariant

Alexander Invariant

$H_1(\tilde{X};\mathbb{C}) = (G'/G'') \otimes_{\mathbb{Z}} \mathbb{C}$ is a Λ-module, $\Lambda = \mathbb{C}[H]$.

Alexander presentation

- Choose X to be a CW-complex and consider the chain complex $C_*(X;\mathbb{C})$.
- $\tilde{X}$ inherits a CW-complex structure
- Given σ an r-cell of X, fix $\tilde{\sigma}$ an r-cell in $\tilde{X}$ such that $\pi(\tilde{\sigma}) = \sigma$.
- The cells of $\tilde{X}$ are $\{\tilde{\sigma}^h | \sigma \ r\text{-cell of } X, h \in H\}$.

Construction of the Alexander Invariant

Alexander Invariant

$H_1(\tilde{X};\mathbb{C}) = (G'/G'') \otimes_{\mathbb{Z}} \mathbb{C}$ is a Λ-module, $\Lambda = \mathbb{C}[H]$.

Alexander presentation

- Choose X to be a CW-complex and consider the chain complex $C_*(X;\mathbb{C})$.
- $\tilde{X}$ inherits a CW-complex structure
- Given σ an r-cell of X, fix $\tilde{\sigma}$ an r-cell in $\tilde{X}$ such that $\pi(\tilde{\sigma}) = \sigma$.
- The cells of $\tilde{X}$ are $\{\tilde{\sigma}^h | \sigma\ r\text{-cell of } X, h \in H\}$.
- $C_*(\tilde{X};\mathbb{C})$ is complex of free Λ-modules.

Twisted Cohomology

Local system of coefficients

- $\xi \in \mathbb{T}_G$.
- Define a *locally constant* sheaf $\underline{\mathbb{C}}_\xi$.
- $\tilde{X} \times \mathbb{C}$ sheaf over X ($\mathbb{C}$ with the discrete topology).
- Action of H given by $(\tilde{x}, t)^h := (\tilde{x}^h, \xi(h^{-1})t)$.
- $\underline{\mathbb{C}}_\xi$ is the quotient of $\tilde{X} \times \mathbb{C}$ by this action.

Twisted Cohomology

Local system of coefficients

- $\xi \in \mathbb{T}_G$.
- Define a *locally constant* sheaf $\underline{\mathbb{C}}_\xi$.
- $\tilde{X} \times \mathbb{C}$ sheaf over X ($\mathbb{C}$ with the discrete topology).
- Action of H given by $(\tilde{x}, t)^h := (\tilde{x}^h, \xi(h^{-1})t)$.
- $\underline{\mathbb{C}}_\xi$ is the quotient of $\tilde{X} \times \mathbb{C}$ by this action.

Definition of twisted cohomology

$H^1(X; \underline{\mathbb{C}}_\xi)$ means *sheaf cohomology*. It depends only on G!

Computation of twisted cohomology

- Fix X a CW-complex with $\pi_1(X) = G$ as before.

Computation of twisted cohomology

- Fix X a CW-complex with $\pi_1(X) = G$ as before.
- Endow $\mathbb{C}$ with a structure $\mathbb{C}_\xi$ of Λ-module: if $t \in H$, $t \cdot z := \xi(t)z$.

Computation of twisted cohomology

- Fix X a CW-complex with $\pi_1(X) = G$ as before.
- Endow $\mathbb{C}$ with a structure $\mathbb{C}_\xi$ of Λ-module: if $t \in H$, $t \cdot z := \xi(t)z$.
- Consider the complexes $C_*(X;\mathbb{C}_\xi) := C_*(\tilde{X};\mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$ and $C^*(X;\mathbb{C}_\xi) := C^*(\tilde{X};\mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$.

Computation of twisted cohomology

- Fix X a CW-complex with $\pi_1(X) = G$ as before.
- Endow $\mathbb{C}$ with a structure $\mathbb{C}_\xi$ of Λ-module: if $t \in H$, $t \cdot z := \xi(t)z$.
- Consider the complexes $C_*(X;\mathbb{C}_\xi) := C_*(\tilde{X};\mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$ and $C^*(X;\mathbb{C}_\xi) := C^*(\tilde{X};\mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$.
- $H^1(X;\underline{\mathbb{C}}_\xi)$ is the cohomology in degree 1 of $C^*(X;\mathbb{C}_\xi)$.

Computation of twisted cohomology

- Fix X a CW-complex with $\pi_1(X) = G$ as before.
- Endow $\mathbb{C}$ with a structure $\mathbb{C}_\xi$ of Λ-module: if $t \in H$, $t \cdot z := \xi(t) z$.
- Consider the complexes $C_*(X; \mathbb{C}_\xi) := C_*(\tilde{X}; \mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$ and $C^*(X; \mathbb{C}_\xi) := C^*(\tilde{X}; \mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$.
- $H^1(X; \underline{\mathbb{C}}_\xi)$ is the cohomology in degree 1 of $C^*(X; \mathbb{C}_\xi)$.
- For computational purposes we will consider $H_1(X; \underline{\mathbb{C}}_\xi)$.

Examples I

Free group

Examples I

Free group

- $\mathbb{F}_k$ the free group generated by $x_1, \dots, x_k$

Examples I

Free group

- $\mathbb{F}_k$ the free group generated by $x_1, \ldots, x_k$
- H is the free abelian group generated by $t_1, \ldots, t_k$ ($t_j \equiv x_j \mod G'$).

Examples I

Free group

- $\mathbb{F}_k$ the free group generated by $x_1, \ldots, x_k$
- H is the free abelian group generated by $t_1, \ldots, t_k$ ($t_j \equiv x_j \mod G'$).
- Let X be the CW-complex associated to the presentation

Examples I

Free group

- $\mathbb{F}_k$ the free group generated by $x_1, \dots, x_k$
- H is the free abelian group generated by $t_1, \dots, t_k$ ($t_j \equiv x_j \mod G'$).
- Let X be the CW-complex associated to the presentation
- Λ-Bases for $C_*(\tilde{X}; \mathbb{C})$.

Examples I

Free group

- $\mathbb{F}_k$ the free group generated by $x_1, \ldots, x_k$
- H is the free abelian group generated by $t_1, \ldots, t_k$ ($t_j \equiv x_j \mod G'$).
- Let X be the CW-complex associated to the presentation
- Λ-Bases for $C_*(\tilde{X}; \mathbb{C})$.

$C_0 \quad (p)$ (p is the unique 0-cell).

Examples I

Free group

- $\mathbb{F}_k$ the free group generated by $x_1, \ldots, x_k$
- H is the free abelian group generated by $t_1, \ldots, t_k$ ($t_j \equiv x_j \mod G'$).
- Let X be the CW-complex associated to the presentation
- Λ-Bases for $C_*(\tilde{X}; \mathbb{C})$.

C_0 (p) (p is the unique 0-cell).
C_1 $(x_1, \ldots, x_k)$.

Examples I

Free group

- $\mathbb{F}_k$ the free group generated by $x_1, \ldots, x_k$
- H is the free abelian group generated by $t_1, \ldots, t_k$ ($t_j \equiv x_j \mod G'$).
- Let X be the CW-complex associated to the presentation
- Λ-Bases for $C_*(\tilde{X}; \mathbb{C})$.

C_0 (p) (p is the unique 0-cell).
C_1 $(x_1, \ldots, x_k)$.
C_2 $\emptyset$.

Examples I

Free group

- $\mathbb{F}_k$ the free group generated by $x_1, \ldots, x_k$
- H is the free abelian group generated by $t_1, \ldots, t_k$ ($t_j \equiv x_j \mod G'$).
- Let X be the CW-complex associated to the presentation
- Λ-Bases for $C_*(\tilde{X}; \mathbb{C})$.

C_0 (p) (p is the unique 0-cell).
C_1 $(x_1, \ldots, x_k)$.
C_2 $\emptyset$.

- The matrix for δ_1 is $(t_1 - 1 \ldots t_k - 1)$.

Examples I

Free group

- $\mathbb{F}_k$ the free group generated by $x_1, \dots, x_k$
- H is the free abelian group generated by $t_1, \dots, t_k$ ($t_j \equiv x_j \mod G'$).
- Let X be the CW-complex associated to the presentation
- Λ-Bases for $C_*(\tilde{X}; \mathbb{C})$.

$$\begin{array}{ll} C_0 & (p) \text{ (} p \text{ is the unique 0-cell).} \\ C_1 & (x_1, \dots, x_k). \\ C_2 & \emptyset. \end{array}$$

- The matrix for δ_1 is $(t_1 - 1 \dots t_k - 1)$.
- $k - 1 \leq \dim H_1(X; \mathbb{C}_\xi) \leq k$ (it equals $k \Leftrightarrow \xi = \mathbf{1}$).

Examples II

Zariski group

Examples II

Zariski group

- $G = \langle x, y | x^2 = y^3 = 1 \rangle$

Examples II

Zariski group

- $G = \langle x, y | x^2 = y^3 = 1 \rangle$
- $H = \langle t | t^6 = 1 \rangle$, $x \mapsto t^3$, $y \mapsto t^2$.

Examples II

Zariski group

- $G = \langle x, y | x^2 = y^3 = 1 \rangle$
- $H = \langle t | t^6 = 1 \rangle$, $x \mapsto t^3$, $y \mapsto t^2$.
- The matrix for δ_1 is $\begin{pmatrix} t^3 - 1 & t^2 - 1 \end{pmatrix}$.

Examples II

Zariski group

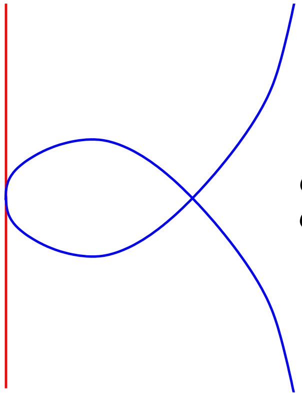
- $G = \langle x, y | x^2 = y^3 = 1 \rangle$
- $H = \langle t | t^6 = 1 \rangle$, $x \mapsto t^3$, $y \mapsto t^2$.
- The matrix for δ_1 is $\begin{pmatrix} t^3 - 1 & t^2 - 1 \end{pmatrix}$.
- (Fox Derivation) δ_2: $\begin{pmatrix} 1 + t^3 & 0 \\ 0 & 1 + t^2 + t^4 \end{pmatrix}$.

Examples II

Zariski group

- $G = \langle x, y | x^2 = y^3 = 1 \rangle$
- $H = \langle t | t^6 = 1 \rangle$, $x \mapsto t^3$, $y \mapsto t^2$.
- The matrix for δ_1 is $\begin{pmatrix} t^3 - 1 & t^2 - 1 \end{pmatrix}$.
- (Fox Derivation) δ_2: $\begin{pmatrix} 1 + t^3 & 0 \\ 0 & 1 + t^2 + t^4 \end{pmatrix}$.
- $\dim H_1(X; \underline{\mathbb{C}}_\xi) = 1$ if $\xi(t)$ is a primitive 6^{th}-root of unity.

A curve group



$$C : (x+z)z(y^2z - x^2z - x^3) = 0$$
$$G = \langle x, y, z | [x, y] = 1, y = z^{-1}xz, [yx, z] = 1 \rangle.$$

Examples III

Curve group

Examples III

Curve group

- $G = \langle x, z | [x, z^2] = 1, [x, zxz] = 1 \rangle$.

Examples III

Curve group

- $G = \langle x, z | [x, z^2] = 1, [x, zxz] = 1 \rangle$.
- $H = \langle t, s | [t, s] = 1 \rangle, x \mapsto t, z \mapsto s$.

Examples III

Curve group

- $G = \langle x, z | [x, z^2] = 1, [x, zxz] = 1 \rangle$.
- $H = \langle t, s | [t, s] = 1 \rangle$, $x \mapsto t$, $z \mapsto s$.
- The matrix for δ_1 is $\begin{pmatrix} t-1 & s-1 \end{pmatrix}$.

Examples III

Curve group

- $G = \langle x, z | [x, z^2] = 1, [x, zxz] = 1 \rangle$.
- $H = \langle t, s | [t, s] = 1 \rangle$, $x \mapsto t$, $z \mapsto s$.
- The matrix for δ_1 is $\begin{pmatrix} t-1 & s-1 \end{pmatrix}$.
- δ_2: $\begin{pmatrix} 1-s^2 & (1-t)(1+s) \\ (1+ts)(1-s) & (t-1)(1+ts) \end{pmatrix}$.

Examples III

Curve group

- $G = \langle x, z | [x, z^2] = 1, [x, zxz] = 1 \rangle$.
- $H = \langle t, s | [t, s] = 1 \rangle$, $x \mapsto t$, $z \mapsto s$.
- The matrix for δ_1 is $\begin{pmatrix} t-1 & s-1 \end{pmatrix}$.
- δ_2: $\begin{pmatrix} 1-s^2 & (1-t)(1+s) \\ (1+ts)(1-s) & (t-1)(1+ts) \end{pmatrix}$.
- $\dim H_1(X; \underline{\mathbb{C}}_\xi) = 1$ if $\xi(t) = 1$, $\xi(s) = -1$.

Deligne's theory I

Deligne extensions

- $X = \bar{X} \setminus \mathcal{D}$, $\bar{X}$ smooth projective, $\mathcal{D}$ normal crossing divisor.
- Flat line bundle (L_ξ, ∇) on X where $L_\xi = \underline{\mathbb{C}}_\xi \otimes \mathcal{O}_X$.
- $(\bar{L}_\xi, \bar{\nabla})$ extension to $\bar{X}$
- A lot of extensions are possible: parametrized by $\mathbb{Z}^n$, n number of irreducible components of $\mathcal{D}$.

Deligne's theory II

Example

- Let $X = \mathbb{C}^*$, x a generator of $\pi_1(X)$ realized by $t \mapsto \exp(2i\pi t)$.
- Let ξ a character such that $\xi(x) = \exp(-2i\pi\alpha)$, for some $\alpha \in \mathbb{C}$
- Take a trivial bundle on $\mathbb{C}$ with section σ such that $\nabla(\tau) = 0$ for $\tau : z \mapsto z^{-\alpha}\sigma$ multivalued section.
- $\bar{\nabla}(\sigma) = \alpha\frac{dz}{z} \otimes \sigma$: α is the residue along $0 \in \mathbb{C}$.

Deligne's theory II

Example

- Let $X = \mathbb{C}^*$, x a generator of $\pi_1(X)$ realized by $t \mapsto \exp(2i\pi t)$.
- Let ξ a character such that $\xi(x) = \exp(-2i\pi\alpha)$, for some $\alpha \in \mathbb{C}$
- Take a trivial bundle on $\mathbb{C}$ with section σ such that $\nabla(\tau) = 0$ for $\tau : z \mapsto z^{-\alpha}\sigma$ multivalued section.
- $\bar{\nabla}(\sigma) = \alpha\frac{dz}{z} \otimes \sigma$: α is the residue along $0 \in \mathbb{C}$.

Definition

The Deligne extension is defined as follows: the real part of the residues are in $[0, 1)$.

Deligne's theory III

Theorem

1. $H^1(X;\underline{\mathbb{C}}_\xi) = H_\xi^{\mathcal{O}} \oplus H_\xi^{\bar{\mathcal{O}}}$.
2. *If the residues are not positive integers then,* $H_\xi^{\mathcal{O}}$ *is the homology of*

$$H^0(\bar{X};\bar{L}_\xi) \xrightarrow{\bar{\nabla}} H^0(\bar{X};\Omega^1_{\bar{X}}(\log\mathcal{D})\otimes\bar{L}_\xi) \xrightarrow{\bar{\nabla}} H^0(\bar{X};\Omega^2_{\bar{X}}(\log\mathcal{D})\otimes\bar{L}_\xi)$$

and $H_\xi^{\bar{\mathcal{O}}}$ *is* $\ker\left(\bar{\nabla} : H^1(\bar{X};\bar{L}_\xi) \to H^1(\bar{X};\Omega^1_{\bar{X}}(\log\mathcal{D})\otimes\bar{L}_\xi\right)$.

3. ξ *is unitary,* $\tilde{L}_\xi$ *is the Deligne extension:*

$$H_\xi^{\mathcal{O}} = H^0(\bar{X};\Omega^1_{\bar{X}}(\log\mathcal{D})\otimes\tilde{L}_\xi), \quad H_\xi^{\bar{\mathcal{O}}} = H^1(\bar{X};\tilde{L}_\xi).$$

Definition of Characteristic Varieties

Definition

The k-th *characteristic variety* of X (or G) is the subvariety of $\mathbb{T}_G$, defined by $\mathcal{V}_k(G) := \{\xi \in \mathbb{T}_G \mid \dim H^1(X, \underline{\mathbb{C}}_\xi) \geq k\}$.

Definition of Characteristic Varieties

Definition

The k-th *characteristic variety* of X (or G) is the subvariety of $\mathbb{T}_G$, defined by $\mathcal{V}_k(G) := \{\xi \in \mathbb{T}_G \mid \dim H^1(X, \underline{\mathbb{C}}_\xi) \geq k\}$.

Properties

Definition of Characteristic Varieties

Definition

The k-th *characteristic variety* of X (or G) is the subvariety of $\mathbb{T}_G$, defined by $\mathcal{V}_k(G) := \{\xi \in \mathbb{T}_G \mid \dim H^1(X, \underline{\mathbb{C}}_\xi) \geq k\}$.

Properties

- Characteristic varieties are algebraic subvarieties of $\mathbb{T}_G$ defined over $\mathbb{Q}$.

Definition of Characteristic Varieties

Definition

The k-th *characteristic variety* of X (or G) is the subvariety of $\mathbb{T}_G$, defined by $\mathcal{V}_k(G) := \{\xi \in \mathbb{T}_G \mid \dim H^1(X, \underline{\mathbb{C}_\xi}) \geq k\}$.

Properties

- Characteristic varieties are algebraic subvarieties of $\mathbb{T}_G$ defined over $\mathbb{Q}$.
- $\mathcal{V}_k$ is the set of zeros of the k^{th}-Fitting ideal of the Alexander invariant (except possibly at $\mathbf{1}$).

Definition of Characteristic Varieties

Definition

The k-th *characteristic variety* of X (or G) is the subvariety of $\mathbb{T}_G$, defined by $\mathcal{V}_k(G) := \{\xi \in \mathbb{T}_G \mid \dim H^1(X, \underline{\mathbb{C}}_\xi) \geq k\}$.

Properties

- Characteristic varieties are algebraic subvarieties of $\mathbb{T}_G$ defined over $\mathbb{Q}$.
- $\mathcal{V}_k$ is the set of zeros of the k^{th}-Fitting ideal of the Alexander invariant (except possibly at $\mathbf{1}$).
- If ξ is torsion then its depth (the maximal k such that $\xi \in \mathcal{V}_k$) is related with jumpings of Betti numbers of finite abelian coverings.

Computations

Hypersurface case: Libgober approach

- Libgober gives a method to compute *most* irreducible components of $\mathcal{V}_k(\mathbb{P}^2 \setminus C)$ without computing the fundamental group.
- Using Sakuma's formula the problem is reduced to compute equivariant Betti numbers of finite abelian coverings of $X := \mathbb{P}^2 \setminus C$:

Computations

Hypersurface case: Libgober approach

- Libgober gives a method to compute *most* irreducible components of $\mathcal{V}_k(\mathbb{P}^2 \setminus C)$ without computing the fundamental group.
- Using Sakuma's formula the problem is reduced to compute equivariant Betti numbers of finite abelian coverings of $X := \mathbb{P}^2 \setminus C$:
  - Let σ a torsion element of T_G of order $\ell > 1$
  - ρ is associated to a cyclic ℓ-fold covering $\rho_\sigma : Y_\sigma \to X$
  - There is a natural decomposition of $H^1(Y_\sigma; \mathbb{C}) = \oplus_{j=0}^{\ell-1} H_{\sigma^j}$
  - $\sigma \in \mathcal{V}_k(G) \Leftrightarrow \dim H_\sigma \geq k$.

Computations

Hypersurface case: Libgober approach

- Libgober gives a method to compute *most* irreducible components of $\mathcal{V}_k(\mathbb{P}^2 \setminus C)$ without computing the fundamental group.
- Using Sakuma's formula the problem is reduced to compute equivariant Betti numbers of finite abelian coverings of $X := \mathbb{P}^2 \setminus C$:
- Using quasiadjunction polytopes and ideals for the singular points, one obtains a finite combinatorial partition; it is enough to compute the twisted cohomology for one character in each partition.
- The *position* of the quasiadjunction ideals on each point is the key point in the computation.

Comments

Epimorphisms

If $G_1 \to G_2$ is an epimorphism, then $\mathbb{T}_{G_2}$ is a subtorus of $\mathbb{T}_{G_1}$ and $\mathcal{V}_k(G_2) \subset \mathcal{V}_k(G_1)$.

Comments

Epimorphisms

If $G_1 \to G_2$ is an epimorphism, then $\mathbb{T}_{G_2}$ is a subtorus of $\mathbb{T}_{G_1}$ and $\mathcal{V}_k(G_2) \subset \mathcal{V}_k(G_1)$.

Theorem (Arapura)

Let Σ be an irreducible component of $\mathcal{V}_1(G)$. Then,

1. *If $\dim \Sigma > 0$ then there exists a surjective morphism $\rho : X \to C$, C algebraic curve, and a torsion element σ such that $\Sigma = \sigma\rho^*(H^1(C;\mathbb{C}^*))$.*
2. *If $\dim \Sigma = 0$ then Σ is unitary.*

In particular, positive dimensional irreducible components are subtori translated by torsion elements.

Comments

Epimorphisms

If $G_1 \to G_2$ is an epimorphism, then $\mathbb{T}_{G_2}$ is a subtorus of $\mathbb{T}_{G_1}$ and $\mathcal{V}_k(G_2) \subset \mathcal{V}_k(G_1)$.

Uncovered Components

- Isolated unitary non torsion points are not visible with Libgober's method.
- Following Deligne, if ξ is torsion then, for the Deligne extension $H^{\mathcal{O}}_\xi = H^0(\bar{X}; \Omega^1_{\bar{X}}(\log \mathcal{D}) \otimes \tilde{L}_\xi)$, $\quad H^{\bar{\mathcal{O}}}_\xi = H^1(\bar{X}; \tilde{L}_\xi)$.
- Libgober proves that if ξ ramifies along any irreducible component of C then $H^0(\bar{X}; \Omega^1_{\bar{X}}(\log \mathcal{D}) \otimes \tilde{L}_\xi) = H^0(\bar{X}; \Omega^1_{\bar{X}} \otimes \tilde{L}_\xi)$.

Theorem

Theorem

Let X be a quasi-projective smooth variety and let $\mathcal{V} := \mathcal{V}_k(X)$ be the k^{th} characteristic variety of X. Let V be an irreducible component of $\mathcal{V}$. Then one of the two following statements holds:

Theorem

Theorem

Let X be a quasi-projective smooth variety and let $\mathcal{V} := \mathcal{V}_k(X)$ be the k^{th} characteristic variety of X. Let V be an irreducible component of $\mathcal{V}$. Then one of the two following statements holds:

1. *There exists a surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component V_1 of $\mathcal{V}_k(\pi_1^{\text{orb}}(C_\varphi))$ such that $V = \rho^*(V_1)$.*

Theorem

Theorem

Let X be a quasi-projective smooth variety and let $\mathcal{V} := \mathcal{V}_k(X)$ be the k^{th} characteristic variety of X. Let V be an irreducible component of $\mathcal{V}$. Then one of the two following statements holds:

1. *There exists a surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component V_1 of $\mathcal{V}_k(\pi_1^{\text{orb}}(C_\varphi))$ such that $V = \rho^*(V_1)$.*
2. *V is an isolated torsion point.*

Theorem

Theorem

Let X be a quasi-projective smooth variety and let $\mathcal{V} := \mathcal{V}_k(X)$ be the k^{th} characteristic variety of X. Let V be an irreducible component of $\mathcal{V}$. Then one of the two following statements holds:

1. *There exists a surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component V_1 of $\mathcal{V}_k(\pi_1^{\text{orb}}(C_\varphi))$ such that $V = \rho^*(V_1)$.*
2. *V is an isolated torsion point.*

Consequence

Behaviour of torsion characters determine characteristic varieties.

Orbifold groups

Orbifold

An orbifold X_φ is a quasiprojective Riemann surface X with a function $\varphi : X \to \mathbb{N}$ with value 1 outside a finite number of points.

Orbifold groups

Orbifold

An orbifold X_φ is a quasiprojective Riemann surface X with a function $\varphi : X \to \mathbb{N}$ with value 1 outside a finite number of points.

Orbifold group

For an orbifold X_φ, let $p_1, \ldots, p_n$ the points such that $\varphi(p_j) := m_j > 1$. Then

$$\pi_1^{\text{orb}} := \pi_1(X \setminus \{p_1, \ldots, p_n\}) / \langle \mu_j^{m_j} = 1 \rangle$$

where μ_j is a meridian of p_j. We denote X_φ by $X_{m_1,\ldots,m_n}$.

Orbifold maps

Definition

A dominant algebraic morphism $\rho : Y \to X$ defines an orbifold morphism $Y \to X_\varphi$ if for all $p \in X$, the divisor $\rho^*(p) = \varphi(p)\mathcal{D}_p$.

Orbifold maps

Definition

A dominant algebraic morphism $\rho : Y \to X$ defines an orbifold morphism $Y \to X_\varphi$ if for all $p \in X$, the divisor $\rho^*(p) = \varphi(p)\mathcal{D}_p$.

Example

If $\rho : X \to C_\varphi$ is an orbifold morphism and C is a rational curve, then ρ comes from a pencil in $\bar{X}$; the multiple points comes from multiple fibers of the pencil outside $\mathcal{D}$.

Orbifold maps

Definition

A dominant algebraic morphism $\rho : Y \to X$ defines an orbifold morphism $Y \to X_\varphi$ if for all $p \in X$, the divisor $\rho^*(p) = \varphi(p)\mathcal{D}_p$.

Example

If $\rho : X \to C_\varphi$ is an orbifold morphism and C is a rational curve, then ρ comes from a pencil in $\bar{X}$; the multiple points comes from multiple fibers of the pencil outside $\mathcal{D}$.

Remark

If $\bar{X}$ is rational only morphisms on rational orbifolds are allowed.

Curves and pencils

Example

Curves and pencils

Example

- $C = \{(x+z)z(y^2z - x^2z - x^3) = 0\}$

Curves and pencils

Example

- $C = \{(x+z)z(y^2z - x^2z - x^3) = 0\}$
- $X := \mathbb{P}^2 \setminus C = \mathbb{C}^2 \setminus \{(x+1)(y^2 - x^2 - x^3) = 0\}$

Curves and pencils

Example

- $C = \{(x+z)z(y^2z - x^2z - x^3) = 0\}$
- $X := \mathbb{P}^2 \setminus C = \mathbb{C}^2 \setminus \{(x+1)(y^2 - x^2 - x^3) = 0\}$
- $\rho : X \to \mathbb{C}$, $\rho(x,y) := \frac{x^2(x+1)}{y^2 - x^2 - x^3}$

Curves and pencils

Example

- $C = \{(x+z)z(y^2z - x^2z - x^3) = 0\}$
- $X := \mathbb{P}^2 \setminus C = \mathbb{C}^2 \setminus \{(x+1)(y^2 - x^2 - x^3) = 0\}$
- $\rho : X \to \mathbb{C}$, $\rho(x,y) := \frac{x^2(x+1)}{y^2 - x^2 - x^3}$
- Two multiple fibers outside $\mathcal{D}$: $y^2z = 0$ and $x^2(x+z) = 0$.

Curves and pencils

Example

- $C = \{(x+z)z(y^2z - x^2z - x^3) = 0\}$
- $X := \mathbb{P}^2 \setminus C = \mathbb{C}^2 \setminus \{(x+1)(y^2 - x^2 - x^3) = 0\}$
- $\rho : X \to \mathbb{C}$, $\rho(x,y) := \frac{x^2(x+1)}{y^2 - x^2 - x^3}$
- Two multiple fibers outside $\mathcal{D}$: $y^2z = 0$ and $x^2(x+z) = 0$.
- $\pi_1^{\text{orb}}(C_\varphi) = \mathbb{Z}/2 * \mathbb{Z}/2$

Curves and pencils

Example

- $C = \{(x+z)z(y^2z - x^2z - x^3) = 0\}$
- $X := \mathbb{P}^2 \setminus C = \mathbb{C}^2 \setminus \{(x+1)(y^2 - x^2 - x^3) = 0\}$
- $\rho : X \to \mathbb{C}$, $\rho(x,y) := \frac{x^2(x+1)}{y^2-x^2-x^3}$
- Two multiple fibers outside $\mathcal{D}$: $y^2z = 0$ and $x^2(x+z) = 0$.
- $\pi_1^{\text{orb}}(C_\varphi) = \mathbb{Z}/2 * \mathbb{Z}/2$
- This case is uncovered by Libgober's method.

Properties of characteristic varieties of orbifolds

- Except **1**, irreducible components of $\mathcal{V}_k$ are connected components of $\mathbb{T}_G$

Properties of characteristic varieties of orbifolds

- Except $\mathbf{1}$, irreducible components of $\mathcal{V}_k$ are connected components of $\mathbb{T}_G$
- For non-projective orbifolds, if $k = \operatorname{rk} H$, then the component through $\mathbf{1}$ is in $\mathcal{V}_{k-1} \setminus \mathcal{V}_k$ (depth $k-1$). The other components are in $\mathcal{V}_k$ (at least).

Properties of characteristic varieties of orbifolds

- Except $\mathbf{1}$, irreducible components of $\mathcal{V}_k$ are connected components of $\mathbb{T}_G$
- For non-projective orbifolds, if $k = \operatorname{rk} H$, then the component through $\mathbf{1}$ is in $\mathcal{V}_{k-1} \setminus \mathcal{V}_k$ (depth $k-1$). The other components are in $\mathcal{V}_k$ (at least).
- For projective orbifolds of genus g, then the component through $\mathbf{1}$ has depth $2g-2$. The other components are in $\mathcal{V}_{2g-1}$ (at least).

Properties of characteristic varieties of quasiprojective groups G

- The irreducible components of $\mathcal{V}_1(G)$ are subtori translated by torsion elements. Given such a subvariety Σ its *shadow* $\operatorname{Sh}\Sigma$ is the paralell subtorus through the origin.

Properties of characteristic varieties of quasiprojective groups G

- The irreducible components of $\mathcal{V}_1(G)$ are subtori translated by torsion elements. Given such a subvariety Σ its *shadow* $\operatorname{Sh}\Sigma$ is the paralell subtorus through the origin.
- Σ_1, Σ_2 irreducible components of $\mathcal{V}_1(G)$, $\dim(\Sigma_1 \cap \Sigma_2) > 0 \Rightarrow \Sigma_1 = \Sigma_2$. The same happens for their shadows.

Properties of characteristic varieties of quasiprojective groups G

- The irreducible components of $\mathcal{V}_1(G)$ are subtori translated by torsion elements. Given such a subvariety Σ its *shadow* $\operatorname{Sh}\Sigma$ is the paralell subtorus through the origin.
- Σ_1, Σ_2 irreducible components of $\mathcal{V}_1(G)$, $\dim(\Sigma_1 \cap \Sigma_2) > 0 \Rightarrow \Sigma_1 = \Sigma_2$. The same happens for their shadows.
- Σ_1, Σ_2 irreducible components of $\mathcal{V}_1(G)$ of positive dimension $\Rightarrow \Sigma_1 \cap \Sigma_2 \subset V_2(G)$ and consists of torsion points. If their shadows are not equal, $\mathbf{1}$ is an isolated intersection point.

Properties of characteristic varieties of quasiprojective groups G

- The irreducible components of $\mathcal{V}_1(G)$ are subtori translated by torsion elements. Given such a subvariety Σ its *shadow* $\operatorname{Sh}\Sigma$ is the paralell subtorus through the origin.
- Σ_1, Σ_2 irreducible components of $\mathcal{V}_1(G)$, $\dim(\Sigma_1 \cap \Sigma_2) > 0 \Rightarrow \Sigma_1 = \Sigma_2$. The same happens for their shadows.
- Σ_1, Σ_2 irreducible components of $\mathcal{V}_1(G)$ of positive dimension $\Rightarrow \Sigma_1 \cap \Sigma_2 \subset V_2(G)$ and consists of torsion points. If their shadows are not equal, $\mathbf{1}$ is an isolated intersection point.
- If Σ has depth k and $\mathbf{1} \notin \Sigma$ then, $\dim \Sigma \leq k+1$ ($\leq k$ if $\dim \Sigma$ odd). Moreover $\operatorname{Sh}\Sigma$ has depth $\dim \Sigma - 2$ or $\dim \Sigma - 1$ (depth $\dim \Sigma - 1$ if $\dim \Sigma$ odd).

More properties quasiprojective groups G

- If Σ has depth k and $\xi \in \Sigma$ belongs to $\mathcal{V}_{k+1}$ then ξ is torsion.

More properties quasiprojective groups G

- If Σ has depth k and $\xi \in \Sigma$ belongs to $\mathcal{V}_{k+1}$ then ξ is torsion.
- If $\mathbf{1} \notin \Sigma$ and $\dim \Sigma > 2$, then its shadow is an irreducible component of $\mathcal{V}_1(G)$.

More properties quasiprojective groups G

- If Σ has depth k and $\xi \in \Sigma$ belongs to $\mathcal{V}_{k+1}$ then ξ is torsion.
- If $\mathbf{1} \notin \Sigma$ and $\dim \Sigma > 2$, then its shadow is an irreducible component of $\mathcal{V}_1(G)$.
- $\mathbf{1} \notin \Sigma$ and $\dim \Sigma = 2$, then its shadow is an irreducible component of $\mathcal{V}_1(G)$ if and only if it is for $\mathcal{V}_2(G)$.

More properties quasiprojective groups G

- If Σ has depth k and $\xi \in \Sigma$ belongs to $\mathcal{V}_{k+1}$ then ξ is torsion.
- If $\mathbf{1} \notin \Sigma$ and $\dim \Sigma > 2$, then its shadow is an irreducible component of $\mathcal{V}_1(G)$.
- $\mathbf{1} \notin \Sigma$ and $\dim \Sigma = 2$, then its shadow is an irreducible component of $\mathcal{V}_1(G)$ if and only if it is for $\mathcal{V}_2(G)$.
- If $\mathbf{1} \notin \Sigma$ and $\dim \Sigma = 1$, then its shadow is not an irreducible component of $\mathcal{V}_1(G)$.

More properties quasiprojective groups G

- If Σ has depth k and $\xi \in \Sigma$ belongs to $\mathcal{V}_{k+1}$ then ξ is torsion.
- If $\mathbf{1} \notin \Sigma$ and $\dim \Sigma > 2$, then its shadow is an irreducible component of $\mathcal{V}_1(G)$.
- $\mathbf{1} \notin \Sigma$ and $\dim \Sigma = 2$, then its shadow is an irreducible component of $\mathcal{V}_1(G)$ if and only if it is for $\mathcal{V}_2(G)$.
- If $\mathbf{1} \notin \Sigma$ and $\dim \Sigma = 1$, then its shadow is not an irreducible component of $\mathcal{V}_1(G)$.
- Let Σ_1 be an irreducible component of $\mathcal{V}_k(G)$ and let Σ_2 be an irreducible component of $\mathcal{V}_\ell(G)$. If $\xi \in \Sigma_1 \cap \Sigma_2$ then it is a torsion point and $\xi \in \mathcal{V}_{k+\ell}(G)$.

An Artin group

Example

Let $G := \langle x, y, z | [x, y] = 1, (yz)^2 = (zy)^2, (xz)^3 = (zx)^3 \rangle$; $\mathcal{V}_2(G) = \emptyset$ and $\mathcal{V}_1(G)$ has 5 irreducible components of dimension 1 Σ_i such that $\Sigma_i \cap \Sigma_{i+1}$ consists of one point (of torsion type). Then G is not quasiprojective.

An Artin group

Example

Let $G := \langle x,y,z | [x,y]=1, (yz)^2=(zy)^2, (xz)^3=(zx)^3\rangle$; $\mathcal{V}_2(G)=\emptyset$ and $\mathcal{V}_1(G)$ has 5 irreducible components of dimension 1 Σ_i such that $\Sigma_i\cap\Sigma_{i+1}$ consists of one point (of torsion type). Then G is not quasiprojective.

Theorem

Let $G_{p,q,r}$ the Artin group associated to a triangle with sides p,q,r

- *If $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\geq 1$ then there exists an affine curve $C_{p,q,r}$ such that $G_{p,q,r}=\pi_1(\mathbb{C}^2\setminus C_{p,q,r})$*
- *If p,q,r are even, not all of them equal and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ then the groups $G_{p,q,r}$ are not quasiprojective.*

Thanks for your attention