On the global monodromy of a fibration of the Fermat surface of odd degree n

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Abstract

The purpose of this paper is to investigate the global topological monodoromy of a certain fibration of the Fermat surface without using numerical analysis by computer.

1 Introduction

Let M be a complex surface and let B be a complex curve. A holomorphic map $f : M \to B$ is *a degeneration map* if f satisfies (1) f is proper and surjective, (2) there exist finite number of critical values $s_i \in B$ $(i = 1, 2, \dots, r)$ and (3) if $s \neq s_i$ then $f^{-1}(s)$ is a compact Riemann surface.

We consider a simple loop $\gamma_i \subset B \backslash \{s_i\}$ surrounding only s_i with a base point s_0. Then $f^{-1}(\gamma_i)$ is a topological mapping torus and we obtain a self-homeomorphism $\rho_i : f^{-1}(s_0) \to f^{-1}(s_0)$ of the *reference fiber* $f^{-1}(s_0)$. We call it a *local monodromy* of the singular fiber $f^{-1}(s_i)$. Choice of γ_i has ambiguity by isotopy and conjugation. Hence a local monodromy ρ_i is determined up to isotopy and conjugation.

The local monodromy is well-studied from both of algebraic and topological aspects. Matsumoto and Montesinos-Amilibia's paper [9] is one of the most important ones because they gave a perfect correspondence between local monodromies and degeneration maps from a topological viewpoint.

On the other hand, if we fix the base point s_0 $(s_0 \neq s_i)$, then the monodromy is given by a homomorphism

$$\rho : \pi_1(B \setminus \{s_i\}, s_0) \to \mathcal{M}(f^{-1}(s_0)),$$

where $\mathcal{M}(f^{-1}(s_0))$ is a mapping class group of the reference fiber $f^{-1}(s_0)$. This ρ is called *a global monodromy*. For a given degeneration map $f : M \to B$, we are much interested in how to calculate ρ concretely, but it is difficult to do that.

Simply because if B and/or f are given by high-degree polynomials, then we have few idea to 'solve' the equations on $f^{-1}(s)$ generally.

Experimental trials of getting global monodromies were done for some examples. Ahara [1], [2] and Matsumoto [7] give the global monodromy of the degeneration map (1.1) from the Fermat surface of degree 5 (and 6) to $\mathbb{C}P^1$. Kuno [6] also determine the global monodromy of another degenaration map on the Fermat surface of degree 4. In both examples, in order to obtain the global monodromies they use numerical analysis by computer.

In this paper we give a way to get the global monodromy 'by hand', without useing computer calculation. The recipe of calculation is the same as those of Matsumoto, Ahara, and Kuno. In this paper, we use lots of tricks to pursue solutions of high-degree equations and succeed in acquiring the results.

We fix a degenaration map $f : V_n \to \mathbb{C}P^1$ from the Fermat surface of degree n to $\mathbb{C}P^1$ and assume n is an odd number. Also in the case that n is an even number, we have similar results but we omit these for simple description. See [4] for detail.

This paper is organized as follows. In the remaining of this section, we prepare some notations and introduce some basic results of the singular fibers. In section 2, we define a branched covering map p_s of each fiber $f^{-1}(s)$. In section 3, we obtain the configuration of branch points of p_{s_0} of the reference fiber $f^{-1}(s_0)$. Finally in section 4, we show our main results.

1.1 Preparation

We set

$$V_n := \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 : z_0^n - z_1^n - z_2^n + z_3^n = 0\}.$$

Then V_n is a complex projective hypersurface, and we call it *the Fermat surface* of degree n. We regard $\mathbb{C}P^1$ as $\mathbb{C} \cup \{\infty\}$ and define a fibration $f : V_n \to \mathbb{C}P^1$ by

$$f([z_0 : z_1 : z_2 : z_3]) := \begin{cases} \dfrac{z_2^{n-1}}{z_0^{n-1}} & \text{if} \quad z_0 = z_1 \quad \text{and} \quad z_2 = z_3, \\ \dfrac{z_0 - z_1}{z_2 - z_3} & \text{otherwise.} \end{cases} \tag{1.1}$$

We take an open covering

$$\mathbb{C}P^3 = U_1 \cup U_2 \cup U_3 \cup U_4,$$

where $U_i := \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 : z_0 \neq z_i\}$ $(i = 1, 2, 3)$ and U_4 is an open neighborhood of $[1 : 1 : 1 : 1]$. Here $U_i \cong \mathbb{C}^3$ $(i = 1, 2, 3)$. Setting

$$X := \frac{z_0}{z_0 - z_1},\ Y := \frac{z_2}{z_0 - z_1},\ Z := \frac{z_3}{z_0 - z_1},$$

then

$$\frac{z_1}{z_0 - z_1} = X - 1.$$

Hence

$$V_n \cap U_1 = \{(X, Y, Z) \in \mathbb{C}^3 : X^n - (X-1)^n - Y^n + Z^n = 0\},$$

and $f : V_n \cap U_1 \to \mathbb{C}P^1$ is expressed as

$$f(X, Y, Z) = \frac{1}{Y - Z}.$$

For a nonzero $s \in \mathbb{C}$, we can express $f^{-1}(s) \cap U_1$ as

$$f^{-1}(s) \cap U_1 = \{(X, Y) \in \mathbb{C}^2 : g_s(X, Y) = 0\},$$

where

$$g_s(X, Y) := X^n - (X-1)^n - Y^n + \left(Y - \frac{1}{s}\right)^n.$$

In order to know the positions of the singularities, we solve the system of equations

$$\frac{\partial g_s}{\partial X} = 0, \quad \frac{\partial g_s}{\partial Y} = 0, \quad g_s(X, Y) = 0.$$

First, from $\frac{\partial g_s}{\partial X} = 0$, we solve the equation

$$\frac{\partial g_s}{\partial X} = nX^{n-1} - n(X-1)^{n-1} = 0,$$

which is rewritten as

$$X^{n-1} = (X-1)^{n-1}.$$

Then we obtain $\nu_k X = (X - 1)$ and

$$X = \frac{1}{1 - \nu_k},$$

where $\nu_k = \exp(\frac{2k\pi i}{n-1})$ $(k = 1, 2, \ldots, n-2)$ is an $(n-1)$st root of unity other than 1. We set

$$X_k := \frac{1}{1 - \nu_k} \tag{1.2}$$

. Next, from $\frac{\partial g_s}{\partial Y} = 0$, we solve the equation

$$\frac{\partial g_s}{\partial Y} = -nY^{n-1} + n\left(Y - \frac{1}{s}\right)^{n-1} = 0,$$

which is rewritten as

$$Y^{n-1} = \left(Y - \frac{1}{s}\right)^{n-1}.$$

Then we have $\tau_l Y = (Y - 1/s)$ and

$$Y = \frac{1}{s(1-\tau_l)},$$

where $\tau_l = \exp(\frac{2l\pi i}{n-1})$ $(l = 1, 2, \dots, n-2)$ is an $(n-1)$st root of unity other than 1. We set

$$Y_l(s) := \frac{1}{s(1-\tau_l)}. \tag{1.3}$$

Substituting X_k, $Y_l(s)$ into $g_s(X, Y)$, then we have

$$g_s(X_k, Y_l(s)) = \frac{1}{(1-\nu_k)^{n-1}} - \frac{1}{s^n(1-\tau_l)^{n-1}}.$$

We solve the equation $g_s(X_k, Y_l(s)) = 0$ in s. Then the critical values of $f : V_n \to \mathbb{C}P^1$ other than 0 or ∞ are the solutions of

$$s^n = \left(\frac{1-\nu_k}{1-\tau_l}\right)^{n-1}.$$

We can rewrite the right hand side of this equation as

$$(-1)^{k-l}\left(\frac{\sin\frac{k\pi}{n-1}}{\sin\frac{l\pi}{n-1}}\right)^{n-1}$$

by using Lemma 3.1.2. We denote the critical value by

$$s_{k,l}^{(j)} \quad (j = 0, 1, \dots, n-1 \text{ and } k, l = 1, 2, \dots, n-2).$$

The singular points are given by

$$(X, Y) = \left(\frac{1}{1-\nu_k}, \frac{1}{s_{k,l}^{(j)}(1-\tau_l)}\right).$$

For a regular value s_0 $(\neq s_{k,l}^{(i)}, 0, \infty)$, a general fiber $f^{-1}(s_0)$ is defined by a polynomial of degree $n-1$. By Plücker's formula, we obtain the following

Proposition 1.1.1. *If s_0 is a regular value of $f : V_n \to \mathbb{C}P^1$, then $f^{-1}(s_0)$ is a complex curve of genus $(n-2)(n-3)/2$.*

We remark that the fibers have some symmetry like

$$f^{-1}(s) \cong f^{-1}(e^{2\pi i/n}s)$$

and

$$f^{-1}(s) \cong f^{-1}(1/s).$$

1.2 The shapes of singular fibers

Matsumoto [8] determined the topological types of all singular fibers of $f : V_n \to \mathbb{C}P^1$.

Theorem 1.2.1 (Matsumoto [8]). *We assume that the degree n is greater than 3. Then the singular fiber is as follows:*

(I) *If n is odd (and if $n \geq 13$, then $n \not\equiv 1 \pmod{6}$), then there appear four types of singular fibers:*

(1) *For $s_0 = 0$ or ∞, $f^{-1}(s_0)$ consists of $n-1$ projective lines. Each projective line intersects the others projective lines at only one point.*

(2) *For s_0 which is an nth root of unity, each fiber $f^{-1}(s_0)$ consists of a plane curve of degree $n-3$ and two projective lines. Each projective line intersects the plane curve at $n-3$ points and intersects the other line at one point.*

(3) *For an integer k $(1 \leq k < \frac{n-3}{2})$, letting s_0 be an nth root of*

$$(-1)^{\frac{n-1}{2}+k}\left(\sin\frac{k\pi}{n-1}\right)^{n-1}$$

or

$$(-1)^{\frac{n-1}{2}+k}\left(\frac{1}{\sin\frac{k\pi}{n-1}}\right)^{n-1},$$

then each fiber $f^{-1}(s_0)$ is an irreducible plane curve of degree $n-1$ with two nodes. Its vanishing cycles corresponding to the two nodes are non-separating simple closed curves and they are not homologous to each other.

(4) *For an ordering pair of integers (k, l) $(1 \leq k, l \leq \frac{n-3}{2})$, letting s_0 be an nth root of*

$$(-1)^{k-l}\left(\frac{\sin\frac{k\pi}{n-1}}{\sin\frac{l\pi}{n-1}}\right)^{(n-1)},$$

then each fiber $f^{-1}(s_0)$ is an irreducible plane curve of degree $n-1$ with four nodes. Its vanishing cycles corresponding to the four nodes are non-separating simple closed curves and they are not homologous to each other.

(II) *If n is even, then there appear three types of singular fibers:*

(1) *For* $s_0 = 0$ *or* ∞, *each fiber* $f^{-1}(s_0)$ *consists of* $n-1$ *projective lines and each projective line intersects the others projective lines at only one point.*

(2) *For* s_0 *which is a* $2n$*th root of unity, each* $f^{-1}(s_0)$ *consists of a plane curve of degree* $n-2$ *and a projective line. The line intersects the plane curve at* $n-2$ *points.*

(3) *For an ordering pair of integers* (k,l) $(1 \leq k,l \leq \frac{n-2}{2})$, *letting* s_0 *be a* $2n$*th root of*

$$\left(\frac{\sin\frac{k\pi}{n-1}}{\sin\frac{l\pi}{n-1}}\right)^{2(n-1)},$$

then each fiber $f^{-1}(s_0)$ *is an irreducible plane curve of degree* $n-1$ *with two nodes. Its vanishing cycles corresponding to the two nodes are non-separating simple closed curves and they are not homologous to each other.*

Moreover, Matsumoto told us that he had a certain result about the singular fibers in case $n \equiv 1$ (mod. 6) in a joint paper with K. Masuda but it is not published yet.

2 Branched covering map

Is this section, we define a branched covering map p_s from a fiber $f^{-1}(s)$ to $\mathbb{C}P^1$ for a general s. This map plays an important role to describe the reference fiber and to determine the topological monodromy around a singular fiber.

2.1 Definition of a branched covering map

Before we define the branched covering map, we note that the following lemma.

Lemma 2.1.1. *If* s *is not zero nor infinity, then* $f^{-1}(s) \cap \{z_0 = z_1\}$ *consists of* $n-1$ *points.*

Proof. From the definition of the map f;

$$f([z_0 : z_1 : z_2 : z_3]) = \begin{cases} 0 & \text{if } z_2 \neq z_3, \\ (z_2/z_0)^{n-1} & \text{if } z_2 = z_3, \end{cases}$$

if $s \neq 0$, then the equation $z_2^{n-1} = s z_0^{n-1}$ has $n-1$ solutions. We solve the equation as $z_2 = y_1, y_2, \ldots, y_{n-1}$. Then we obtain

$$f^{-1}(s) \cap \{z_0 = z_1\} = \{[z_0 : z_1 : y_1 : y_1], [z_0 : z_1 : y_2 : y_2], \ldots, [z_0 : z_1 : y_{n-1} : y_{n-1}]\}.$$

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Now, we define a branched covering map $p_s : f^{-1}(s) \to \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ by

$$p_s([z_0 : z_1 : z_2 : z_3]) := \frac{z_0}{z_0 - z_1}.$$

Since the inverse image $p_s^{-1}(\infty)$ of the infinity point consists of $n-1$ points from Lemma 2.1.1, ∞ is not a branch point of p_s. Hence we consider a branched covering map p_s from $f^{-1}(s) \setminus \{n-1 \text{ points}\}$ to $\mathbb{C}$ defined by

$$p_s : f^{-1}(s) \setminus \{n-1 \text{ points}\} \to \mathbb{C}, \quad p_s(X, Y) := X.$$

Here $f^{-1}(s) \cap \{z_0 \neq z_1\} = \{(X, Y) \in \mathbb{C}^2 : g_s(X, Y) = 0\}$. Let s_0 be a regular value of $f : V_n \to \mathbb{C}P^1$. Then this map is an $(n-1)$-fold branched covering map from a smooth complex curve $f^{-1}(s_0)$ of genus $(n-2)(n-3)/2$ to $\mathbb{C}P^1$. Hereafter, for simplicity, we denote $f^{-1}(s_0) \setminus \{n-1\text{points}\}$ by $f^{-1}(s_0)$.

2.2 Branch points and ramification points

We determine the branch points of $p_{s_0} : f^{-1}(s_0) \to \mathbb{C}$. For a general point $X_0 \in \mathbb{C}$, the number of the solutions of the equation in Y

$$X_0^n - (X_0 - 1)^n - Y^n + \left(Y - \frac{1}{s_0}\right)^n = 0 \tag{2.1}$$

is $n-1$. The Y-coordinate of the ramification points are the multiple roots of the equation (2.1). Solving $\frac{\partial g_{s_0}}{\partial Y} = 0$, then we have

$$Y_l = Y_l(s_0) = \frac{1}{s_0(1 - \tau_l)}, \quad l = 1, 2, \ldots, n-2.$$

The branch points of $p_{s_0} : f^{-1}(s_0) \to \mathbb{C}$ is the solutions of the equation in X

$$X^n - (X-1)^n - Y_l^n + \left(Y_l - \frac{1}{s_0}\right)^n = 0, \quad l = 1, 2, \ldots, n-2. \tag{2.2}$$

Let $X_j^{(l)}$ $(j = 1, 2, \ldots, n-1)$ be the solution of (2.2). That is, the branch points are the solutions of the equation in X

$$\prod_{l=1}^{n-2} \left\{ X^n - (X-1)^n - Y_l^n + \left(Y_l - \frac{1}{s_0}\right)^n \right\} = 0.$$

As in Lemma 3.1.2.

$$\begin{aligned} Y_l^n - (Y_l - \tfrac{1}{s_0})^n &= \frac{1}{s_0^n(1-\tau_l)^{n-1}} \\ &= \frac{i^{n-1}(-1)^l}{2^{n-1}s_0^n(\sin(l\pi/(n-1)))^{n-1}}. \end{aligned}$$

If $l' = n-1-l$, then $Y_l^n - (Y_l - 1/s_0)^n = Y_{l'}^n - (Y_{l'} - 1/s_0)^n$ and $\{X_j^{(l)}\}_j = \{X_j^{(l')}\}_j$. Hence we can reduce the running number l and obtain

$$\prod_{l=1}^{[(n-1)/2]} \left\{ X^n - (X-1)^n - \frac{1}{s_0^n(\tau_l - 1)^{n-1}} \right\} = 0, \tag{2.3}$$

where $[\cdot]$ is Gauss symbol. Generally in order to identify $X_j^{(l)}$ to $X_j^{(l')}$, we need to permute the index j. But if $l' = n-1-l$, then the equations (2.2) coincide for l and l', so we may identify $X_j^{(l)}$ to $X_j^{(l')}$ naturally.

Hence if s_0 is a regular value of $f : V_n \to \mathbb{C}P^1$, the number of the branch points is

$$\begin{cases} (n-1)^2/2 & \text{if } n \text{ is odd,} \\ (n-1)(n-2)/2 & \text{if } n \text{ is even.} \end{cases}$$

Lemma 2.2.1. *For a general fiber $f^{-1}(s_0)$, the ramification index of each ramification point of the branched covering map $p_{s_0} : f^{-1}(s_0) \to \mathbb{C}$ is two.*

Proof. There exist no solutions of the system of equations

$$\begin{cases} g_{s_0}(X_0, Y) = X_0^n - (X_0 - 1)^n - Y^n + \left(Y - \dfrac{1}{s_0} \right)^n = 0, \\ \dfrac{\partial g_s}{\partial Y} = 0, \\ \dfrac{\partial^2 g_s}{\partial Y^2} = 0. \end{cases}$$

This leads to the assertion.

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It is easy to check that if s_0 is a regular value of $f : V_n \to \mathbb{C}P^1$, then the equation (2.3) does not have any multiple roots. If s is a critical value of $f : V_n \to \mathbb{C}P^1$, then the equation (2.3) has multiple roots. (Precisely speaking, they are double roots from Lemma 2.2.1.)

Moreover, we can determine the positions of all branch points. See Figure 6. In order to draw the positions of branch points, we need more discussions. Hence we leave the conclusion to subsection 3.3.

In order to determine the topology of a reference fiber $f^{-1}(s_0)$, we want to know the permutation of the solutions of $g_{s_0}(X, Y_l) = 0$ when we move X from X_0 to the branch point $X_j^{(l)}$. We investigate the branched covering map p_{s_0} in detail and determine the permutation in section 3. In order to determine the monodromy around the singular fiber $f^{-1}(s_{k,l}^{(j)})$, we want to know the trace of the branch points $X_j^{(l)}$ when s moves from s_0 to the singular value $s_{k,l}^{(j)}$ and we determine it in section 4.

3 Determination of the reference fiber

We keep the notation as above. In order to determine topological structure of the reference fiber $f^{-1}(s_0)$, we need some technical theorems. We have to separate into two cases that (i) n is odd and (ii) n is even. In this article, we only state the case that n is odd, but we can get similar results for even n. See [4].

3.1 Technical theorems

For "good" s_0 and X_0, we want a good configuration of the solutions of the equation $g_{s_0}(X_0, Y) = 0$ and that of the branch points of p_{s_0}. The key theorem is

Theorem 3.1.1. *Let $X_0 = 1/2$ and let $Y^{(1)}, Y^{(2)}, \ldots, Y^{(n-1)}$ be the solutions of $g_{s_0}(X_0, Y) = 0$. If s_0 is a sufficiently small positive real number, then $Y^{(1)}, Y^{(2)}, \ldots, Y^{(n-1)}$ lie on a line $\{Y \in \mathbb{C} \mid \mathrm{Re}\ Y = 1/2s_0\}$ ($\mathrm{Im}\ Y^{(1)} > \mathrm{Im}\ Y^{(2)} > \cdots > \mathrm{Im}\ Y^{(n-1)}$). Moreover, there exists Y_l between $Y^{(l)}$ and $Y^{(l+1)}$ on the line. See Figure 1.*

Before we proceed the proof of Theorem 3.1.1, we show technical lemmas.

Lemma 3.1.2. *Let $\theta = \pi/(n-1)$. The following equalities hold:*

(i) $\tau_l + 1 = 2e^{l\theta i}\cos l\theta$.

(ii) $1 - \tau_l = -2ie^{l\theta i}\sin l\theta$.

(iii) $Y_l(s)^n - \left(Y_l(s) - \dfrac{1}{s}\right)^n = \dfrac{1}{s^n(1-\tau_l)^{n-1}}$.

(iv) $Y_l(s) = \dfrac{1}{s(1-\tau_l)} = \dfrac{1}{2s} + i\dfrac{\sin 2l\theta}{2s(1-\cos 2l\theta)}$.

Proof.

(i)
$$\begin{aligned}\tau_l + 1 &= e^{2l\theta i} + 1\\ &= e^{l\theta i}\{e^{l\theta i} + e^{-l\theta i}\}\\ &= 2e^{l\theta i}\cos l\theta.\end{aligned}$$

(ii) $1-\tau_l = 1-e^{2l\theta i}$

$$= e^{l\theta i}\{e^{-l\theta i}-e^{l\theta i}\}$$

$$= -2ie^{l\theta i}\sin l\theta.$$

(iii) $$Y_l(s)^n - \left(Y_l(s)-\frac{1}{s}\right)^n = \left(\frac{1}{s(1-\tau_l)}\right)^n - \left(\frac{\tau_l}{s(1-\tau_l)}\right)^n$$

$$= \frac{(1-\tau_l)}{s^n(1-\tau_l)^n} = \frac{1}{s^n(1-\tau_l)^{n-1}}.$$

(iv) $$\frac{1}{s(1-\tau_l)} = \frac{1}{s(1-\cos 2l\theta - i\sin 2l\theta)}$$

$$= \frac{1-\cos 2l\theta + i\sin 2l\theta}{2s(1-\cos 2l\theta)}$$

$$= \frac{1}{2s} + i\frac{\sin 2l\theta}{2s(1-\cos 2l\theta)}.$$

■

Corollary 3.1.3. *The real part of* $1/(1-\tau_l)$ *is* $1/2$.

From Corollary 3.1.3, if we take s a real number, then not only the real part of $Y_l(s)$ is $1/2s$ but also the real part of X_k is $1/2$. (We note that $Y_l(s)$ is the Y-coordinate of the ramification point of p_s.) We remark that the real part of $Y_l(s)$ is independent of n. It depends only on s.

Let s_0 be a regular value of f and let X_0 be a regular value of p_{s_0}, that is, X_0 is not a branch point. We investigate the solutions of the equation

$$X_0^n - (X_0-1)^n - Y^n + \left(Y-\frac{1}{s_0}\right)^n = 0. \tag{3.1}$$

Lemma 3.1.4. *The equation* (3.1) *has solutions of the form*

$$Y = \frac{1}{2s_0} \pm \beta_j,$$

where $\beta_j \in \mathbb{C}$ $(j = 1, 2, \ldots, (n-1)/2)$.

Remark 3.1.5. *This lemma implies that the configuration of the solutions of* (3.1) *has symmetry on* $1/2s_0$. *If all* β_j *are purely imaginary numbers and* s_0 *is a real number, then the solutions of* (3.1) *lie on the line* $\{Y \in \mathbb{C} \mid \mathrm{Re}\, Y = 1/2s_0\}$. (*See Figure* 1.)

Proof. We set $Y' := Y - 1/2s_0$. Then the equation (3.1) is rewritten as

$$X_0^n - (X_0 - 1)^n - \left(Y' + \frac{1}{2s_0}\right)^n + \left(Y' - \frac{1}{2s_0}\right)^n = 0.$$

The left hand side of this equation is

$$-2{}_nC_1(Y')^{n-1}\left(\frac{1}{2s_0}\right) - 2{}_nC_3(Y')^{n-3}\left(\frac{1}{2s_0}\right)^3 - \cdots - 2{}_nC_{n-2}(Y')^2\left(\frac{1}{2s_0}\right)^{n-2} \\ -2\left(\frac{1}{2s_0}\right)^n + X_0^n - (X_0 - 1)^n. \tag{3.2}$$

Since n is odd (and $n-1$ is even), the polynomial (3.2) has only the terms of even degree. Hence there exist some complex numbers β_j $(j = 1, 2, \ldots, (n-1)/2)$, we have solutions as

$$(Y')^2 = \beta_j^2,$$

and we have

$$Y' = \pm\beta_j.$$

Substituting this into $Y = Y' + 1/2s_0$, we can solve

$$Y = \frac{1}{2s_0} \pm \beta_j, \quad \beta_j \in \mathbb{C}, \quad j = 1, 2, \ldots, \frac{n-1}{2}.$$

■

We set

$$\Psi(X) := X^n - (X - 1)^n$$

and

$$\phi(Y') := \Psi(X_0) - \left(Y' + \frac{1}{2s_0}\right)^n + \left(Y' - \frac{1}{2s_0}\right)^n.$$

Expanding $\phi(Y')$, we have

$$\phi(Y') = -2{}_nC_1(Y')^{n-1}\left(\frac{1}{2s_0}\right) - 2{}_nC_3(Y')^{n-3}\left(\frac{1}{2s_0}\right)^3 - \cdots - 2{}_nC_{n-2}(Y')^2\left(\frac{1}{2s_0}\right)^{n-2} \\ -2\left(\frac{1}{2s_0}\right)^n + \Psi(X_0).$$

Now, we prove that all solutions of $\phi(Y') = 0$ are purely imaginary numbers for $X_0 = 1/2$ and a sufficiently small positive number s_0. We obviously obtain

Lemma 3.1.6. *Let $Y' = vi$ be a purely imaginary number and let s_0 be a real number. If $\Psi(X_0) \in \mathbb{R}$, then $\phi(Y') \in \mathbb{R}$.*

Under the assumption of Lemma 3.1.6, we can define a function $\overline{\phi}(v) : \mathbb{R} \to \mathbb{R}$ by $\overline{\phi}(v) := \phi(vi)$. That is,

$$\begin{aligned}\overline{\phi}(v) := \; & -(-1)^{(n-1)/2} 2_n C_1 v^{n-1} - (-1)^{(n-3)/2} 2_n C_3 v^{n-3} \left(\frac{1}{2s_0}\right)^2 - \cdots \\ & -(-1) 2_n C_{n-2} v^2 \left(\frac{1}{2s_0}\right)^{n-2} - 2\left(\frac{1}{2s_0}\right)^n + \Psi(X_0).\end{aligned}$$

We draw a graph of $w = \overline{\phi}(v)$. In order to know the extreme points of $w = \overline{\phi}(v)$, we solve $\frac{d\phi}{dY'}(Y') = 0$. (Equivalently $\frac{dg_{s_0}(X_0,Y)}{dY} = 0$.) From (1.3),

$$Y' = Y_l - \frac{1}{2s_0} = \frac{\tau_l + 1}{2s_0(1-\tau_l)}, \quad l = 1, 2, \ldots, n-2.$$

We denote this by Y_l'. From Lemma 3.1.2, we obtain

$$Y_l' = \frac{i \cot l\theta}{2s_0}. \tag{3.3}$$

We set $b_l := \operatorname{Im} Y_l' = (\cot l\theta)/2s_0$. Then the following inequalities hold:

Lemma 3.1.7. $b_1 > b_2 > \cdots > b_{(n-1)/2} = 0 > b_{(n+1)/2} > \cdots > b_{n-2}$.

We note that $\overline{\phi}(b_l)$ is the extremum. Now we investigate the sign of $\overline{\phi}(b_l)$. We compute $\overline{\phi}(b_l)$:

$$\overline{\phi}(b_l) = \phi(Y_l') = g_{s_0}(X_0, Y_l) = \Psi(X_0) - \frac{1}{s_0^n(1-\tau_l)^{n-1}}.$$

From Lemma 3.1.2 (ii), we have

$$\begin{aligned}\overline{\phi}(b_l) &= \Psi(X_0) - \frac{1}{s_0^n(2ie^{l\theta i}\sin l\theta)^{n-1}} \\ &= \Psi(X_0) - \frac{(-1)^{(n-1)/2+l}}{s_0^n 2^{n-1}(\sin l\theta)^{n-1}}.\end{aligned}$$

For $X_0 = 1/2$, $\Psi(X_0) = X_0^n - (X_0 - 1)^n = 1/2^{n-1}$ and we have

$$\overline{\phi}(b_l) = \frac{1}{2^{n-1}} - \frac{(-1)^{(n-1)/2+l}}{s_0^n 2^{n-1}(\sin l\theta)^{n-1}}.$$

Obviously, for any n and l, there exists an small positive real number s_0 such that the following inequalities hold:

$$0 < s_0^n < 1. \tag{3.4}$$

Therefore we deduce

Lemma 3.1.8. *Suppose that $X_0 = 1/2$ and s_0 is sufficiently small positive real number satisfying* (3.4). *Then*

(I) *If $(n-1)/2$ is even, then*

$$\overline{\phi}(b_l) = \begin{cases} \text{negative} & \text{if } l \text{ is even,} \\ \text{positive} & \text{if } l \text{ is odd.} \end{cases}$$

(II) *If $(n-1)/2$ is odd, then*

$$\overline{\phi}(b_l) = \begin{cases} \text{positive} & \text{if } l \text{ is even,} \\ \text{negative} & \text{if } l \text{ is odd.} \end{cases}$$

From this lemma, it follows that the graph $w = \overline{\phi}(v)$ is as Figure 2. Since $\overline{\phi}(v)$ is a polynomial of degree $n-1$, the number of solutions of the equation $\overline{\phi}(v) = 0$ is $n-1$ and we obtain

Proposition 3.1.9. *Let s_0 be a positive real number satisfying* (3.4). *Setting $X_0 := 1/2$, then any solution of the equation $\phi(Y') = 0$ is a purely imaginary number.*

We denote the solutions of $\overline{\phi}(v) = 0$ by $v^{(j)}$ $(j = 1, 2, \ldots, n-1)$ such that $v^{(1)} > v^{(2)} > \cdots > v^{(n-1)}$. Then $\phi(v^{(j)}i) = 0$ and hence the solution $Y^{(j)}$ of $\phi(Y) = 0$ is expressed as

$$Y^{(j)} = \frac{1}{2s_0} + v^{(j)}i,$$

By Lemma 3.1.4 and Proposition 3.1.9, we conclude

Corollary 3.1.10. *Let s_0 and X_0 be as in Proposition* 3.1.9. *Then $Y^{(1)}, Y^{(2)}, \ldots, Y^{(n-1)}$ lie on the line defined by Re $Y = 1/2s_0$. See Figure* 1.

As seen in Figure 2, the inequalities

$$v^{(n-1)} < b_{n-2} < v^{(n-2)} < b_{n-1} < \cdots < v^{(2)} < b_1 < v^{(1)}$$

hold. Then we conclude Theorem 3.1.1 for the case n is odd.

3.2 The curve C defined by $\mathrm{Im}\ \Psi(X) = 0$

In order to investigate the permutation of $Y^{(l)}$'s when we move X from the base point X_0 to the branch points $X_j^{(l)}$, we find a "good" path from X_0 to $X_j^{(l)}$. In this section, we assume that s_0 is a real number.

We set

$$\begin{aligned}\overline{\phi}_X(v) \quad &:= g_{s_0}(X, \tfrac{1}{2s_0} + vi) \\ &= -(-1)^{(n-1)/2} 2_n C_1 v^{n-1} \left(\frac{1}{2s_0}\right) - (-1)^{(n-3)/2} 2_n C_3 v^{n-3} \left(\frac{1}{2s_0}\right)^3 \\ &\quad - \cdots - (-1) 2_n C_{n-2} v^2 \left(\frac{1}{2s_0}\right)^{n-2} - 2\left(\frac{1}{2s_0}\right)^n + \Psi(X).\end{aligned}$$

We remark that $\overline{\phi}_{X_0} = \overline{\phi}$. If $X = X_j^{(l)}$, then $\overline{\phi}_X(b_l) = 0$. It follows that the graph $w = \overline{\phi}_{X_j^{(l)}}(v)$ is tangent to the v-axis at $(b_l, 0)$. If X moves along a path satisfying that $\Psi(X)$ is a real number, that is Im $\Psi(X) = 0$, then we can see the movement of the graph $w = \overline{\phi}_X(v)$ and how the intersection points $v^{(l)}$ and $v^{(l+1)}$ converse to b_l. Here we consider $v^{(j)}$ as a continuous function of X, whenever they exist. In this subsection, we investigate a curve defined by Im $\Psi(X) = 0$.

We set $X := x + iy$. Then $\Psi(X) = (x + iy)^n - (x + iy - 1)^n$. We often denote $\Psi(X)$ by $\Psi(x, y)$ and we define the curve

$$C := \{X \in \mathbb{C} : \text{Im } \Psi(X) = 0\} (= \{(x, y) \in \mathbb{R}^2 : \text{Im } \Psi(x, y) = 0\}).$$

Proposition 3.2.1. *The notation is as above. Then the imaginary part of* $\Psi(x, y)$ *is factorized as* Im $\Psi(x, y) = y(x - 1/2)h(x, y)$. *Moreover the curve* C *passes through the points* $X_j^{(l)}$ *and* $X_k = 1/(1 - \nu_k)$.

We remark that $h(x, y)$ is a polynomial of degree $n - 3$ in y.

In order to show this proposition, we need three lemmas.

Lemma 3.2.2. *If* $x = 1/2$, *then* $\Psi(1/2, y)$ *is a real number.*

Proof. Substituting $X = 1/2 + iy$ into $\Psi(X)$, then

$$\begin{aligned}\Psi\left(\frac{1}{2} + iy\right) \quad &= \left(\frac{1}{2} + iy\right)^n - \left(\frac{1}{2} + iy - 1\right)^n \\ &= \left(iy + \frac{1}{2}\right)^n - \left(iy - \frac{1}{2}\right)^n.\end{aligned}$$

Since n is odd, this is a polynomial of $(iy)^2$. Hence $\Psi(1/2 + iy)$ is a real number. ■

Lemma 3.2.3. *If* $y = 0$, *then* $\Psi(x, 0)$ *is a real number.*

Proof. Let X be a real number. Then

$$\Psi(x, 0) = x^n - (x - 1)^n,$$

and it is obviously a real number.

■

From Lemma 3.2.2 and 3.2.3, it follows that Im $\Psi(x,y)$ factorize as Im $\Psi(x,y) = y(x-1/2)h(x,y)$. Moreover all the branch points $X_j^{(l)}$ and X_k are on the curve C from the next lemma.

Lemma 3.2.4. (i) $\Psi(X_j^{(l)}) \in \mathbb{R}$ *and* (ii) $\Psi(X_k) \in \mathbb{R}$.

Proof. (i) Since $X_j^{(l)}$ is a solution of the equation

$$g_{s_0}(X, Y_l) = \Psi(X) - \frac{1}{s_0^n(1-\tau_l)^{n-1}} = 0,$$

$\Psi(X_j^{(l)}) = 1/s_0^n(1-\tau_l)^{n-1}$. Now from Lemma 3.1.2,

$$\Psi(X_j^{(l)}) = \frac{(-1)^{(n-1)/2+l}}{s_0^n 2^{n-1}(\sin l\theta)^{n-1}}.$$

Clearly, $\Psi(X_j^{(l)})$ is a real number.

(ii) Since the real part of X_k is $1/2$ from Lemma 3.1.3, $\Psi(X_k) \in \mathbb{R}$ from Lemma 3.2.2.

■

Therefore we obtain Proposition 3.2.1.

For convenience, we set

$$L := \left\{(x,y) \in \mathbb{R}^2 : x = \frac{1}{2}\right\} \subset \mathbb{R}^2 = \mathbb{C}$$

and

$$H := \{(x,y) \in \mathbb{R}^2 : yh(x,y) = 0\}.$$

We note that $C = L \cup H$. We next show

Proposition 3.2.5. *The line L and the curve H intersect at X_k. Moreover, the number of the intersection points is $n-2$.*

In order to show Proposition 3.2.5, we first show

Lemma 3.2.6. *Let y_k be a solution of $h(1/2, y) = 0$. Then $1/2 + iy_k = X_k$, a solution of $\frac{d\Psi}{dX} = 0$.*

Proof. We separate the holomorphic function $\Psi(X)$ into the real part and the imaginary part: $\Psi(X) := u(x,y) + iv(x,y)$ where $X = x + iy$. Since $\Psi(X)$ is a holomorphic function, Cauchy-Riemann formula is followed:

$$\frac{\partial \Psi}{\partial X} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}. \tag{3.5}$$

From Proposition 3.2.1, the imaginary part $v(x,y) = y(x-1/2)h(x,y)$. We compute the derivatives $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$:

$$\frac{\partial v}{\partial x} = y\left\{h(x,y) + \left(x - \frac{1}{2}\right)\frac{\partial h}{\partial x}\right\}, \tag{3.6}$$

$$\frac{\partial v}{\partial y} = \left(x - \frac{1}{2}\right)\left\{h(x,y) + y\frac{\partial h}{\partial y}\right\}. \tag{3.7}$$

The conditions $h(1/2, y_k) = 0$ and $x = 1/2$ imply that $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Hence $\frac{d\Psi}{dX}(1/2, y_k) = 0$.

■

Next we show the converse.

Lemma 3.2.7. *Let $X_k = 1/2 + iy_k$. Then $y_k h(1/2, y_k) = 0$.*

Proof. We note that $X_k = 1/2 + iy_k$ is a solution of $\frac{d\Psi}{dX} = 0$. Then from (3.5),

$$\frac{\partial v}{\partial x}(\frac{1}{2}, y_k) = 0.$$

From (3.6), we obtain

$$y_k h(\frac{1}{2}, y_k) = 0.$$

■

Moreover, we have

Proposition 3.2.8. *The curve C is symmetric about the line L and the x-axis.*

Proof. Easily we can show that

$$\begin{aligned}
&\mathrm{Im}\ \Psi(x,y) \\
&= y\{{}_nC_1[x^{n-1} - (x-1)^{n-1}] - {}_nC_3[x^{n-3} - (x-1)^{n-3}]y^2 \\
&\quad + \cdots + (-1)^{(n-3)/2}{}_nC_{n-2}[x^2 - (x-1)^2]y^{n-3}\}.
\end{aligned}$$

Therefore we have $\mathrm{Im}\ \Psi(x,y) = -\mathrm{Im}\ \Psi(x,-y)$ and $\mathrm{Im}\ \Psi(x,y) = -\mathrm{Im}\ \Psi(1-x,y)$. These are followed by the conclusion.

$\blacksquare$

We can move X from X_0 to $X_j^{(l)}$ along the curve C by the following theorem.

Theorem 3.2.9. *The branch points $X_j^{(l)}$, the base point $X_0 = 1/2$ and X_k are on the same connected component of the curve C for any l, j, k.*

In order to show Theorem 3.2.9, we need following two lemmas. For simplicity, We denote

$$a_l(s) := \frac{(-1)^{(n-1)/2+l}}{2^{n-1}s^n(\sin l\theta)^{n-1}}.$$

Lemma 3.2.10. *If s^n is a real number, then the solutions of $\Psi(X) - a_l(s) = 0$ are on the curve C.*

Proof. Let X be a solution of $\Psi(X) - a_l(s) = 0$. Then $\Psi(X) = a_l(s)$ and $\Psi(X)$ is a real number. It follows that the solution X is on the curve C.

$\blacksquare$

Lemma 3.2.11. *There exists a positive real number s such that the solutions of $\Psi(X) - a_l(s) = 0$ are on the line L.*

Proof. We denote $X = x + iy$. If $a_l(s) \in \mathbb{R}$, then (1) the solutions of $\Psi(X) - a_l(s) = 0$ are on the curve C, and (2) if $X = 1/2 + iy \in L$, then $\Psi(X) - a_l(s) \in \mathbb{R}$. Therefore we can define the function $\overline{\psi} : \mathbb{R} \to \mathbb{R}$ defined by

$$\overline{\psi}(y) := \Psi(\frac{1}{2} + iy) - a_l(s).$$

We set $y_k := \mathrm{Im}\, X_k$ as in Lemma 3.2.6. Then $\overline{\psi}(y_k)$ is an extremum.

$$\overline{\psi}(y_k) = \frac{1}{2^{n-1}(-1)^{(n-1)/2+k}} \left\{ \frac{1}{(\sin k\theta)^{n-1}} - \frac{(-1)^{k-l}}{s^n(\sin l\theta)^{n-1}} \right\}.$$

If s is enough large, then

$$\left| \frac{1}{(\sin k\theta)^{n-1}} \right| > \left| \frac{1}{s^n(\sin l\theta)^{n-1}} \right|.$$

Hence the sign of $\overline{\psi}(y_k)$ is determined whether $(n-1)/2$ and k are odd or even. Therefore, we summarize

(I) If $(n-1)/2$ is odd, then

$$\overline{\psi}(y_k) = \begin{cases} \text{positive} & k \text{ is odd,} \\ \text{negative} & k \text{ is even.} \end{cases}$$

(II) If $(n-1)/2$ is even, then

$$\overline{\psi}(y_k) = \begin{cases} \text{negative} & k \text{ is odd,} \\ \text{positive} & k \text{ is even.} \end{cases}$$

In the same way as in the proof of Proposition 3.1.9, there exist $n-1$ zero points of $\overline{\psi}(y)$. The function $\overline{\psi}(y) = \Psi(1/2+iy) - a_l(s)$ is a polynomial of degree $n-1$ in y. Hence all solutions of $\Psi(X) - a_l(s) = 0$ lie on the line L.

■

Proof of Theorem 3.2.9

Let s_1 be a real number satisfying the condition of Lemma 3.2.11. When we move s from s_0 to s_1 along the real axis, the solutions of $\Psi(X) - a_l(s) = 0$ move from $X_j^{(l)}$ to the point on the line L along the curve C from Lemma 3.2.10 and 3.2.11. Hence we obtain the assertion.

■

From Theorem 3.2.9, we can choose a path γ from X_0 to $X_j^{(l)}$ such that $\Psi(X)$ is a real number. If we move X along the path γ, then we can see the movement of the Y-coordinate $Y^{(l)}$ of the ramification points of $p_{s_0} : f^{-1}(s_0) \to \mathbb{C}P^1$.

3.3 The positions of the branch points

We determine the positions of the branch points of $p_{s_0} : f^{-1}(s_0) \to \mathbb{C}P^1$. In this subsection, we assume that $(n-1)/2$ is odd. In the case that $(n-1)/2$ is even, similar discussion holds. Hence we omit the case that $(n-1)/2$ is even. We keep notation and take s_0 a sufficiently small positive real number satisfying (3.4)

We recall that

$$\overline{\phi}_{X_0}(b_l) = \frac{1}{2^{n-1}} - \frac{(-1)^{(n-1)/2+l}}{s_0^n 2^{n-1}(\sin l\theta)^{n-1}}$$

is an extremum of the graph $w = \overline{\phi}_{X_0}(v)$. Then we obvious obtain

Lemma 3.3.1. *The following inequalities hold: If $(n-1)/2$ is odd, then*

(i) $\overline{\phi}_{X_0}(b_1) < \overline{\phi}_{X_0}(b_3) < \cdots < \overline{\phi}_{X_0}(b_{(n-1)/2}) < 0,$ *and*

(ii) $\overline{\phi}_{X_0}(b_2) > \overline{\phi}_{X_0}(b_4) > \cdots > \overline{\phi}_{X_0}(b_{(n-3)/2}) > 0.$

Hence the graph $w = \overline{\phi}_{X_0}(v)$ is concretely drawn as Figure 2. Now we investigate increase (or decrease) of the value $\Psi(X)$ on the curve C.

Lemma 3.3.2. *Let $X(t) \in C = \{\mathrm{Im}\ \Psi(X) = 0\}$ be a path such that* (i) $X(t)$ *is of class* C^1, (ii) $|\frac{d}{dt}X(t)| \neq 0$ *and* (iii) *for each* t, $X(t) \neq X_k$. *Then* $\frac{d}{dt}\Psi(X(t)) \neq 0$.

Proof. Since the condition (ii) means $\frac{d}{dt}X(t) \neq 0$ for any t and the condition (iii) means $\frac{d}{dX}\Psi(X) \neq 0$,

$$\frac{d}{dt}\Psi(X(t)) = \frac{d}{dX}\Psi(X) \times \frac{d}{dt}X(t) \neq 0.$$

■

From Lemma 3.3.2, $\Psi(X(t)) \in C$ is monotone increase or monotone decrease on the path $X(t)$ which does not pass through X_k.

If $X = 1/2 + iy$, then $\Psi(X) \in \mathbb{R}$. Hence we can define the function

$$\overline{\Psi} : \mathbb{R} \to \mathbb{R}, \quad \overline{\Psi}(y) := \Psi(\frac{1}{2} + iy).$$

The function $\overline{\Psi}(y)$ has extremums at y_k ($=$ Im X_k). The function $\overline{\Psi}(y)$ is a degree $(n-1)$ polynomial in y and there exist $n-2$ extreme points. Then the increase/decrease table of $\overline{\Psi}(y)$ is as follows:

y	$\cdots$	y_{n-2}	$\cdots$	y_{n-3}	$\cdots$	$\cdots$	$\cdots$	y_3	$\cdots$	y_2	$\cdots$	y_1	$\cdots$
$\overline{\Psi}(y)$	$\nearrow$		$\searrow$		$\nearrow$		$\nearrow$		$\searrow$		$\nearrow$		$\searrow$

(3.8)

Hence we have

Lemma 3.3.3. *The direction of increase of $\overline{\Psi}(y)$ on the line L is as Table* (3.8) *and Figure* 3.

From the maximum principle of a holomorphic function, we have

Lemma 3.3.4. *Around the point X_k, the direction of increase is either of two cases in Figure* 4.

From Lemma 3.3.3 and 3.3.4, we deduce

Proposition 3.3.5. *The direction of the increasing of $\Psi(X)$ around the line L is as Figure* 5.

We investigate the positions of the branch points $X_j^{(l)}$ on C. First, we show that for small $s_0 > 0$, there exists no branch point $X_j^{(l)}$ on the segment from X_1 to X_{n-2}. In order to show this, we assume that $X_j^{(l)}$ is on the segment from X_1 to X_{n-2}. Since

$$\Psi(X_k) = \frac{(-1)^{(n-1)/2+k}}{2^{n-1}(\sin k\theta)^{n-1}},$$

we have

$$|\Psi(X_j^{(l)})| < |\Psi(X_1)| = \frac{1}{2^{n-1}(\sin\theta)^{n-1}}.$$

On the other hand, $\overline{\phi}_{X_j^{(l)}}(b_l) = \Psi(X_j^{(l)}) - a_l(s_0) = 0$ is followed by

$$|\Psi(X_j^{(l)})| = |a_l(s_0)| = \frac{1}{2^{n-1}|s_0|^n(\sin l\theta)^{n-1}} > \frac{1}{2^{n-1}|s_0|^n}.$$

Hence if s_0 satisfies

$$|s_0|^n < (\sin\theta)^{n-1}, \tag{3.9}$$

then

$$|\Psi(X_j^{(l)})| > \frac{1}{2^{n-1}|s_0|^n} > \frac{1}{2^{n-1}(\sin\theta)^{n-1}} > |\Psi(X_j^{(l)})|$$

and this is a contradiction.

We fix $s_0 > 0$ satisfying (3.9). Let B_k be a divisor of the curve $H = \{yh(x,y) = 0\}$ intersecting $L = \{x = 1/2\}$ at X_k and set $B_{k,+} := \{X \in B_k : \text{Re } X > 1/2\}$, $B_{k,-} := \{X \in B_k : \text{Re } X < 1/2\}$, $B_+ := \{X \in L : \text{Im } X > \text{Im } X_1\}$ and $B_- := \{X \in L : \text{Im } X < \text{Im } X_{n-2}\}$. We call B_k, $B_{k,+}$, $B_{k,-}$, B_+ and B_- *branches* of the curve C. If k is odd, then when X moves from X_k along the branch $B_{k,+}$ or $B_{k,-}$, $\Psi(X)$ is positive and is monotone increasing. On the other hand, if k is even, then when X moves from X_k along the branch $B_{k,+}$ or $B_{k,-}$, $\Psi(X)$ is negative and is monotone decreasing. (See Figure 5.)

Lemma 3.3.6. *Each branch $B_{k,+}$, $B_{k,-}$ B_+ or B_- of the curve C does not intersect with other branches.*

Proof. In the case $(n-1)/2$ is odd, from Proposition 3.3.5,

$$\Psi(B_k) \subset \begin{cases} \{w \in \mathbb{R} : \Psi(X_k) \leq w\} \ (k: \ \text{odd}) \\ \{w \in \mathbb{R} : \Psi(X_k) \geq w\} \ (k: \ \text{even}). \end{cases}$$

The value $\Psi(X_k)$ is positive (resp. negative) when k is odd (resp. even). Hence B_k and B_{k+1} never intersect. From Proposition 3.3.5 again,

$$\Psi(B_\pm) \subset \{w \in \mathbb{R} : \Psi(X_1) \geq w\}.$$

(Remark that $\Psi(X_1) = \Psi(X_{n-2})$.) Therefore $B_\pm$ and B_1 ($B_\pm$ and B_{n-2}) never intersect.

Around the line L, the curve H intersects to L at X_k. Then there exist at least $n-2$ divisors in $yh(x,y) = 0$. Since the degree of $yh(x,y) = 0$ in y is $n-1$, $yh(x,y) = 0$ is factorized into a product of analytic functions: $\prod_k (y - h_k(x)) = 0$.

■

We assume that $(n-1)/2$ is odd. For any l,

$$\Psi(X_j^{(l)}) = \frac{(-1)^{l+1}}{2^{n-1}s_0^n(\sin l\theta)^{n-1}}.$$

Hence

$$\Psi(X_j^{(1)}) > \Psi(X_j^{(3)}) > \cdots > \Psi(X_j^{((n-1)/2)}) > 0$$

and

$$\Psi(X_j^{(2)}) < \Psi(X_j^{(4)}) < \cdots < \Psi(X_j^{((n-3)/2)}) < 0.$$

From the assumption (3.9),

$$\Psi(X_j^{((n-1)/2)}) > \Psi(X_1)\ (> \Psi(X_3) > \cdots > \Psi(X_{(n-1)/2}) > 0)$$

and

$$\Psi(X_j^{((n-3)/2)}) < \Psi(X_2)\ (< \Psi(X_4) \cdots < \Psi(X_{(n-3)/2}) < 0).$$

Hence for each odd k, $X_j^{((n-1)/2)}, X_j^{((n-5)/2)}, \cdots, X_j^{(3)}, X_j^{(1)}$ lie on $B_{k,\pm}$ in this order and for each even k, $X_j^{((n-3)/2)}, X_j^{((n-7)/2)}, \cdots, X_j^{(4)}, X_j^{(2)}$ lie on $B_{k,\pm}$ in this order. Similarly, $X_j^{((n-3)/2)}, X_j^{((n-7)/2)}, \cdots, X_j^{(4)}, X_j^{(2)}$ lie on $B_\pm$ in this order. We renumber the indices j and we summarize as follows:

Proposition 3.3.7. *The position of branch points $X_j^{(l)}$ and X_k is as follows:*

(I) *If k is odd, then $X_k^{((n-3)/2)}$, $X_k^{((n-7)/2)}$, $\ldots$, $X_k^{(3)}$, $X_k^{(1)}$ (resp. $X_{k+1}^{((n-3)/2)}$, $X_{k+1}^{((n-7)/2)}$, $\ldots$, $X_{k+1}^{(3)}$, $X_{k+1}^{(1)}$) is on the kth branch $B_{k,+}$ (resp. $B_{k,-}$) in this order.*

(II) *If k is even, then $X_k^{((n-1)/2)}$, $X_k^{((n-5)/2)}$, $\ldots$, $X_k^{(4)}$, $X_k^{(2)}$ (resp. $X_{k+1}^{((n-1)/2)}$, $X_{k+1}^{((n-5)/2)}$, $\ldots$, $X_{k+1}^{(4)}$, $X_{k+1}^{(2)}$) is on the kth branch $B_{k,+}$ (resp.$B_{k,-}$) in this order.*

(III) *$X_1^{((n-1)/2)}$, $X_1^{((n-5)/2)}$, $\ldots$, $X_1^{(4)}$, $X_1^{(2)}$ (resp. $X_{n-1}^{((n-1)/2)}$, $X_{n-1}^{((n-5)/2)}$, $\ldots$, $X_{n-1}^{(4)}$, $X_{n-1}^{(2)}$) on B_+ (resp. B_-) in this order.*

And the outline of the curve C is as Figure 6.

3.4 The monodromy permutations of the branch covering map p_{s_0}

In subsection 3.3, we get the configuration of the branch loci of p_{s_0}. Next we determine its monodromy permutations.

Let $\overline{\pi} := \pi_1(\mathbb{C} \setminus \{X_j^{(l)}\}, X_0)$ be the fundamental group of non-branched locus domain of p_{s_0}. The map

$$p_{s_0} : p_{s_0}^{-1}(\mathbb{C} \setminus \{X_j^{(l)}\}) \to \mathbb{C} \setminus \{X_j^{(l)}\}$$

is a covering map and any path $[\gamma] \in \overline{\pi}$ gives a permutation of $p_{s_0}^{-1}(X_0) = \{Y^{(1)}, Y^{(2)}, \ldots, Y^{(n-1)}\}$ through the liftings of γ. We denote this permutation by $\overline{\gamma}$ and we call it the *monodromy permutation*.

For $l = 1, 2, \ldots, (n-1)/2$, $j = 1, 2, \ldots, n-1$, we define a path $\gamma_j^{(l)}$ as follows: The path $\gamma_j^{(l)}$ starts at X_0 and goes (almost) along C toward near $X_j^{(l)}$ and turns around $X_j^{(l)}$ once and goes back on the coming path. See Figure 7. Here, around the the branch points $X_{j'}^{(l')}$ and $X_j^{(l)}$, the path $\gamma_j^{(l)}$ goes along ε-circles.

Theorem 3.4.1. *The monodromy permutation $\overline{\gamma_j}^{(l)}$ is as follows:*

(I) *For $1 \leq l < (n-1)/2$, $\overline{\gamma_j}^{(l)} = (l, l+1)(n-1-l, n-l)$.*

(II) *For $l = (n-1)/2$, $\overline{\gamma_j}^{(l)} = ((n-1)/2, (n-1)/2+1)$.*

Proof. We assume that l, k are odd integers other than $l = (n-1)/2$. If we move X from X_0 via X_k to $X_k^{(l)}$, strictly along the curve C, then we can strictly pursue the movement of $Y^{(l)}$. Indeed a real solution v of $\overline{\phi}_X(v) = 0$ gives a solution $Y = 1/2s_0 + iv$ of $g_{s_0}(X, Y) = 0$. The function

$$\overline{\phi}_X(v) = \Psi(X) - \Psi(X_0) + \overline{\phi}(v)$$

has extreme points $b_{l'}$ and extremums $\Psi(X) - a_{l'}(s_0)$ ($l' = 1, 2, \ldots, n-2$). We note that while we move X from X_0 to X_k, the number of real solutions of $\overline{\phi}_X(v) = 0$ does not change, because of the discussion around the condition (3.9). On the other hand, while we move from X_k to $X_j^{(l)}$, on the branch $B_{k,+}$, $\Psi(X)$ is monotone increasing, and the extremum

$$\Psi(X_k^{(l)}) - a_l(s_0) = 0 \quad (\text{also } \Psi(X_k^{(n-1-l)}) - a_{n-1-l}(s_0) = 0).$$

This means if we pursue the movement of $Y^{(l)}$'s when we make X at $X_j^{(l)}$, $Y^{(l)}$ meets $Y^{(l+1)}$ at a point Y_l, and $Y^{(n-1-l)}$ meets $Y^{(n-l)}$ at a point Y_{n-1-l}. This is the result of the halfway of $\gamma_j^{(l)}$ with $\varepsilon \to 0$. This means that $\overline{\gamma}_j^{(l)} = (l, l+1)(n-1-l, n-l)$. For other l, k, the statements are shown in the same way.

■

From Theorem 3.4.1, the reference fiber $f^{-1}(s_0)$ is obtained by the following way: (I) Prepare $n-1$ projective lines with $(n-1)(n-2)$ holes and $(n-1)(n-2)/2$ annuli. (II) Paste projective lines and annuli along the hole with rules in Theorem 3.4.1. We can construct a smooth complex curve of genus $(n-2)(n-3)/2$.

4 Determination of the global monodromy

In this section, we determine the global monodromy. We investigate the movement of branch points of $p_s : f^{-1}(s) \to \mathbb{C}P^1$ when we move s from s_0 to the singular value $s^{(j)}_{k,l}$.

4.1 Recipe for the global monodromy

We set $\Psi(X) := X^n - (X-1)^n$ and consider the equation

$$g_s(X, Y_l(s)) = \Psi(X) - a_l(s) = 0,$$

where $a_l(s) := 1/s^n(1-\tau_l)^{n-1}$. First we note that the solutions of $g_s(X, Y_l(s)) = \Psi(X) - a_l(s) = 0$ $(l = 1, 2, \ldots, (n-1)/2)$ give all branch points of the branched covering map $p_s : f^{-1}(s) \to \mathbb{C}$. We investigate the movement of the solutions of $g_s(X, Y_l) = 0$ when we move s. In the case that n is odd, if $a_l(s) \in \mathbb{R}$ then the solutions of $\Psi(X) - a_l(s) = 0$ lie on the curve $C = \{\mathrm{Im}\ \Psi(X) = 0\}$. Since $n-1$ is even, $(1-\tau_l)^{n-1}$ is a real number by Lemma 3.1.2 and $a_l(s)$ is a real number precisely when s is an nth root of a real number. Hence we obtain

Lemma 4.1.1. *If n is an odd number and s^n is a real number, then every solution of the equation $g_s(X, Y_l) = 0$ is on the curve C. That is, all branch points of p_s are on the curve C.*

We recall that $X^{(l)}_j$ $(j = 1, 2, \ldots, n-1)$ are all solutions of $g_{s_0}(X, Y_l(s_0)) = 0$ and X_k satisfies $g_{s^{(j)}_{k,l}}(X_k, Y_l(s^{(j)}_{k,l})) = 0$ (See subsection 1.1). For every s, there exist solutions of $g_s(X, Y_l(s)) = 0$ and X is continuous with respect to s. Then we conclude

Proposition 4.1.2. *We fix k and l. If we move s from s_0 to $s^{(0)}_{k,l}$ along the real axis, then some of the branch points $X^{(l)}_j$ of p_{s_0} move to X_k along the curve C.*

Proof. Since $X^{(l)}_j$ (resp. X_k) is a solution of $g_{s_0}(X, Y_l) = 0$ (resp. $g_{s^{(j)}_{k,l}}(X, Y_l) = 0$), we obtain the assertion from Lemma 4.1.1.

■

For simplicity, we put $S = 1/s^n$ and set $A_l(S) := S/(\tau_l - 1)^{n-1}$, $g_S(X, Y_l) := \Psi(X) - A_l(S)$, $S_0 := 1/s_0^n$ and $S_{k,l} := 1/(s^{(j)}_{k,l})^n$.

We discuss how to obtain the global monodromy. For details, see [2], [6], [7]. In our case, we know that there occur single nodes except on $f^{-1}(0)$ or $f^{-1}(\infty)$.

Each single node is correspondent to a vanishing cycle, so it is sufficient to know how to obtain the vanishing cycles.

Let γ be a path in s-plane as in Figure 8. Our goal is getting vanishing cycles with respect to γ. We push out γ into S-plane as in Figure 9 (Note: $S = 1/s^n$). We denote by $\overline{\gamma}$ the path in S-plane induced from γ. Let δ be a half path of $\overline{\gamma}$ in S-plane, that is, δ is a path from S_0 to $S_{k,l}$ almost along $\overline{\gamma}$. We set the end point of δ as $S_{k,l}$ itself (Figure 10). We move the parameter S along the path δ and observe movement of solutions of

$$\prod_l (\Psi(X) - A_l(S)) = 0.$$

For example, we suppose that $X_{k_1}^{(l)}$ meets $X_{k_2}^{(l)}$ at X_k and other $X_{k'}^{(l')}$'s never meet together (Figure 11). We draw a loop ζ surrounding the trace of $X_{k_1}^{(l)}$ and $X_{k_2}^{(l)}$ (Figure 12), and let $\overline{\zeta_1}, \overline{\zeta_2}, \ldots, \overline{\zeta_r}$ be non-zero-homologous liftings of ζ over p_{s_0}. The liftings $\overline{\zeta_1}, \overline{\zeta_2}, \ldots, \overline{\zeta_r}$ are the vanishing cycles at $S_{k,l}$ with respect to the loop γ. Using this procedure, in order to obtain the global monodromy, it is sufficient for us to know movement and meetings (encounters) of $X_{k'}^{(l')}$'s for any half path δ in S-plane.

In S-plane, critical value $S_{k,l}$ are on the real axis, hence we consider a half path $\delta_{k,l}$ to $S_{k,l}$ consisting of some segments on the real axis and of some half (or full) circles of radius $\varepsilon > 0$ (Figure 13). The equation $\prod_l(\Psi(X) - A_l(S)) = 0$ has multiple solutions if and only if $S = 0, \infty, S_{k,l}$. We denote by $\{X_j^{(l)}\}$ the set of the solutions of $\prod_l(\Psi(X) - A_l(S_0)) = 0$. These facts are followed that there exist unique liftings (traces) of $\delta_{k,l}$ with start point $X_j^{(l)}$ for each l and j.

When the parameter S goes to the end point $S_{k,l}$ of $\delta_{k,l}$ (as in Figure 13), there happens an encounter of $X_{k_1}^{(l)}$ and $X_{k_2}^{(l)}$ at X_k for some k_1 and k_2 (from Proposition 4.1.2). On the other hand, if $\varepsilon > 0$ is very small, then the liftings (traces) of $X_k^{(l)}$'s are almost on the curve C (from Lemma 4.1.1). In the next subsection, we determine k_1 and k_2 for each $\delta_{k,l}$, and pursue the movement of $X_{k_1}^{(l)}$ and $X_{k_2}^{(l)}$ (almost) on the curve C.

4.2 Behavior of the solutions of $g_s(X, Y_l) = 0$ around the critical value 0 and $S_{k,l}$

Let $Q_1, Q_2, \ldots, Q_{n-1}$ be the solutions of the equation $\Psi(X) = 0$. Then we have the following lemma.

Lemma 4.2.1. *The points $Q_1, Q_2, \ldots, Q_{n-1}$ are on the line $L = \{\mathrm{Re}\ X = 1/2\}$. Moreover on the line L, there are $Q_1, X_1, Q_2, X_2, \ldots, X_{n-2}, Q_{n-1}$ in this order.* (*See Figure* 14).

Proof. The value $\overline{\Psi}(y_k)$ $(k = 1, 2, \ldots, n-2)$ are extremums of $\overline{\Psi}$ and $\overline{\Psi}(y_k)\overline{\Psi}(y_{k+1}) < 0$ for any k. Hence there exist $n-1$ real solutions for $\overline{\Psi}(y) = 0$ and they give $n-1$ solutions of $\Psi(X) = 0$ on the line L. Recalling that $y_k = \mathrm{Im}\, X_k$, it is clear that $Q_1, X_1, Q_2, X_2, \ldots, X_{n-2}, Q_{n-1}$ are in this order.

■

From Lemma 4.2.1, if S goes near 0, then $X_j^{(l)}$'s go toward the points $Q_1, Q_2, \ldots, Q_{n-1}$ on the line L since $A_l(0) = 0$ for any l. We remark that for one point Q_j, there are just $(n-2)/2$ of $X_j^{(l)}$'s that converge to Q_j. If a path $\delta_{k,l}$ contains a half circle of radius $\varepsilon > 0$ around 0, the movement of $X_j^{(l)}$'s are given by Figure 15, since a lifting map $S \mapsto X_j^{(l)}$ is a holomorphic (and conformal) map.

When the parameter S goes near $S_{k',l'}$, it is sufficient for us to pay attention to the branch points $X_j^{(l')}$ $(j = 1, 2, \ldots, n-1)$ (and also $X_j^{(n-1-l')} = X_j^{(l')}$). Since any singularities are single nodes, just two of $X_j^{(l')}$'s converge to $X_{k'}$. Therefore, if a path $\delta_{k,l}$ contains a half circle of radius $\varepsilon > 0$ around $S_{k',l'}$, the movement of the two of $X_j^{(l')}$'s looks like in Figure 16. This behavior is just the same as in the case $y^2 = x^2 - s$, standard single node.

4.3 The global monodromy for $S_{k,l}$

From now on, we assume that n is odd and $(n-1)/2$ is odd. In other cases, similar results hold. We determine how a branch point encounters another one. Recall that

$$S_{k,l} = (-1)^{l-k} \left(\frac{\sin l\theta}{\sin k\theta} \right)^{n-1},$$

where $\theta = \pi/(n-1)$, and $S_{k,l} = S_{k,n-1-l}$, $S_{k,l} = S_{n-1-k,l}$. Then we obtain

Lemma 4.3.1. *For a fixed k, the following inequalities hold:*

(I) *If l is odd, then*

   (i) $0 < S_{(n-1)/2,l} < S_{(n-5)/2,l} < \cdots < S_{3,l} < S_{1,l}$.

   (ii) $S_{2,l} < S_{4,l} < \cdots < S_{(n-3)/2,l} < 0$.

(II) *If l is even, then*

   (i) $0 < S_{(n-3)/2,l} < S_{(n-7)/2,l} < \cdots < S_{4,l} < S_{2,l}$.

   (ii) $S_{1,l} < S_{3,l} < \cdots < S_{(n-1)/2,l} < 0$.

The condition (3.4) and (3.9) are followed by

$$S_0 > \frac{1}{(\sin\theta)^{n-1}} = |S_{1,(n-1)/2}| = \max_{k,l} |S_{k,l}|.$$

We indexing of $X_j^{(l)}$ is as in Figure 6. Let $\delta_{k,l}$ (resp. δ_0) be a half path from S_0 to $S_{k,l}$ (resp. 0) such as in Figure 13. Our final goal is the following theorem.

Theorem 4.3.2. *The encounter of $X_j^{(l)}$'s with respect to $\delta_{k,l}$ or δ_0 is as follows:*

(I) *If $S_{k,l} > 0$, that is k and l are both odd (or both even), then two branch points $X_k^{(l)}$ and $X_{k+1}^{(l)}$ (resp. $X_{n-1-k}^{(l)}$ and $X_{n-2-k}^{(l)}$) on the branches $B_{k,+}$ and $B_{k,-}$ (resp. $B_{n-1-k,+}$ and $B_{n-1-k,-}$) converge to X_k (resp. X_{n-1-k}). See Figure* 17.

(II) *If k ($\neq 1$, $\neq n-2$) is odd and l is even ($S_{k,l} < 0$), then two branch points $X_{k+1}^{(l)}$ and $X_k^{(l)}$ (resp. $X_{n-k}^{(l)}$ and $X_{n-1-k}^{(l)}$) on the branches $B_{k+1,+}$ and $B_{k-1,-}$ (resp. $B_{n-k,+}$ and $B_{n-k-2,-}$) converge to X_k (resp. X_{n-k-1}). See Figure* 18.

(III) *If k is even and l is odd then the branch points $X_{k+1}^{(l)}$ and $X_k^{(l)}$ (resp. $X_{n-k}^{(l)}$ and $X_{n-1-k}^{(l)}$) on the branches $B_{k+1,+}$ and $B_{k-1,-}$ (resp. $B_{n-k,+}$ and $B_{n-k-2,-}$) converge to X_k (resp. X_{n-k-1}). See Figure* 18.

(IV) *If $k = 1$ (resp. $n-2$) and l is even then the branch points $X_1^{(l)}$ and $X_2^{(l)}$ (resp. $X_{n-2}^{(l)}$ and $X_{n-1}^{(l)}$) on the branches B_+ and $B_{2,+}$ (resp. $B_{n-3,-}$ and B_-) converge to X_1 (resp. X_{n-2}). See Figure* 19.

(V) *If $S = 0$, then the movement of the branch points is as Figure* 20.

Proof. Let k and l be odd numbers and let k' be an even number. If we move S from S_0 to $S_{k,l}$, then there exists no singular value $S_{k',l}$ between S_0 and $S_{k,l}$ from Lemma 4.3.1. The solutions of the equation $\prod_l(\Psi(X) - A_l(S)) = 0$ on the branch other than B_k do not go to X_k, because if $X_{k''}^{(l)}$ on another branch $B_{k'}$ for even k' goes to X_k, then it must pass through $X_{k'}$. The solutions on B_k are $X_k^{(l)}$ and $X_{k+1}^{(l)}$, and they must encounter each other when S goes to $S_{k,l}$. When $X_k^{(l)}$ meets $X_{k+1}^{(l)}$, $X_k^{(l')}$ and $X_{k+1}^{(l')}$ ($l' > l$) move on B_k toward X_k, turn right at X_k, and finally go to a point on L. The other $X_k^{(l')}$ and $X_{k+1}^{(l')}$ ($l' < l$) move on B_k toward X_k and finally go to a point on B_k. (See Figure 21.) From the definition of $S_{k,l}$, we have $S_{k,l} = S_{n-1-k,l}$. Hence if S goes to $S_{k,l}$, then $X_{n-1-k}^{(l)}$ (on $B_{n-1-k,+}$) encounters $X_{n-k}^{(l)}$ (on $B_{n-1-k,-}$) at X_{n-1-k}.

Similarly, let k and l be even numbers and let k' be an odd number. If we move S from S_0 to $S_{k,l}$, then there exists no singular value $S_{k',l}$. The solutions of the equation $\prod_l(\Psi(X) - A_l(S)) = 0$ on the branch other than B_k do not go to X_k Thus we have (I).

Suppose that k is odd, l is even, $k \neq 1$ and $k \neq n-2$. Then $S_{k,l}$ is negative and $\delta_{k,l}$ pass near 0 once before arriving at $S_{k,l}$. Hence if $X_j^{(l)}$ goes to X_k, then it must pass the points Q_k or Q_{k+1} once (Figure 22). Thus $X_j^{(l)}$ must be on the branch B_{k-1} or B_{k+1} at the start. As in Figure 16, $X_j^{(l)}$ turns right when it visit a crossroad X_{k-1} (or X_{k+1}). This means that $X_j^{(l)}$ must be on $B_{k-1,-}$ or $B_{k+1,+}$ at the start. It follows that $X_k^{(l)}$ (on $B_{k-1,-}$) encounters $X_{k+1}^{(l)}$ (on $B_{k+1,+}$) at X_k. Thus we have (II).

Suppose that k is even and l is odd. Then $S_{k,l}$ is negative and $S_{k,l}$ pass near 0 once. In the same reason as (II), $X_j^{(l)}$ must pass the solution Q_{k-1} or Q_k, and hence $X_k^{(l)}$ (on $B_{k-1,-}$) encounters $X_{k+1}^{(l)}$ (on $B_{k+1,+}$) at X_k. Thus we have (III).

Suppose that $k = 1$ and l is even. Then $S_{k,l}$ is negative and $X_j^{(l)}$ must pass the solution Q_1 or Q_2. Hence $X_1^{(l)}$ (on B_+) encounters $X_2^{(l)}$ (on $B_{2,+}$) at X_1. In case that $k = n-2$ and l is even, we can show in the same way. Thus we have (IV).

In case (V), as in Figure 15, $X_k^{(l)}$ turn right at X_{k-1} or X_k. Therefore every $X_k^{(l)}$ $(l = 1, 2, \ldots, n-2)$ meet together at Q_k.

■

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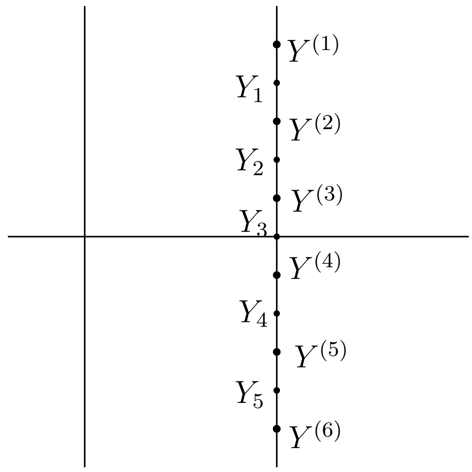


Figure 1: The solutions $Y^{(j)}$ of $g_{s_0}(X_0, Y) = 0$ and the solutions Y_l of $\frac{\partial g_{s_0}}{\partial Y} = 0$ in the case that $n = 7$.

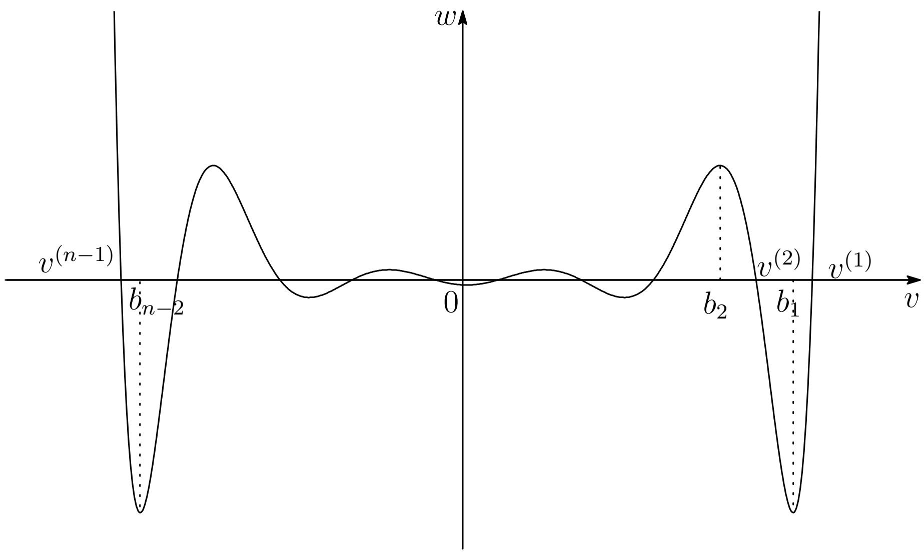


Figure 2: The graph $w = \overline{\phi}_{X_0}(v)$ in the case $n = 11$: The extremums decrese in order.

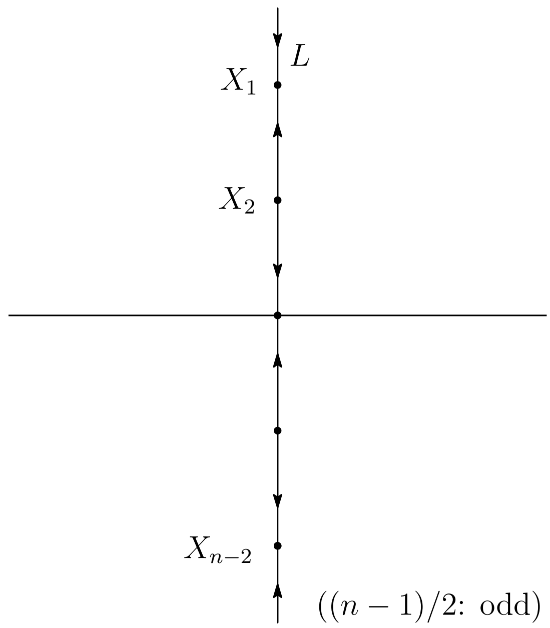


Figure 3: The direction of increase of $\Psi(X)$ on L

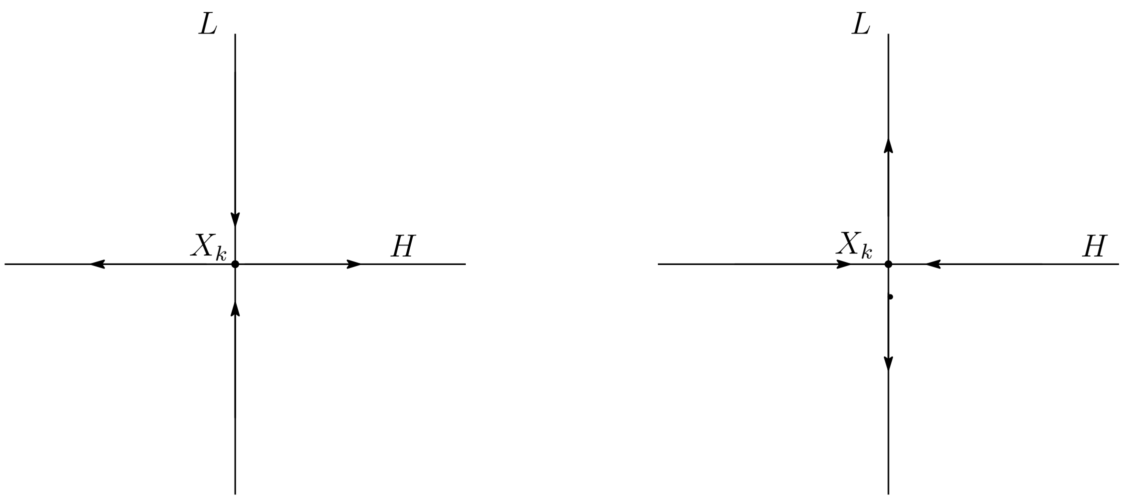
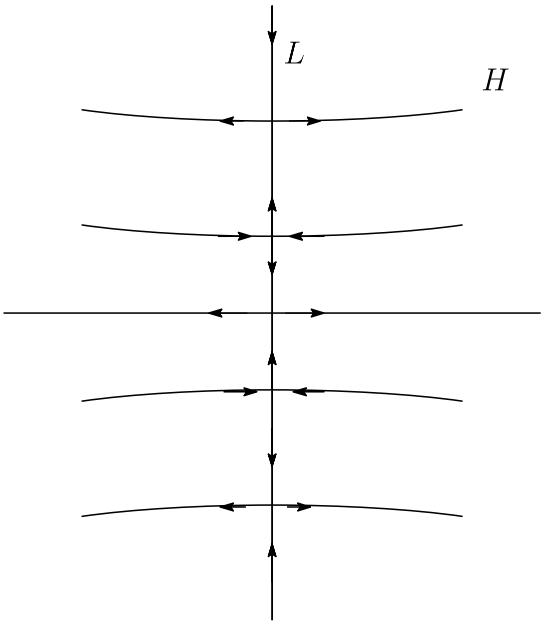


Figure 4: The direction of increase of $\Psi(X)$ around X_k



$((n-1)/2$: odd)

Figure 5: The direction of increase of $\Psi(X)$ on C around L

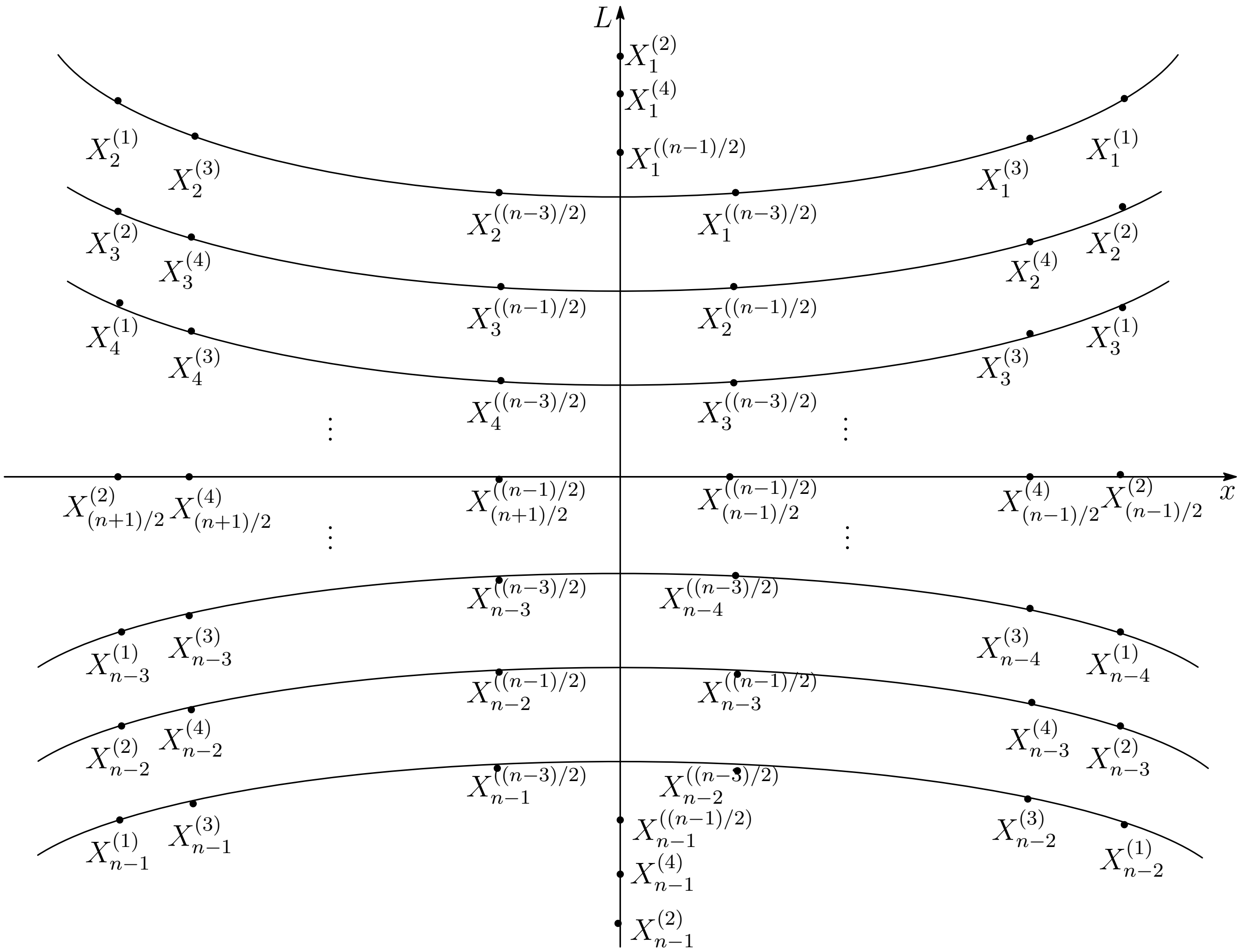


Figure 6: The curve C and the positions of the branch points of p_{s_0}

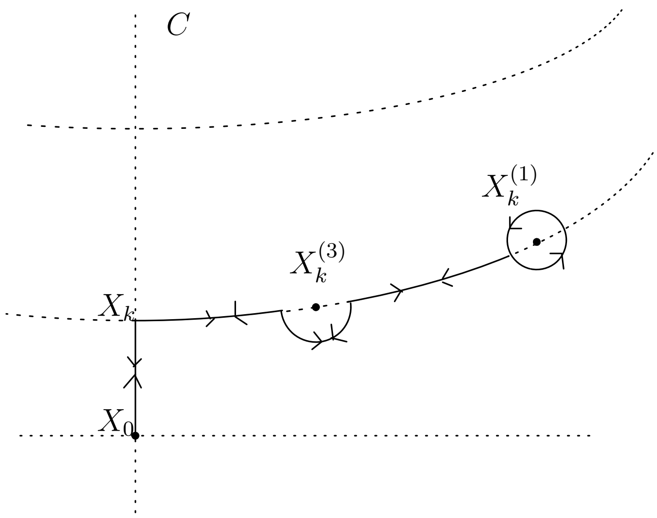


Figure 7: The path $\gamma_k^{(l)}$ along the curve C

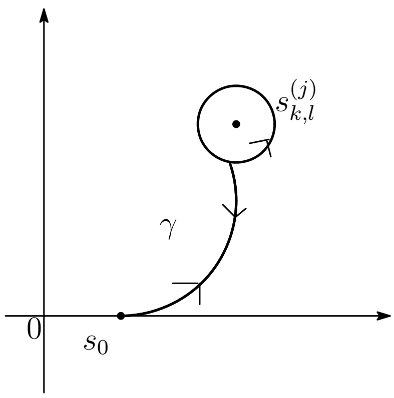


Figure 8: The path γ starting point s_0 such that go around $s_{k,l}^{(j)}$.

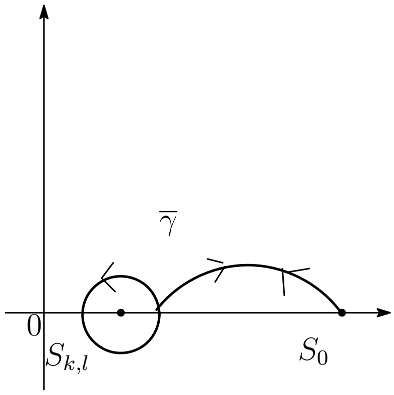


Figure 9: The path $\overline{\gamma}$ starting point S_0 such that go around $S_{k,l}$.

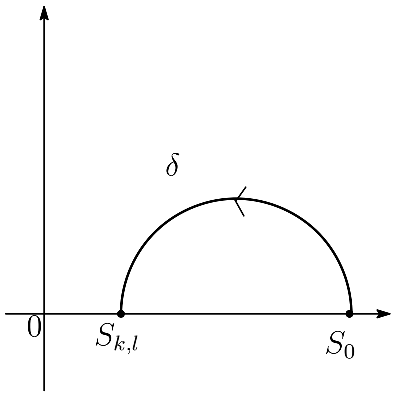


Figure 10: The path δ starting point S_0 to $S_{k,l}$.

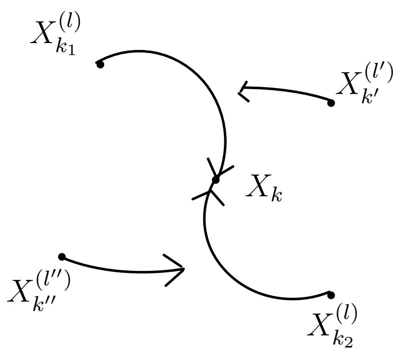


Figure 11: The movement of the branch points

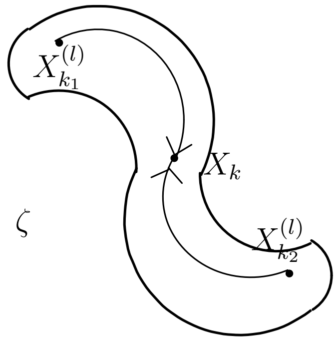


Figure 12: The path ζ surrouding the trace

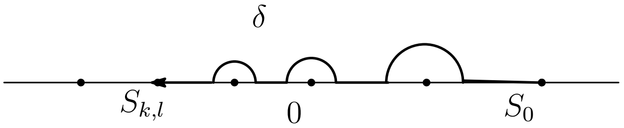


Figure 13:

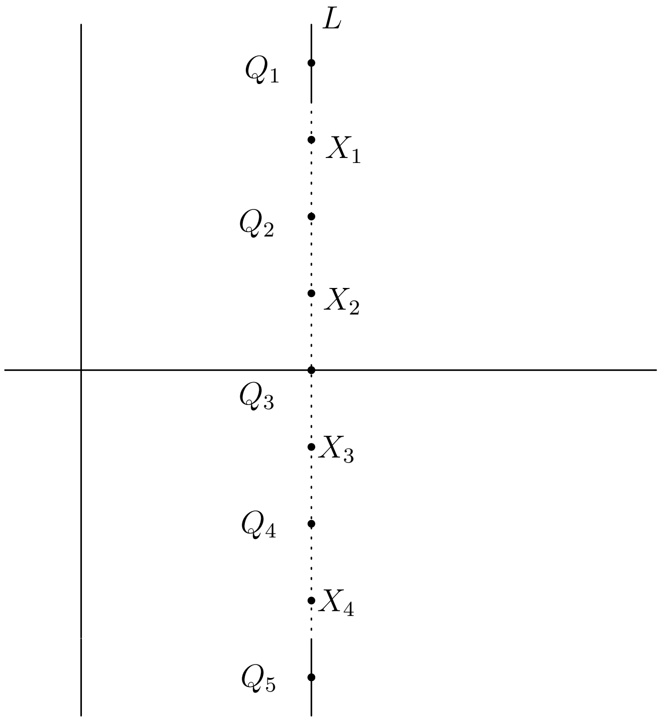


Figure 14: $Q_1, Q_2, \ldots, Q_{n-1}$ are the solutions of $\Psi(X) = 0$ in the case that $n = 6$.

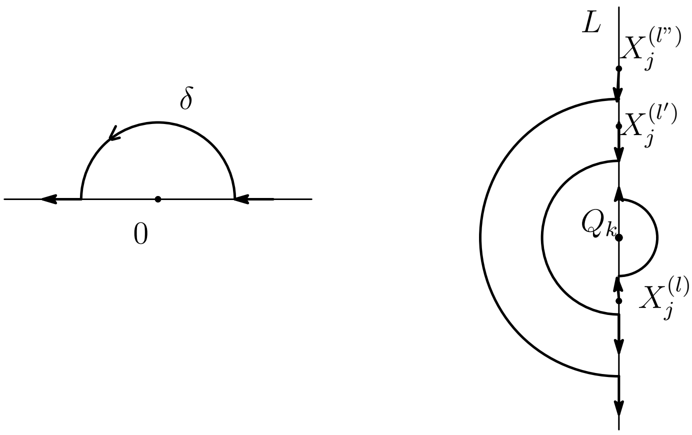


Figure 15:

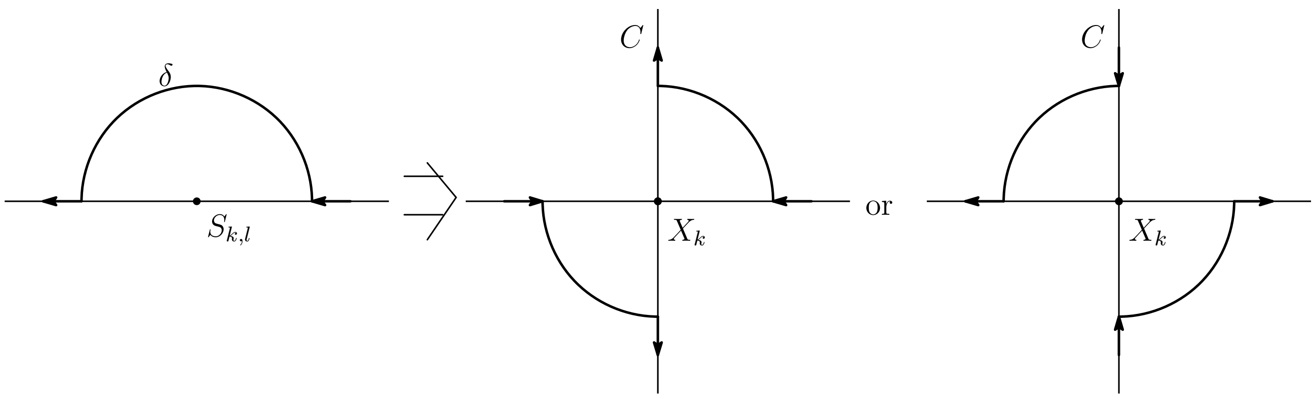


Figure 16: The movement of S and X

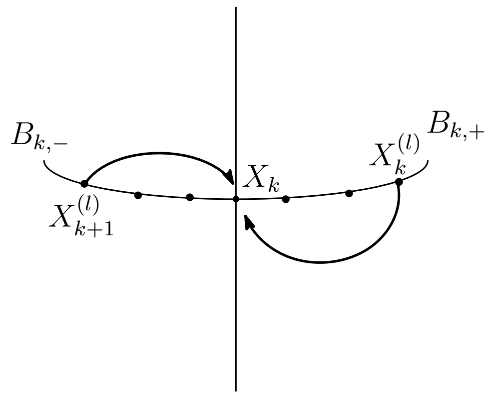


Figure 17: (I) The movement of the branch points: The bold arrow lines are homotopically rearranged.

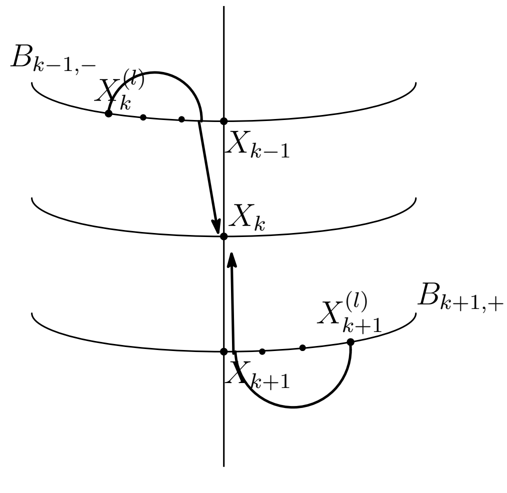


Figure 18: (II), (III) The movement of the branch points

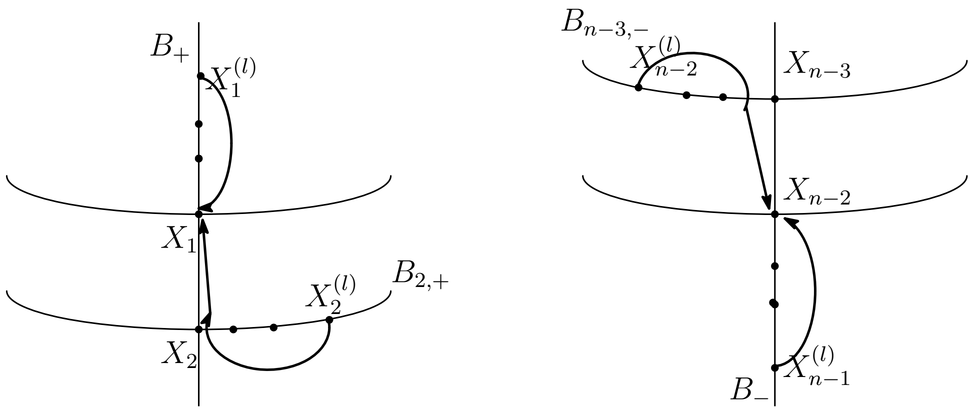


Figure 19: (IV) The movement of the branch points

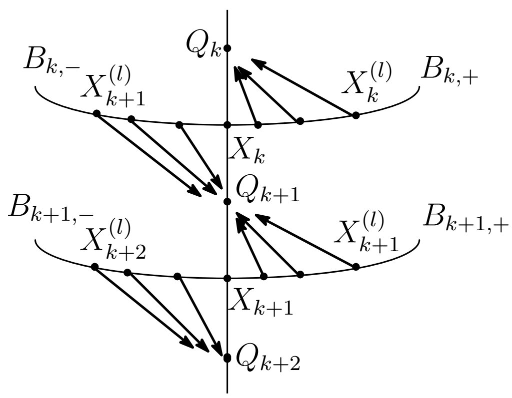


Figure 20: (V) The movement of the branch points: Q_k and Q_{k+1} are the solutions of $\Psi(X) = 0$.

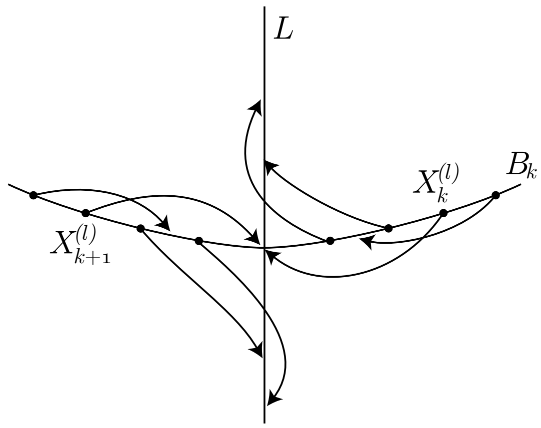


Figure 21:

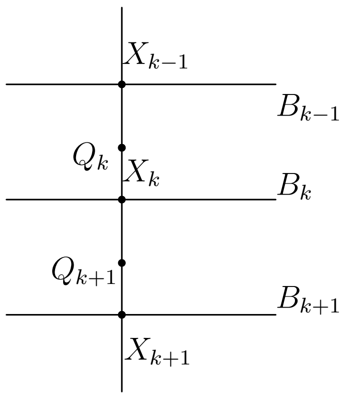


Figure 22: