Johnson-Morita homomorphism and homological fibered knots

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Motivation

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In this talk, we focus on the Johnson-Morita homomorphism.

We give a method to compute it (using computers), and show concrete examples. In particular, we will use knots in the 3-sphere.

Program

$\S 1.$ Homology cylinders and Homological fibered knots

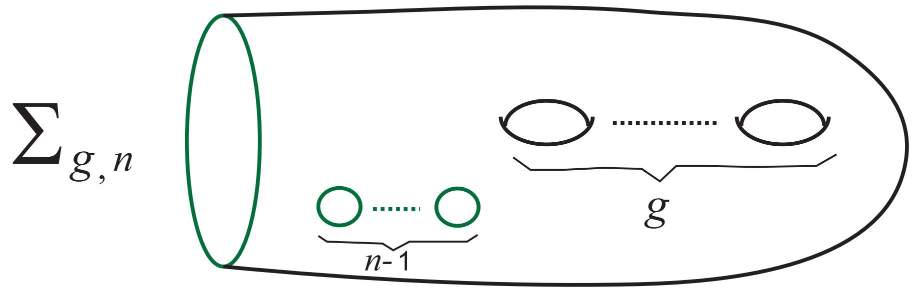
$\S 2.$ Homology cylinders as an enlargement of the mapping class group

$\S 3.$ The Johnson-Morita homomorphism

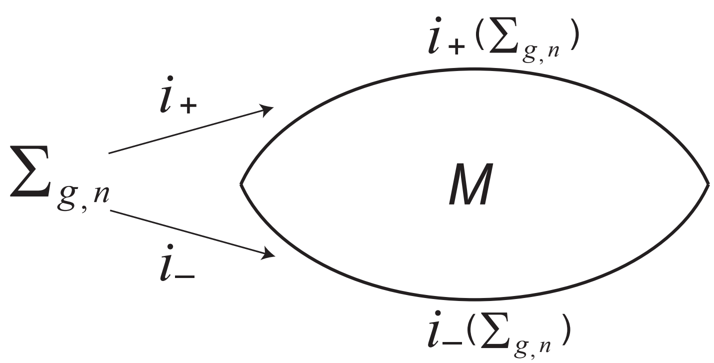
$\S 4.$ Computations

Notations

$\Sigma_{g,n}$: a compact oriented genus g surface with n-comp. boundaries



M: a compact 3-manifold
$i_\pm$: $\Sigma_{g,n} \hookrightarrow \partial M$: embeddings



Homology cylinder

Definition. A homology cylinder (M, i_+, i_-) over $\Sigma_{g,n}$ consists of a compact oriented 3-manifold M with two embeddings $i_+, i_- : \Sigma_{g,n} \hookrightarrow \partial M$ such that:

(i) i_+ is orientation-preserving and i_- is orientation-reversing;

(ii) $\partial M = i_+(\Sigma_{g,n}) \cup i_-(\Sigma_{g,n})$ and $i_+(\Sigma_{g,n}) \cap i_-(\Sigma_{g,n}) = i_+(\partial\Sigma_{g,n}) = i_-(\partial\Sigma_{g,n})$;

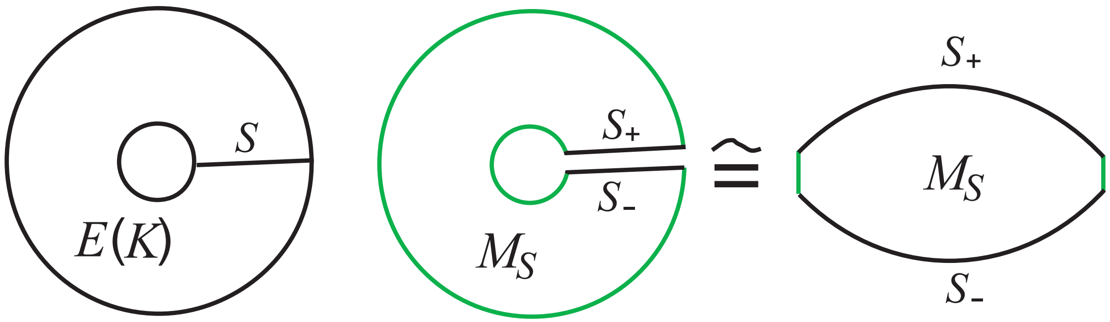
(iii) $i_+|_{\partial\Sigma_{g,n}} = i_-|_{\partial\Sigma_{g,n}}$; and

(iv) $i_+, i_- : H_*(\Sigma_{g,n};\mathbb{Z}) \to H_*(M;\mathbb{Z})$ are isomorphisms.

Sutured manifold

Definition. $K \subset S^3$: a knot, S : a Seifert surface of K.
The manifold M_S that is obtained from $E(K)$ by cutting along S is called a sutured manifold for S.

($E(K) = S^3 - \mathring{N}(K)$, the knot exterior)



Question. When does M_S become a homology cylinder ?

Homological fibered knot

Proposition [Crowell-Trotter,. . . , Sakasai-G]. There is a minimal genus Seifert surface S such that M_S becomes a homology cylinder

$\Longleftrightarrow$

- $2g(K) =$the degree of the Alexander polynomial $\Delta_K(t)$ and
- $\Delta_K(t)$ is monic.

(This knot is called a homological fibered knot, denote by HFKnot.)

Remark. The same result holds for the link case. The knot case corresponds to $\Sigma_{g,1}$.

Homological fibered knot

Further, for any minimal genus Seifert surface S' of HFKnot, $M_{S'}$ becomes a homology cylinder.

We denote by HC a homology cylinder.

Remark.
(1) A fibered knot is a HFKnot.
(2) [Murasugi] For an alternating knot K,

$$K \text{ is fibered} \iff K \text{ is a HFKnot.}$$

(3) For a prime knot K with $\leq$ 11-crossings,

$$K \text{ is fibered} \iff K \text{ is a HFKnot.}$$

Question. Is there a HFKnot which is not fibered ?

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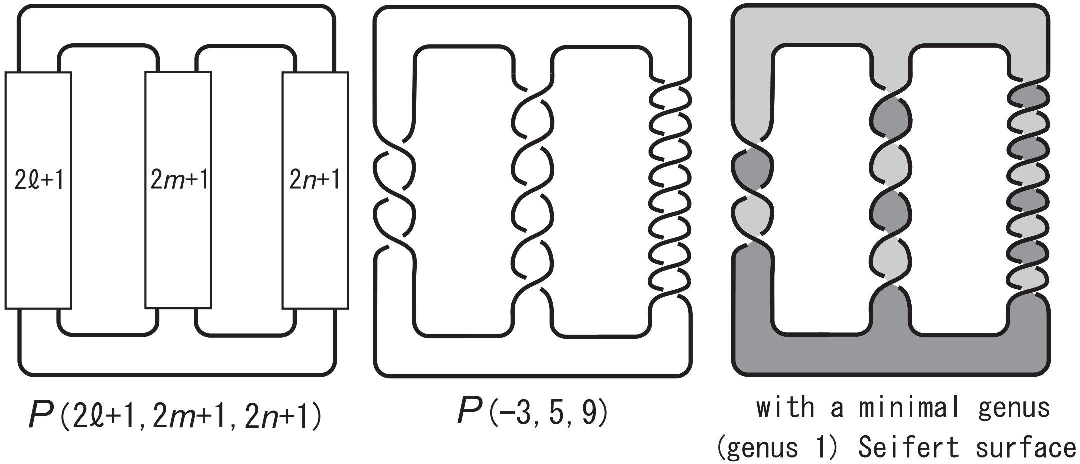
Question. Is there a HFKnot which is not fibered ?
Answer. Yes.

Homological fibered knot

Example 1. Pretzel knots of type
$(-3, 5, 5),\ (-3, 5, 9),\ (-5, 7, 15), \cdots$



Homological fibered knot

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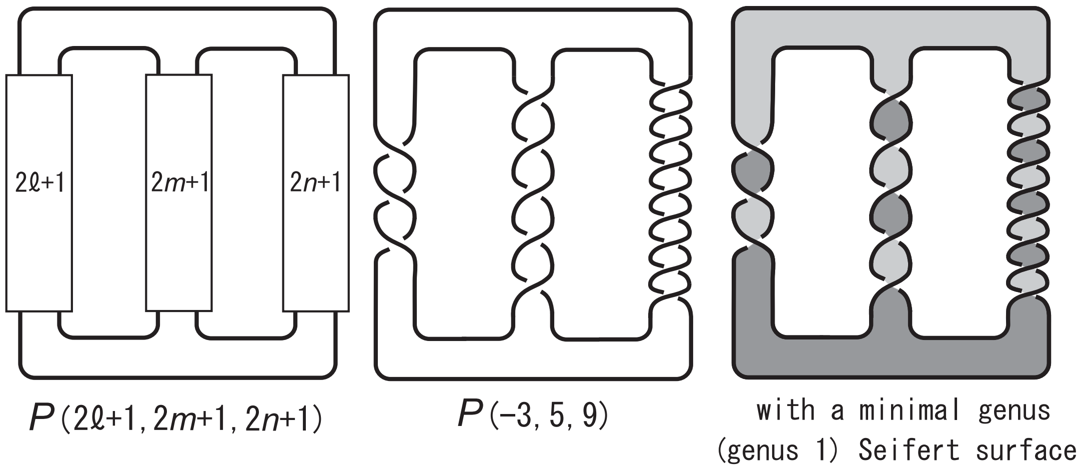
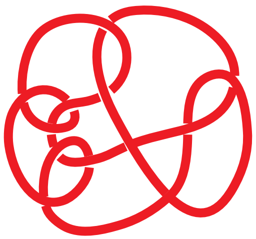
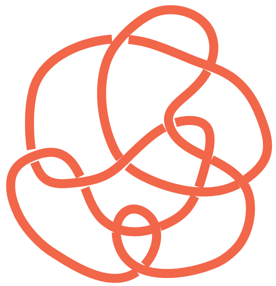


Example 2 [Friedl-Kim]. There are 13 non-fibered HFKnots with 12-crossings. (We focus on these knots in this talk.)

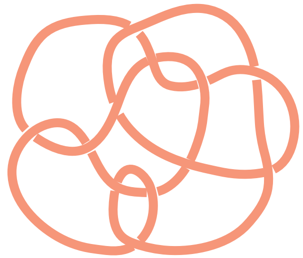
HFKnot with 12-crossings



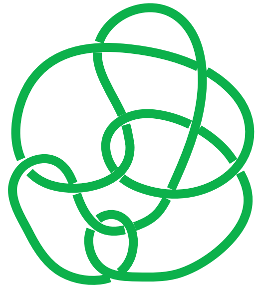
0057



0210



0214



0258

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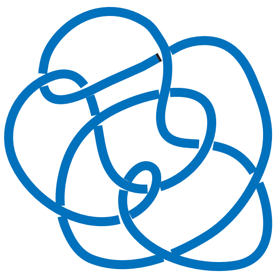
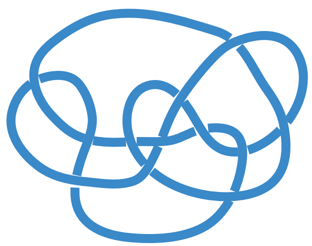


0382



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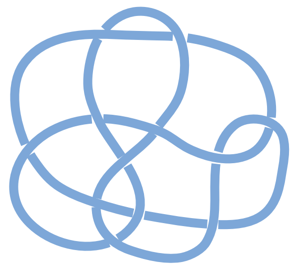

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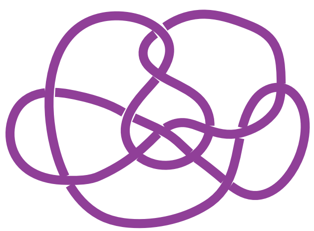

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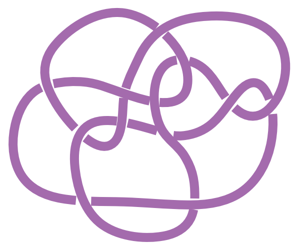


0801

0815

Background

Pure Braid $\longleftrightarrow$ surface $\times[0,1]/_{0\sim 1}$ $\longleftrightarrow$ fibered knot

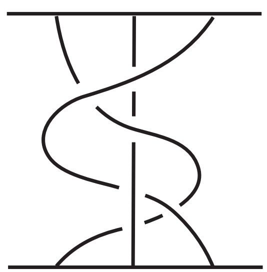
$\cap$ $\qquad\qquad$ $\cap$ $\qquad\qquad$ $\cap$

Pure String link $\overset{\text{Levine}}{\longleftrightarrow}$ Homology cylinder $\longleftrightarrow$ HFKnot

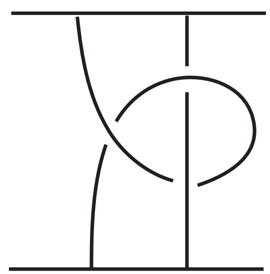
(Habegger-Lin) $\qquad$ (Goussarov, Habiro)

$\uparrow$ $\qquad\qquad$ $\uparrow$

Invariants by Kirk-Livingston-Wang $\qquad$ Invariants by Sakasai



pure braid



pure string link

Definition of $\mathcal{M}_{g,1}$

$$\begin{array}{ccccc}
\text{fibered knot} & \longleftrightarrow & \text{surface } \times[0,1]/_{0\sim 1} & \longleftrightarrow & \text{the mapping class group } \mathcal{M}_{g,1} \\
\cap & & \cap & & \cap \\
\text{HFKnot} & \longleftrightarrow & \text{Homology cylinder} & \longleftrightarrow & \text{the monoid of Homology cylinder } \mathcal{C}_{g,1}
\end{array}$$

Definition.
$\mathrm{Diff}(\Sigma_{g,1}) := \{f : \Sigma_{g,1} \overset{\cong}{\to} \Sigma_{g,1} \mid$ diffeomorphism such that $f|_{\partial\Sigma_{g,1}} = \mathrm{id}_{\partial\Sigma_{g,1}}\}$

Definition. The mapping class group of $\Sigma_{g,1}$

$$\mathcal{M}_{g,1} := \mathrm{Diff}(\Sigma_{g,1})/\text{isotopy}$$

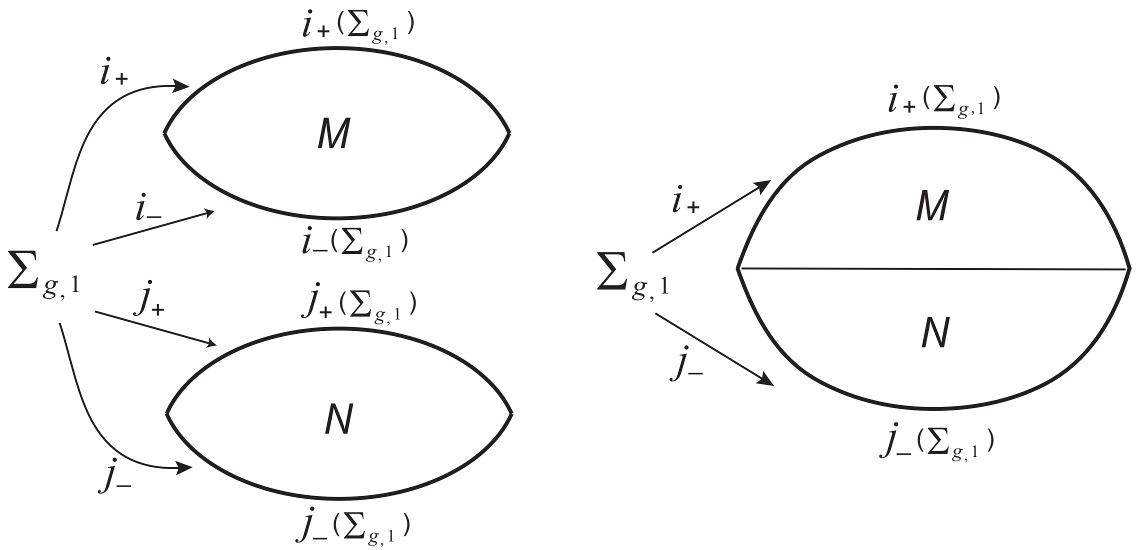
Monoids of Homology cylinders

Definition. Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,1}$ are said to be isomorphic if there exists an orientation-preserving diffeomorphism $f : M \xrightarrow{\cong} N$ satisfying $j_+ = f \circ i_+$ and $j_- = f \circ i_-$.
We denote by $\mathcal{C}_{g,1}$ the set of all isomorphism classes of homology cylinders over $\Sigma_{g,1}$.
We define a product operation on $\mathcal{C}_{g,1}$ by

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-)$$

for (M, i_+, i_-), $(N, j_+, j_-) \in \mathcal{C}_{g,1}$.

Monoids of Homology cylinders



Then $\mathcal{C}_{g,1}$ becomes a monoid with the unit

$$(\Sigma_{g,1} \times [0,1], \mathrm{id} \times 1, \mathrm{id} \times 0).$$

Monoids of Homology cylinders

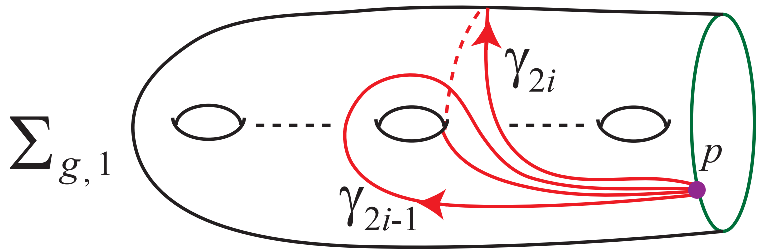
$$\begin{array}{ccccc} \mathcal{M}_{g,1} & \hookrightarrow & \mathcal{C}_{g,1} & & \text{(monoid hom.)} \\ \cup & & \cup & & \\ f & \mapsto & (\Sigma_{g,1} \times [0,1], \mathrm{id} \times 1, f \times 0) & & \end{array}$$

Thus

$\mathcal{C}_{g,1}$ may be regarded as an enlargement of $\mathcal{M}_{g,1}$.

We will consider the actions on $\pi_1(\Sigma_{g,1})$ or the Nilpotent quotient of $\pi_1(\Sigma_{g,1})$ through $i_+^{-1} \circ i_-$.

Dehn-Nielsen-Zieschang Theorem



$\pi := \pi_1(\Sigma_{g,1}, p) = \langle \gamma_1, \gamma_2, \ldots, \gamma_{2g} \rangle$: free group

Theorem [Dehn-Nielsen-Zieschang].

$\mathcal{M}_{g,1} \cong \mathrm{Aut}(\pi, \zeta) = \{\varphi \in \mathrm{Aut}\,(\pi) \mid \zeta = \Pi_{i=1}^{g}[\gamma_{2i-1}, \gamma_{2i}], \varphi(\zeta) = \zeta\}$

The '0th' Johnson homo.

$H := H_1(\Sigma_{g,1};\mathbb{Z}) \cong \mathbb{Z}^{2g}$ with basis $[\gamma_1], [\gamma_2], \ldots, [\gamma_{2g}]$.
$\omega : H \otimes H \to \mathbb{Z}$: intersection pairing

A basis of H is symplectic if it is of the form

- $\omega([\gamma_i], [\gamma_i]) = 0$, and
- $\omega([\gamma_{2i-1}], [\gamma_{2i}]) = -\omega([\gamma_{2i}], [\gamma_{2i-1}]) = 1$.

An automorphism of H is symplectic if it preserves the form; these form a group which we denote by $Sp(2g,\mathbb{Z})$.

Fact. There exists the exact sequence :

$$1 \to \mathcal{T}_{g,1} \to \mathcal{M}_{g,1} \xrightarrow{\sigma} Sp(2g,\mathbb{Z}) \to 1$$

$\mathcal{T}_{g,1} = \mathrm{Ker}\,\sigma$ is called Torelli group.

Johnson's first work

Johnson exploit the action of $\mathcal{M}_{g,1}$ on π to get an abelian quotient of $\mathcal{T}_{g,1}$.

Consider the exact sequence :

$$1 \to [\pi,\pi]/[\pi,[\pi,\pi]] \to \pi/[\pi,[\pi,\pi]] \to \pi/[\pi,\pi] = H \to 1$$

Lemma [Johnson].

(1) For $f \in \mathcal{T}_{g,1}$, $x \in H$ and $e(\in \pi/[\pi,[\pi,\pi]])$ a lifting of x, the element $f(e)\cdot e^{-1}$ is in $[\pi,\pi]/[\pi,[\pi,\pi]]$ and independent of the lifting.

(2) If we put $\tau f(x) = f(e)\cdot e^{-1} \in [\pi,\pi]/[\pi,[\pi,\pi]]$, we have $\tau f : H \to [\pi,\pi]/[\pi,[\pi,\pi]]$ is a homomorphism.

(3) (folklore) $[\pi,\pi]/[\pi,[\pi,\pi]] \cong \Lambda^2 H$.

Johnson's first work

Theorem [Johnson].

(1) The function $\tau : \mathcal{T}_{g,1} \to \mathrm{Hom}(H, \Lambda^2 H)$ is a homomorphism.

(2) $\mathrm{Image}\,\tau = \Lambda^3 H \quad (\subset \Lambda^2 H \otimes H \cong \mathrm{Hom}(H, \Lambda^2 H))$

We may generalize this situation using the lower central series:

$$\pi = \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \cdots ,$$

where $\Gamma_{i+1} = [\pi, \Gamma_i] \quad (i = 1, 2, \ldots)$. Then, we have:

$$1 \to \Gamma_k/\Gamma_{k+1} \to \Gamma_1/\Gamma_{k+1} \to \Gamma_1/\Gamma_k \to 1.$$

$$(1 \to \Gamma_2/\Gamma_3 \to \Gamma_1/\Gamma_3 \to \Gamma_1/\Gamma_2 = H \to 1)$$

Johnson-Morita homo.

Note. $\mathcal{M}_{g,1}$ acts on Γ_i since $\mathcal{M}_{g,1} \subset \mathrm{Aut}(\pi)$.
Then, we have a filtration:

$$\mathcal{M}_{g,1} = \mathcal{M}_{g,1}[1] \supset \mathcal{M}_{g,1}[2] \supset \mathcal{M}_{g,1}[3] \supset \cdots ,$$

where $\mathcal{M}_{g,1}[k] := \mathrm{Ker}(\mathcal{M}_{g,1} \xrightarrow{\sigma_k} \mathrm{Aut}(\Gamma_1/\Gamma_k))$.

$$(\mathcal{M}_{g,1}[2] = \mathrm{Ker}(\mathcal{M}_{g,1} \xrightarrow{\sigma_2 = \sigma} \mathrm{Aut}(\Gamma_1/\Gamma_2)) = \mathcal{T}_{g,1}.)$$

By the similar argument to $\tau : \mathcal{T}_{g,1} \to \mathrm{Hom}(H, \Lambda^2 H)$, we have the k-th Johnson-Morita homomorphism:

$$\tau_k : \mathcal{M}_{g,1}[k+1] \to \mathrm{Hom}(H, \mathcal{L}_{k+1})$$

with $\mathrm{Ker}\, \tau_k = \mathcal{M}_{g,1}[k+2]$.

Johnson-Morita homo.

$\mathcal{L}_i$: the degree i part of the free Lie algebra gene. by H.
Fact.

$$\begin{array}{ccc} \Gamma_i/\Gamma_{i+1} & \xrightarrow{\cong} & \mathcal{L}_i \\ \Cup & & \Cup \\ [\alpha_1,[\alpha_2,\ldots,\alpha_i]]\cdots] & \mapsto & [a_1,[a_2,\ldots,a_i]\cdots] \\ (\alpha_i \in \pi) & & (a_i := [\alpha_i] \in H) \end{array}$$

Example.
Case $i = 1$

$$\Gamma_1/\Gamma_2 = \pi/[\pi,\pi] \xrightarrow{\cong} \mathcal{L}_1 = H.$$

Johnson-Morita homo.

Case $i = 2$

$$\Gamma_2/\Gamma_3 = [\pi, \pi]/[\pi, [\pi, \pi]] \xrightarrow{\cong} \mathcal{L}_2 = \Lambda^2 H.$$

$$[a, b] = -[b, a]$$

Case $i = 3$

$$\Gamma_3/\Gamma_4 \xrightarrow{\cong} \mathcal{L}_3 = H \otimes \Lambda^2 H/\text{Jacobi identity}.$$

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

Theorems of Morita and Yokomizo

$$\mathcal{M}_{g,1} \xrightarrow{\tau_k} \text{Hom}(H, \mathcal{L}_{k+1}) \cong H^* \otimes \mathcal{L}_{k+1} \cong H \otimes \mathcal{L}_{k+1}$$

Set $\mathfrak{H}(k) = \mathrm{Ker}(H \otimes \mathcal{L}_{k+1} \xrightarrow{[\ ,\ ]} \mathcal{L}_{k+2})$.
Note. $\tau_1 : \mathcal{T}_{g,1} \to \mathfrak{H}(1) = \Lambda^3 H$: surjective (Johnson)

Theorem [Morita].

(1) $\mathrm{Image}\,\tau_k \subset \mathfrak{H}(k)$.

(2) $\tau_2(\mathcal{M}_{g,1}[3]) \subsetneqq \mathfrak{H}(2)$ and $\mathfrak{H}(2)/\tau_2(\mathcal{M}_{g,1}[3])$ is a 2-torsion group.

Theorem [Yokomizo].
$\mathfrak{H}(2)/\tau_2(\mathcal{M}_{g,1}[3]) \cong \mathbb{Z}_2^{(g-1)(2g+1)}$ with an explicit basis.

Results of Garoufalidis-Levine

By the similar method to the case of the mapping class group, we have:

Theorem [Garoufalidis-Levine].

(1) We have a similar filtration to $\mathcal{M}_{g,1}$:

$$\mathcal{C}_{g,1} = \mathcal{C}_{g,1}[1] \supset \mathcal{C}_{g,1}[2] \supset \cdots$$

(2) We have a homomorphism (Johnson-Morita hom. for HC)

$$\tau_k : \mathcal{C}_{g,1}[k+1] \longrightarrow \mathfrak{H}(k).$$

(3) τ_k is surjective for all k.

Remark. $\mathfrak{H}(2)/\tau_2(\mathcal{M}_{g,1}[3])$ is a product obstruction.

Computations

Recipe

(1) Find a minimal genus Seifert surface S and obtain $(M_s, i_+, i_-) =: M (\in \mathcal{C}_{g,1})$. (by hand)

(2) Calculate an 'admissible' presentation of $\pi_1(M)$. (by hand)

(3) Find $f \in \mathcal{M}_{g,1}$ such that $M \cdot F \in \mathcal{C}_{g,1}[2]$ $(F = (S \times [0,1], id \times 1, f^{-1} \times 0)$. (by hand and Computer program 'Teruaki')

(4) Find $g \in \mathcal{T}_{g,1}$ such that $M \cdot F \cdot G \in \mathcal{C}_{g,1}[3]$ $(G = (S \times [0,1], id \times 1, g^{-1} \times 0)$. (by hand and 'Teruaki')

(5) Compute $\tau_2(M \cdot F \cdot G)$ in $\mathfrak{H}(2)/\operatorname{Image}\tau_2$. (Mathematica)

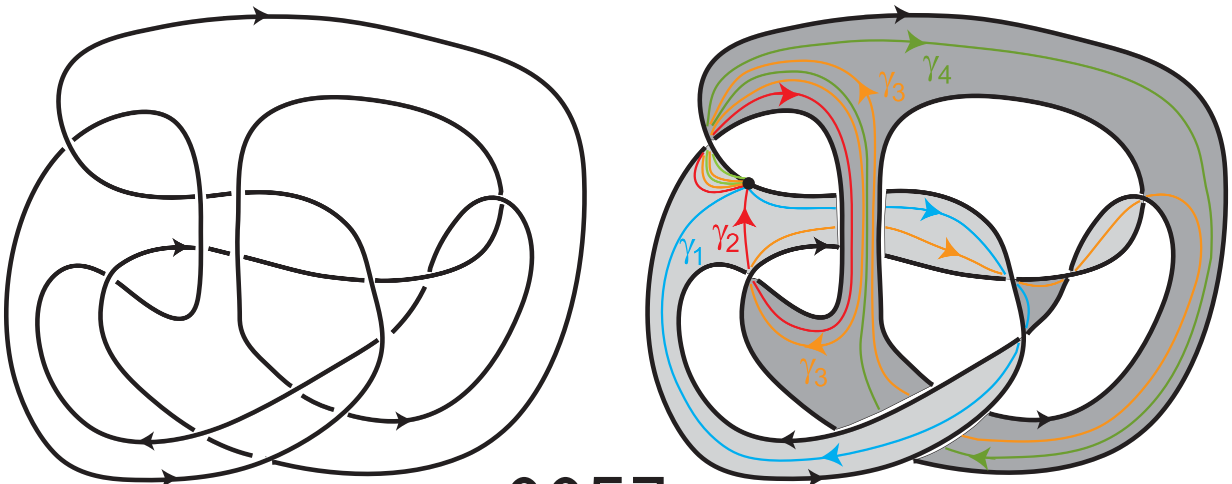
Computations

Note. Suppose $f_1 \in \mathcal{M}_{g,1},\ g_1 \in \mathcal{T}_{g,1}$ be another choice, and set $(F_1 = (S \times [0,1], id \times 1, f_1^{-1} \times 0)$ and
$(G_1 = (S \times [0,1], id \times 1, g_1^{-1} \times 0)$, then

$$\begin{aligned}\tau_2(M \cdot F_1 \cdot G_1) &= \tau_2(M \cdot F \cdot G \cdot G^{-1} \cdot F^{-1} \cdot F_1 \cdot G_1) \\ &= \tau_2(M \cdot F \cdot G) + \tau_2(G^{-1} \cdot F^{-1} \cdot F_1 \cdot G_1) \\ &= \tau_2(M \cdot F \cdot G) \qquad \in \mathfrak{H}(2)/\operatorname{Image} \tau_2(\mathcal{M}_{g,1}[3])\end{aligned}$$

$$\begin{pmatrix} \tau_2(G^{-1} \cdot F^{-1} \cdot F_1 \cdot G_1) \in \tau_2(\mathcal{M}_{g,1}[3]) \\ F_1^{-1} = (S \times [0,1], id \times 1, f_1 \times 0) \\ G_1^{-1} = (S \times [0,1], id \times 1, g_1 \times 0) \end{pmatrix}$$

Example



0057

Generators $\quad i_-(\gamma_1), \ldots, i_-(\gamma_4),\ z_1, \ldots, z_{10},\ i_+(\gamma_1), \ldots, i_+(\gamma_4)$

Relations

$$z_1 z_5 z_6^{-1},\ z_2 z_3 z_4 z_1,\ z_3 z_9^{-1} z_5^{-1},\ z_7 z_4 z_8^{-1},\ z_8 z_{10} z_6,$$
$$z_2 z_5 z_7^{-1} z_5^{-1},\ z_9 z_4 z_{10}^{-1} z_4^{-1},\ i_-(\gamma_1) z_1^{-1} z_5^{-1},\ i_-(\gamma_2) z_2,$$
$$i_-(\gamma_3) z_4 z_8 z_7 z_5^{-1},\ i_-(\gamma_4) z_4,\ i_+(\gamma_1) z_5^{-1},\ i_+(\gamma_2) z_9^{-1} z_6^{-1},$$
$$i_+(\gamma_3) z_6 z_4 z_7 z_5^{-1} z_3^{-1} z_5 z_6^{-1},\ i_+(\gamma_4) z_6 z_7^{-1} z_6^{-1}$$

Example

$\mathrm{Image}\,\tau_2$ of (M_S, i_+, i_-) for 0057 (in $\mathrm{Hom}(H, \mathcal{L}_3)$).

$$\begin{aligned}
\gamma_1 \mapsto & [\gamma_4,[\gamma_1,\gamma_4]] + [\gamma_2,[\gamma_1,\gamma_4]] - [\gamma_1,[\gamma_3,\gamma_4]] - 2[\gamma_1,[\gamma_2,\gamma_4]] \\
& -[\gamma_1,[\gamma_1,\gamma_4]] - 2[\gamma_1,[\gamma_1,\gamma_2]], \\
\gamma_2 \mapsto & [\gamma_4,[\gamma_2,\gamma_4]] - [\gamma_4,[\gamma_1,\gamma_4]] + [\gamma_3,[\gamma_1,\gamma_4]] - [\gamma_2,[\gamma_3,\gamma_4]] \\
& -[\gamma_2,[\gamma_2,\gamma_4]] - 2[\gamma_2,[\gamma_1,\gamma_4]] - 2[\gamma_2,[\gamma_1,\gamma_2]] + [\gamma_1,[\gamma_2,\gamma_4]] \\
\gamma_3 \mapsto & 4[\gamma_4,[\gamma_2,\gamma_4]] + [\gamma_3,[\gamma_1,\gamma_4]] - [\gamma_2,[\gamma_1,\gamma_4]] - [\gamma_2,[\gamma_1,\gamma_3]] \\
& -[\gamma_2,[\gamma_1,\gamma_2]] - [\gamma_1,[\gamma_2,\gamma_4]] + [\gamma_1,[\gamma_2,\gamma_3]] - [\gamma_1,[\gamma_1,\gamma_4]] \\
& -[\gamma_1,[\gamma_1,\gamma_2]] \\
\gamma_4 \mapsto & -4[\gamma_4,[\gamma_2,\gamma_4]] + 3[\gamma_3,[\gamma_1,\gamma_4]] - [\gamma_2,[\gamma_1,\gamma_4]] + [\gamma_1,[\gamma_2,\gamma_4]]
\end{aligned}$$

The red parts survive in $\mathfrak{H}(2)/\,\mathrm{Image}\,\tau_2$. (This means the knot 0057 is not fibered.)

Results

By this procedure, we can detect the non-fiberedness of : 0057, 0210, 0214, 0258, 0382, 0394, 0464, 0535, 0650, 0815.

However, we cannot detect the non-fiberedness of : 0279, 0483, 0801.

$\rightsquigarrow$ Next obstruction using the 3rd Johnson-Morita homo. (In progress).