Holomorphic maps between Riemann surfaces of small genera

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History

- Automorphism groups of Riemann surface

A. Hurwitz (1891) $|Aut(S_g)| \leq 84(g-1), g > 1$

F. Klein showed that upper bound is attained for genus $g = 3$

A. Wiman (1896) tried $g = 4, 5, 6$ It resulted in $< 84(g-1)$

A. M. Macbeath (1961) proved that the upper bound is attained for infinitely many values of $g = 3, 7, \ldots$

Thousands papers appeared on this subject since Hurwitz's time up to now.

M. Conder (2008) created a computer list of all finite groups acting on Riemann surfaces S_g up to genus $g = 101$

History

- Holomorphic maps of Riemann surfaces

M. de Franchis (1913) $|Hol(S_g, S_{g'})| < \infty$, $g \geq g' > 1$

Rough upper bounds for $|Hol(S_g, S_{g'})|$ were obtained by A. Howard, J. Sommese (1983) and A. Alzati, G. P. Pirola (1990)

The best estimates were given by M. Tanabe (1999) and, recently, by M. Ito and H. Yamamoto (2009)

All upper bound asymptotically looks like $(c_1 g)^{c_2 g}$ for some constants c_1 and c_2.

Main Result

The main result of this report is the following theorem.

Theorem

Let S_3 and S_2 be any two Riemann surfaces of genera 3 and 2, respectively. Then the number of holomorphic maps $|Hol(S_3, S_2)| \leq 48$. The obtained upper bound is sharp.

Auxiliary results

- Hyperelliptic Riemann surfaces

  Recall that a genus g Riemann surface is hyperelliptic if there exists an involution $\tau_g \in Aut(S_g)$, $\tau_g^2 = Id$ such that $S_g/\langle\tau_g\rangle$ is the sphere. We indentify $S_g/\langle\tau_g\rangle$ with orbifold $O_g = S^2(\underbrace{2,\ldots,2}_{2g+2})$ whose underlying space the sphere S^2 and the singular set consists of $2g+2$ points labeled by 2.
  The orbifold O_g is endowed by complex structure induced from S_g. $Aut(O_g)$ is formed by elements of $Aut(\overline{\mathbb{C}})$ leaving the set of singular points of O_g invariant. For $g > 1$ the group $Aut(O_g)$ is finite and coincides with one of polyhedral groups $A_5, \mathbb{S}_4, \mathbb{S}_3, \mathbb{D}_n$ and $\mathbb{Z}_n$.

Auxiliary results

The following results are well-known.

Proposition 1

Any Riemann surface S_2 of genus 2 is hyperelliptic.

Proposition 2 (R.D.M. Accola (1994))

Let S_3 be a two-fold unbranched covering of S_2. Then S_3 is hyperelliptic.

Corollary

Let the set of surjective holomorphic maps $Hol(S_3, S_2) \neq \emptyset$. Then S_3 is hyperelliptic.

Auxiliary results

- Automorphism groups of hyperelliptic Riemann surfaces

  The automorphism groups of genus 2 Riemann surfaces were classified by Bolza (1888) There are 8 different cases.

  The list of $Aut(S_3)$ for hyperelliptic Riemann surfaces S_3 was created by A. Kuribayashi (1966) and Magaard, Shaska, Shpectorov and Voelklein (2002). There are 11 possible cases.

  For the sake of convenience, we represent these results in the following two tables.

Table of Riemann surfaces of genus 2

$Aut\ S_2$	$Aut\ O_2$	Signature of orbifold $S_2/Aut\ S_2$	Riemann surface S_2 $y^2 =$
48	$\mathbb{S}_4$	(2,3,8)	$x(x^4-1)$
24	$\mathbb{D}_6$	(2,4,6)	x^6-1
$\mathbb{Z}_{10}$	$\mathbb{Z}_5$	(2,5,10)	x^5-1
$\mathbb{D}_6$	$\mathbb{D}_3$	$(2^3\,3)$	$(x^3-a^3)(x^3-a^{-3})$
$\mathbb{D}_4$	$\mathbb{D}_2$	$(2^3\,4)$	$x(x^2-a^2)(x^2-a^{-2})$
$\mathbb{D}_2$	$\mathbb{Z}_2$	(2^5)	$(x^2-1)(x^2-a^2)(x^2-b^2)$
$\mathbb{Z}_2$	1	(2^6)	$x(x-1)(x^3+ax^2+bx+c)$

Table of Hyperelliptic Riemann surfaces of genus 3

$Aut\ S_3$	$Aut\ O_3$	Signature of orbifold $S_3/Aut\ S_3$	Riemann surface S_3 $y^2 =$
48	$\mathbb{S}_4$	(2,4,6)	$x^8 + 14x^4 + 1$
32	$\mathbb{D}_8$	(2,4,8)	$x^8 - 1$
24	$\mathbb{D}_6$	(2,4,12)	$x(x^6 - 1)$
$\mathbb{Z}_{14}$	$\mathbb{Z}_7$	(2,7,14)	$x^7 - 1$
16	$\mathbb{D}_4$	(2,2,2,4)	$x^8 + ax^4 + 1$
$\mathbb{D}_6$	$\mathbb{S}_3$	(2,2,2,6)	$x(x^6 + ax^3 + 1)$
8	$\mathbb{D}_2$	(2,2,4,4)	$x(x^2 - 1)(x^4 + ax^2 + 1)$
$\mathbb{Z}_2^3$	$\mathbb{D}_2$	(2^5)	$(x^4 + ax^2 + 1)(x^4 + bx^2 + 1)$
$\mathbb{Z}_4$	$\mathbb{Z}_2$	(2,2,2,4,8)	$x(x^2 - 1)(x^4 + ax^2 + b)$
$\mathbb{D}_2$	$\mathbb{Z}_2$	(2^6)	$(x^2 - 1)(x^6 + ax^4 + bx^2 + c)$
$\mathbb{Z}_2$	1	(2^8)	$x(x-1)(x^5 + ax^4 + bx^3 + cx^2 + dx + e)$

Auxiliary results

- Holomorphic map of hyperelliptic Riemann surfaces

Let S_g and $S_{g'}$ be hyperelliptic Riemann surfaces and τ_g, $\tau_{g'}$ are the corresponding hyperelliptic involutions.

Denote by $W(S_g)$ the set of Weierstrass points on S_g.
Consider a surjective holomorphic map $f \in Hol(S_g, S_{g'})$.

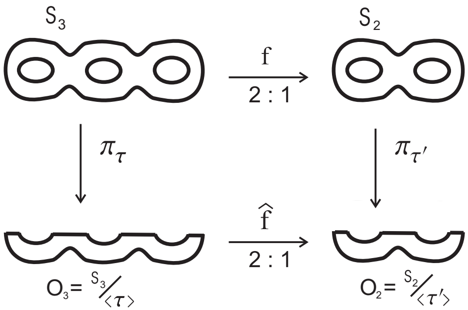
The following results are important for our consideration
(H. Martens (1988), M. Tanabe (2005))

Properties of holomorphic maps

1. $f(W(S_g) \subset W(S_{g'}))$
2. If $f|_{W(S_g)} = h|_{W(S_{g'})}$, *where* $f, h \in Hol(S_g, S_{g'})$ *then either* $f = h$ *or* $f = h \circ \tau_g$
3. $f \circ \tau_g = \tau_{g'} \circ h$

Auxiliary results

The above mentioned results makes us sure that the following diagram is commutative:



Scheme of coverings

- Equivalence relation

Two maps $f, h : S_g \to S_{g'}$ are said to be equivalent if there are automorphisms $\alpha \in Aut(S_{g'})$ and $\beta \in Aut(S_g)$ such that $h = \alpha \circ f \circ \beta$.

The following lemma gives a classification of maps from $Hol(S_3, S_2)$ up to equivalence. Denote by $\widehat{\gamma}_f$ the covering involution of two-fold map $\widehat{f} : O_3 \to O_2$.

Lemma

Two maps $f, h \in Hol(S_3, S_2)$ are equivalent $\Leftrightarrow$ $\widehat{\gamma}_f$ and $\widehat{\gamma}_h$ are conjugated in $Aut(O_3)$.

Sketch of the proof

- The main idea

  Consider a holomorphic map $f : S_3 \to S_2$. Since both S_3 and S_2 are hyperelliptic there is a projection on orbifolds $\widehat{f} : O_3 \to O_2$.
  As it was mentioned before there are 11 possible cases for the structure of $Aut(O_3)$.
  Let us consider only one case. Namely, $Aut(O_3) = \mathbb{D}_8$. The ten remained cases can be treated in a similar way.

  In this case Riemann surface S_3 is defined by equation $y^2 = x^8 - 1$.
  Eight singular points of O_3 are roots of the equation $x^8 - 1 = 0$.
  Up to conjugation in $Aut(O_3)$ we have just two possibilities for covering involution $\widehat{\gamma}_f$:

$$(i)\ \ \widehat{\gamma}_f : x \to -x \ and \ (ii)\ \ \widehat{\gamma}_f : x \to \frac{\varepsilon^2}{x},\ \varepsilon = \exp(\frac{2\pi i}{16}).$$

Sketch of the proof

Solving the equation $\widehat{f} \circ \widehat{\gamma}_f = \widehat{f}$ up to Möbius transformation we obtain

$$(i) \;\; \widehat{f}(z) = x^2 \; and \; (ii) \;\; \widehat{f}(x) = \frac{1}{2}\left(\frac{x}{\varepsilon} + \frac{\varepsilon}{x}\right).$$

We determine $f : S_3 \to S_2$ as a lifting of $\widehat{f}$.
(Two liftings are possible f and $f \circ \tau$!!!)
By straightforward calculation we have

$$(i) \; f : (x, y) \to (u, v) = (x^2, xy)$$

$$(ii) \; f : (x, y) \to (u, v) = \left(\frac{1}{2}\left(\frac{x}{\varepsilon} + \frac{\varepsilon}{x}\right), \frac{x^2 - \varepsilon^2}{8x^3\varepsilon^3} y\right)$$

Sketch of the proof

- Representation of the target surface S_2

  As the result we find the following representation for Riemann surface $S_2 = f(S_3)$.

$$(i) \quad S_2 : v^2 = u(u^4 - 1)$$

$$(ii) \quad S_2 : v^2 = (u^2 - 1)(u^4 - u^2 + 0.125)$$

Sketch of the proof

- Counting holomorphic maps

In the case (i) the surface S_2 is the Bolza curve with maximal possible group of automorphisms $|Aut(S_2)| = 48$. It could be easily verified that each automorphism of S_3 can be projected via map f to an element of $Aut(S_2)$. Hence,

$$|Hol(S_3,\, S_2)| = |\{\alpha \circ f :\ \alpha \in Aut(S_2)\}| = |Aut(S_2)| = 48.$$

In the case (ii) by Bolza classification we have $Aut(S_2) = \mathbb{D}_2$. One can easy check that all automorphisms of S_2 can be lifted via map f to automorphisms of S_3. Hence,

$$|Hol(S_3,\, S_2)| = |\{f \circ \beta :\ \beta \in Aut(S_3)\}| = |Aut(S_3)| = 32.$$

Conclusion

Examining the remained possible cases for $Aut(S_3)$ in a similar way we conclude that the following inequality is valid

$$|Hol(S_3,\ S_2)| \leq 48.$$

The same approach leads to the complete classification of holomorphic maps from a genus 3 Riemann surface S_3 onto a genus 2 Riemann surface S_2.

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