The volume and the Chern-Simons invariant of a PSL(2,C)-representation and quandle homology

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Introduction

M : an oriented closed 3-manifold

$\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$: a rep. of the fund. group of M

$\mathrm{Vol}(M,\rho) \in \mathbb{R}$ and $\mathrm{CS}(M,\rho) \in \mathbb{R}/\pi^2\mathbb{Z}$ are invariants of the representation ρ.

When ρ is a discrete faithful rep. of a hyperbolic mfd M, then Vol and CS are the volume and the Chern-Simons invariant of the hyperbolic metric.

The definition of Vol and CS are generalized to the case of manifolds with torus boundary e.g. knot complements.

A formula of $i(\mathrm{Vol} + i\mathrm{CS}) \in \mathbb{C}/\pi^2\mathbb{Z}$ was given by Neumann in terms of triangulations of 3-manifolds.

We give a formula in terms of knot diagrams by using the *quandle* formed by parabolic elements of $\mathrm{PSL}(2, \mathbb{C})$.

The *quandle homology* plays an important role in our description.

Quandle

The definition of quandles was introduced by Joyce in 1982.

A quandle X is a set with a binary operation $* : X \times X \to X$ satisfying

1. $x * x = x$ for any $x \in X$,
2. the map $*y : X \to X : x \mapsto x * y$ is bijective for any y,
3. $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

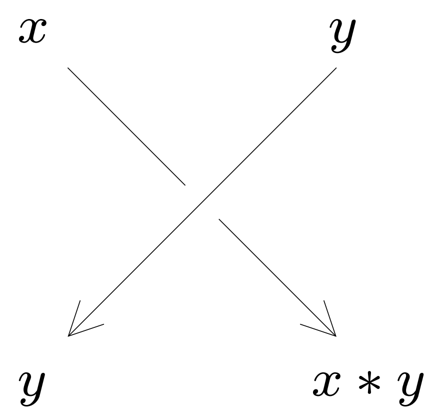
Example

G : a group, $S \subset G$: a subset closed under conjugation.
S has a quandle structure by conjugation $x * y = y^{-1}xy$.

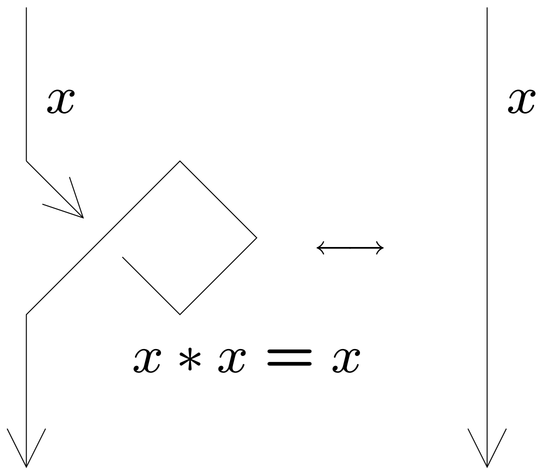
$$(x * y) * z = z^{-1}y^{-1}xyz = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz) = (x * z) * (y * z)$$

Relation with knot theory

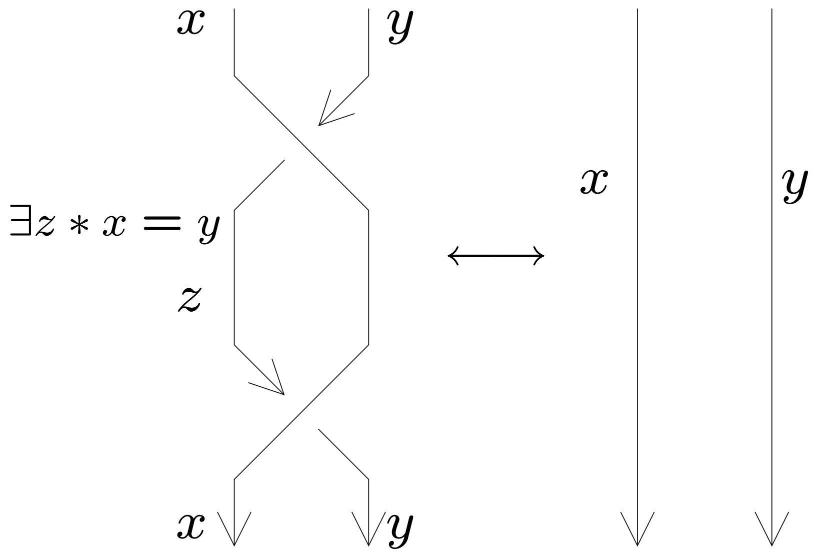
Assign an element of a quandle X for each arc of a knot diagram satisfying the following relation at each crossing. Then the axioms correspond to the Reidemeister moves:



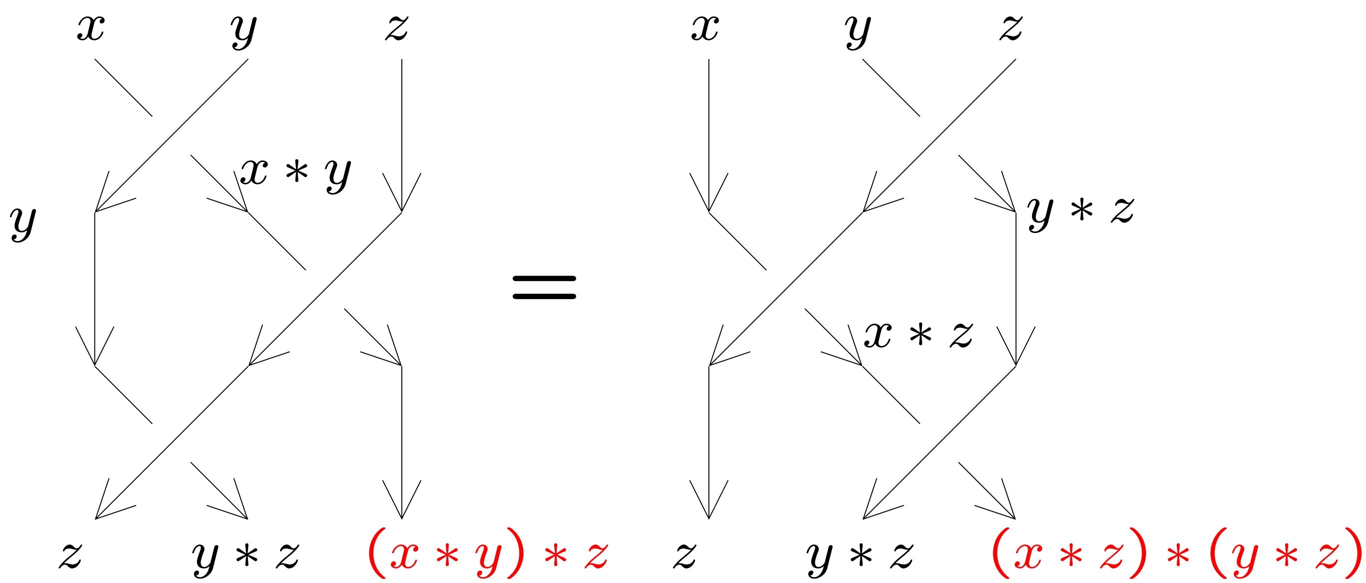
(I)



(II)



Relation with knot theory

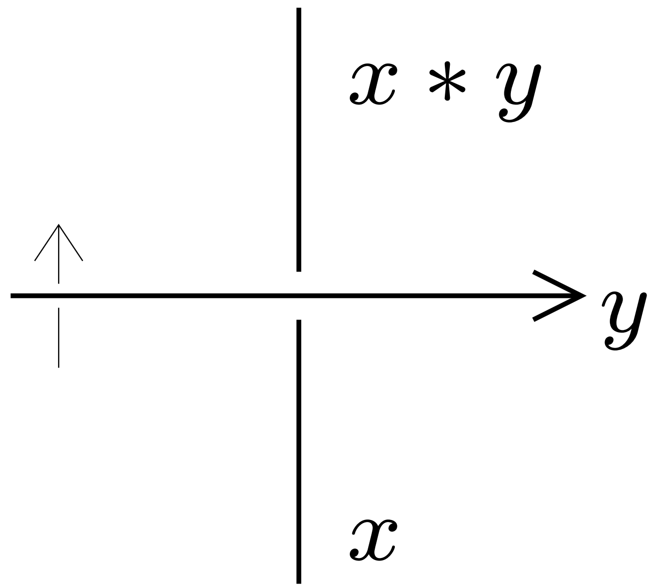


(III)

Arc coloring

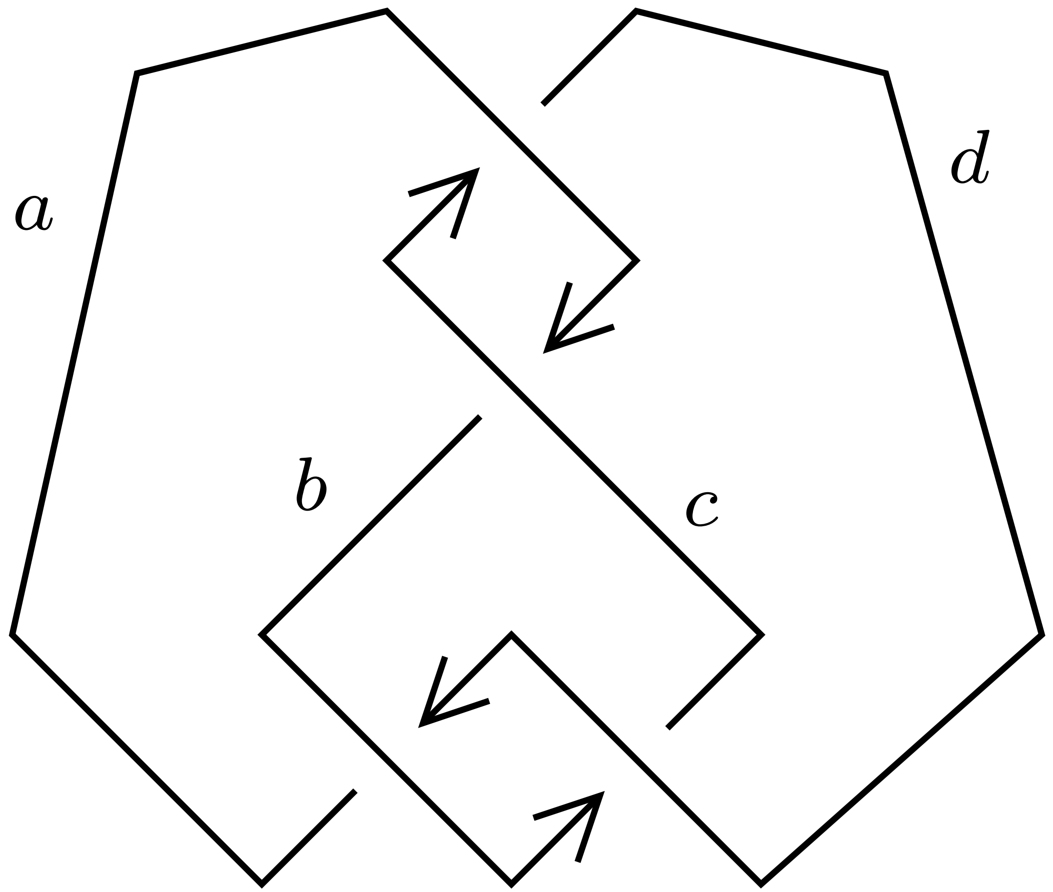
Let D be a diagram of a knot K.

We call a map $\mathcal{A} : \{\text{arcs of } D\} \to X$ *arc coloring* if it satisfies the following relation at each crossing.



x, y and $x * y \in X$

Arc coloring of the figure eight knot



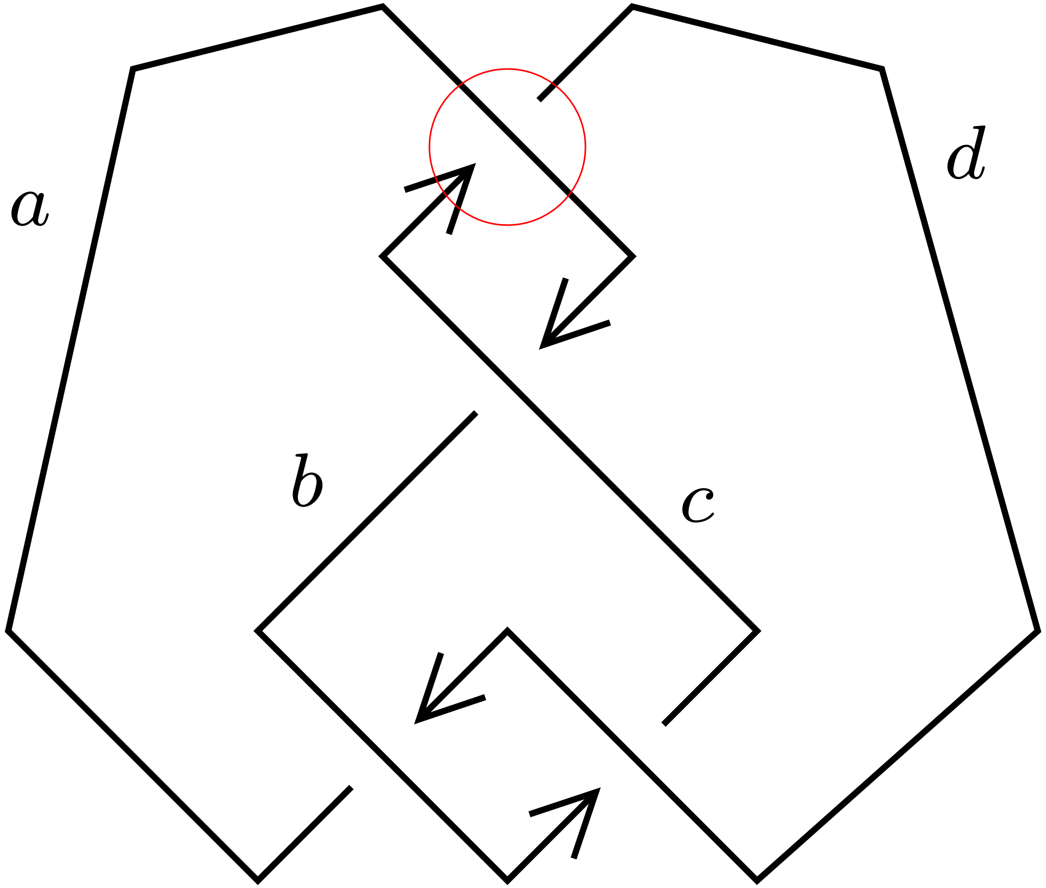
$$c * a = d,$$

$$a * c = b,$$

$$a * b = d,$$

$$c * d = b.$$

Arc coloring of the figure eight knot



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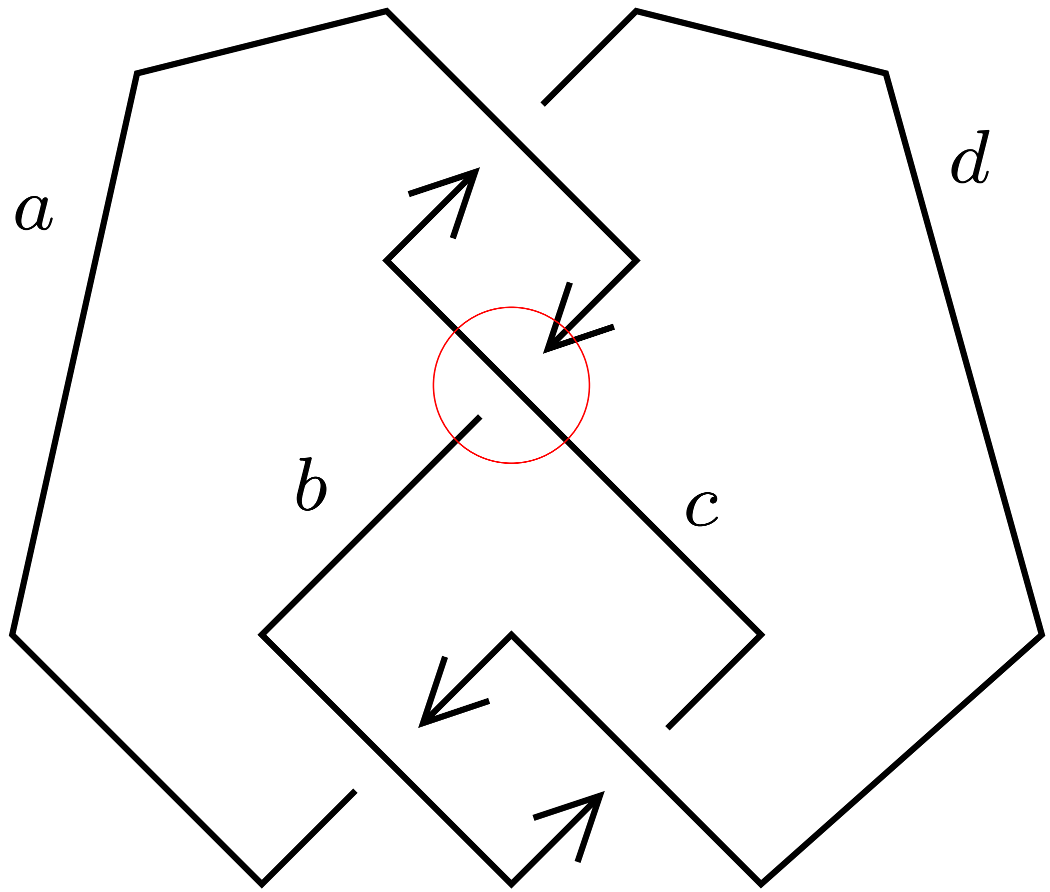

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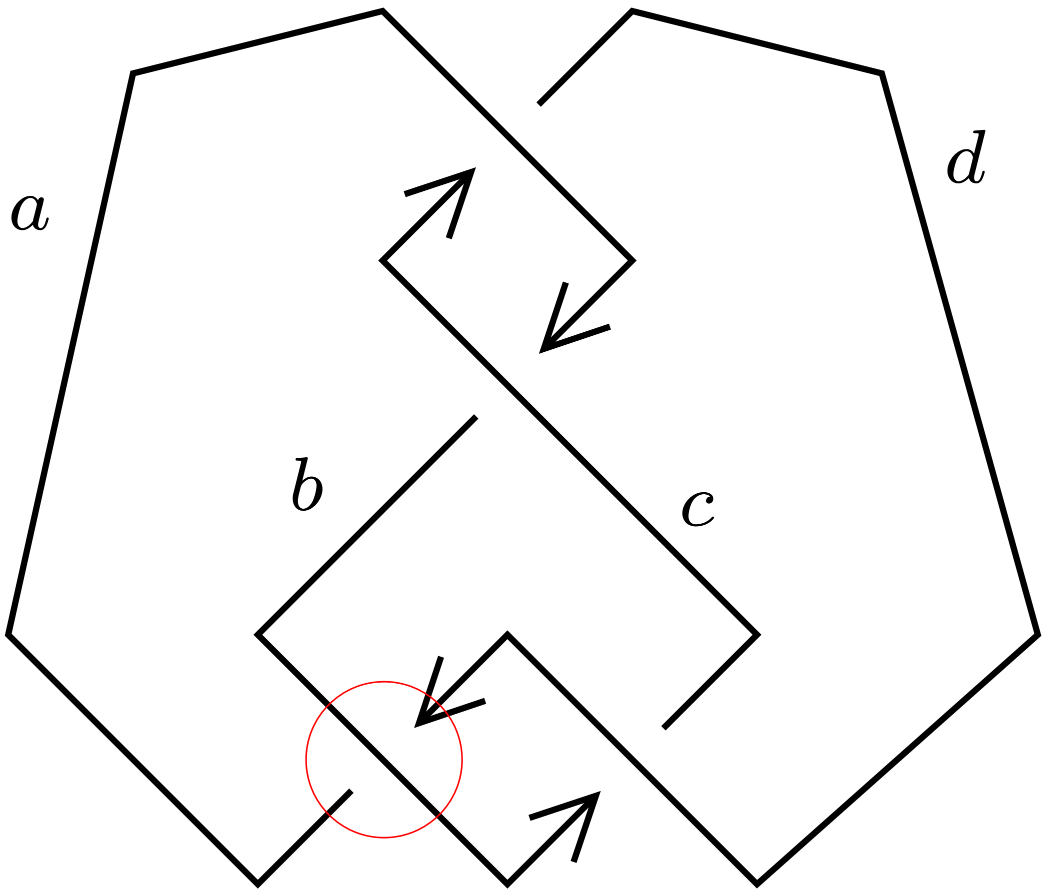

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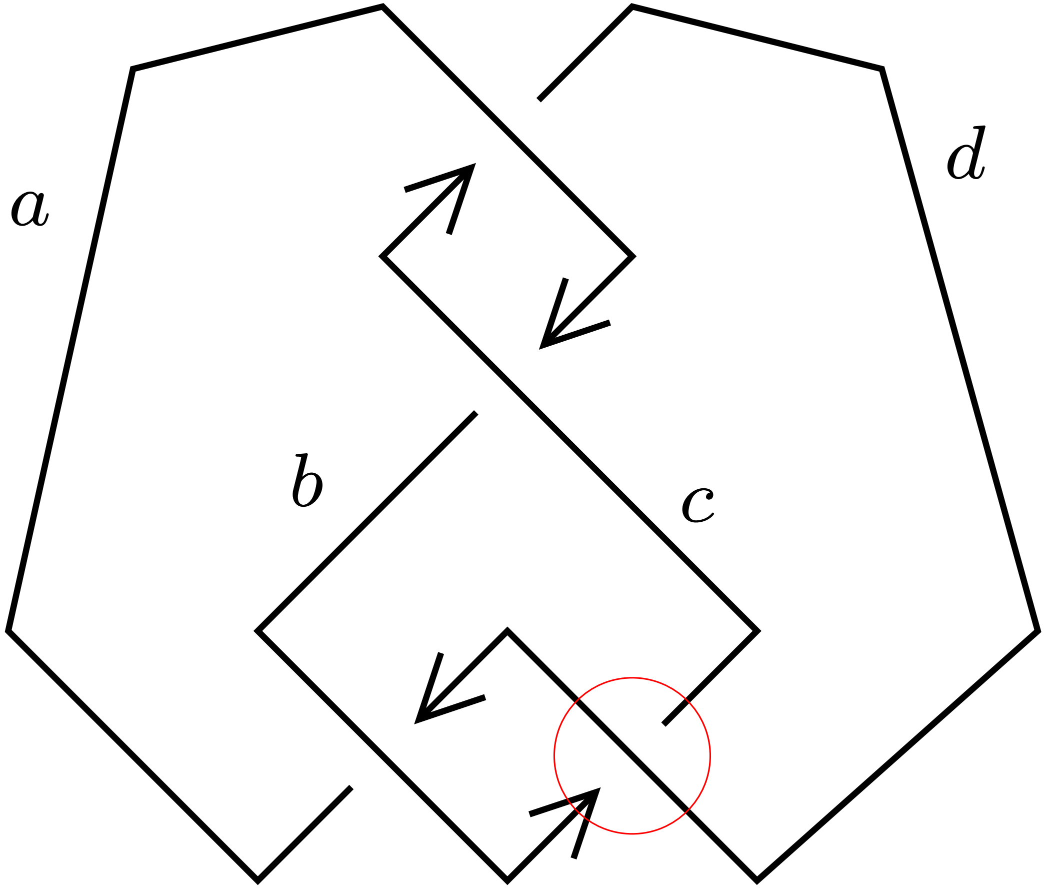
Arc coloring of the figure eight knot



$$c * a = d,$$

$$a * c = b,$$

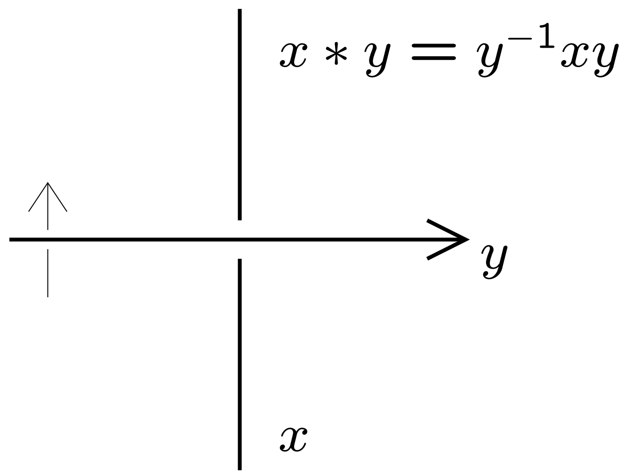
$$a * b = d,$$

$$c * d = b.$$

Associated group

For a quandle X, define the group G_X by $\langle x \in X | x * y = y^{-1}xy \rangle$. This is called the *associated group* of X.

An arc coloring by X gives a representation $\pi_1(S^3 \setminus K) \to G_X$ which sends each meridian to its color. This is a consequence of the Wirtinger presentation of a knot group.



When a quandle is given by a conjugation quandle $S \subset G$, an arc coloring by S induces a representation into G.

Quandle structure on $\mathbb{C}^2 \setminus \{0\}$

Define a binary operation $*$ on $\mathbb{C}^2 \setminus \{0\}$ by

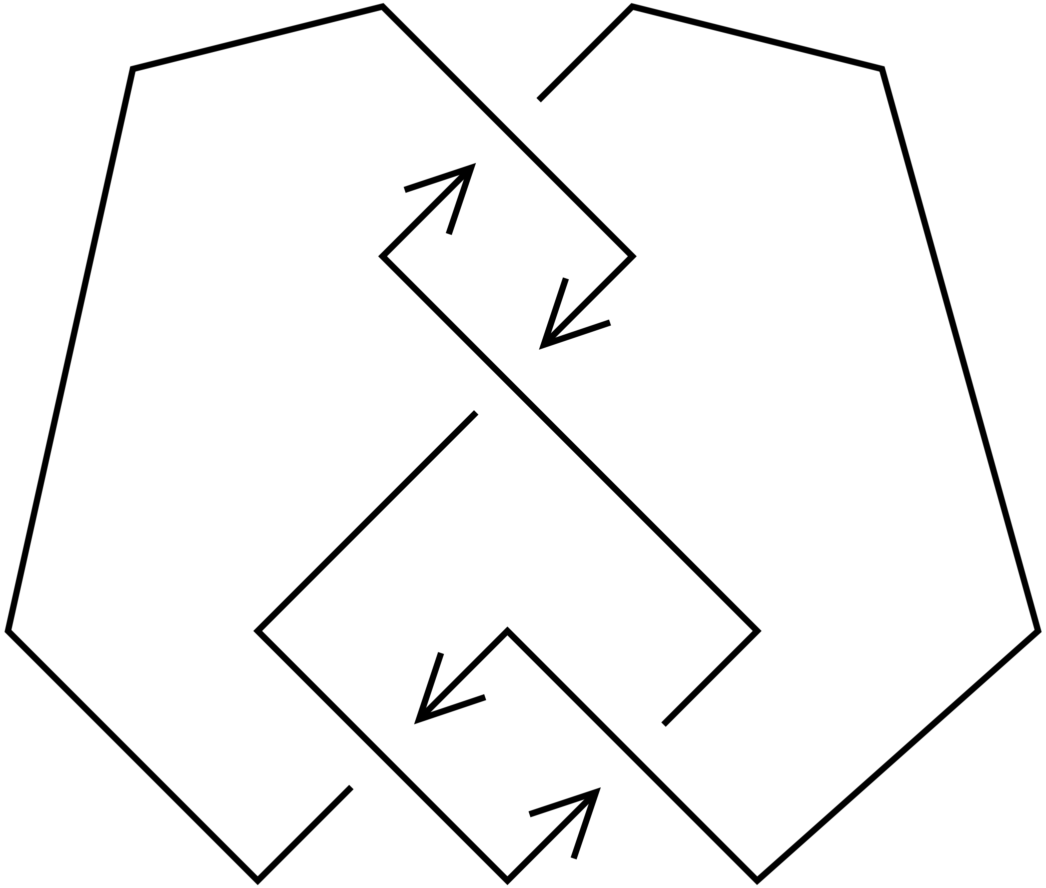
$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} * \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \begin{pmatrix} \overset{1 + x\_2\ y\_2}{\cancel{1 - x_2 y_2}} & -x_2^2 \\ y_2^2 & \underset{1- x\_2\ y\_2}{\cancel{1 + x_2 y_2}} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

This satisfies the quandle axioms. Let $\mathcal{P}$ be the quandle formed by parabolic elements of $\mathrm{PSL}(2, \mathbb{C})$. Define a map $\mathbb{C}^2 \setminus \{0\} \xrightarrow{2:1} \mathcal{P}$ by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - xy & \overset{x\text{\^{}}2}{\cancel{-x^2}} \\ \underset{-\ y\text{\^{}}2}{\cancel{y^2}} & 1 + xy \end{pmatrix}$$

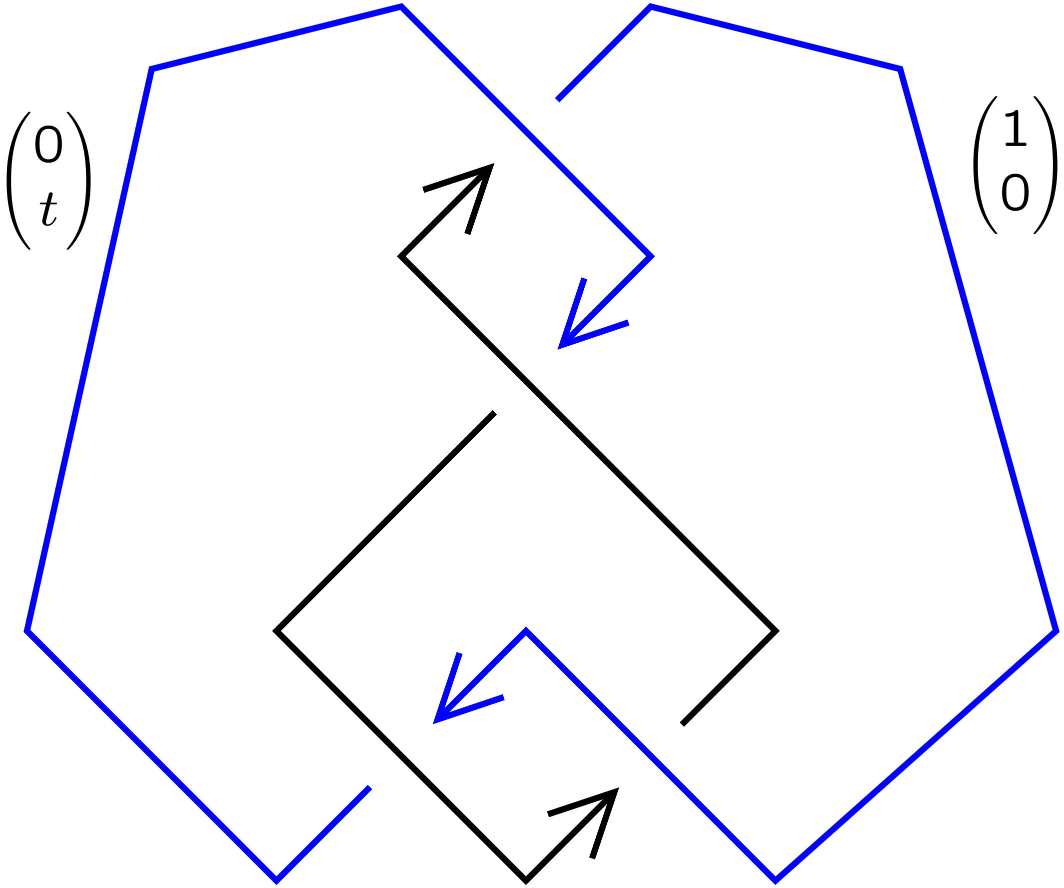
This map induces a quandle isomorphism $(\mathbb{C}^2 \setminus \{0\})/\pm \cong \mathcal{P}$.

Arc coloring of the figure eight knot by $\mathcal{P}$



This is the figure eight knot.

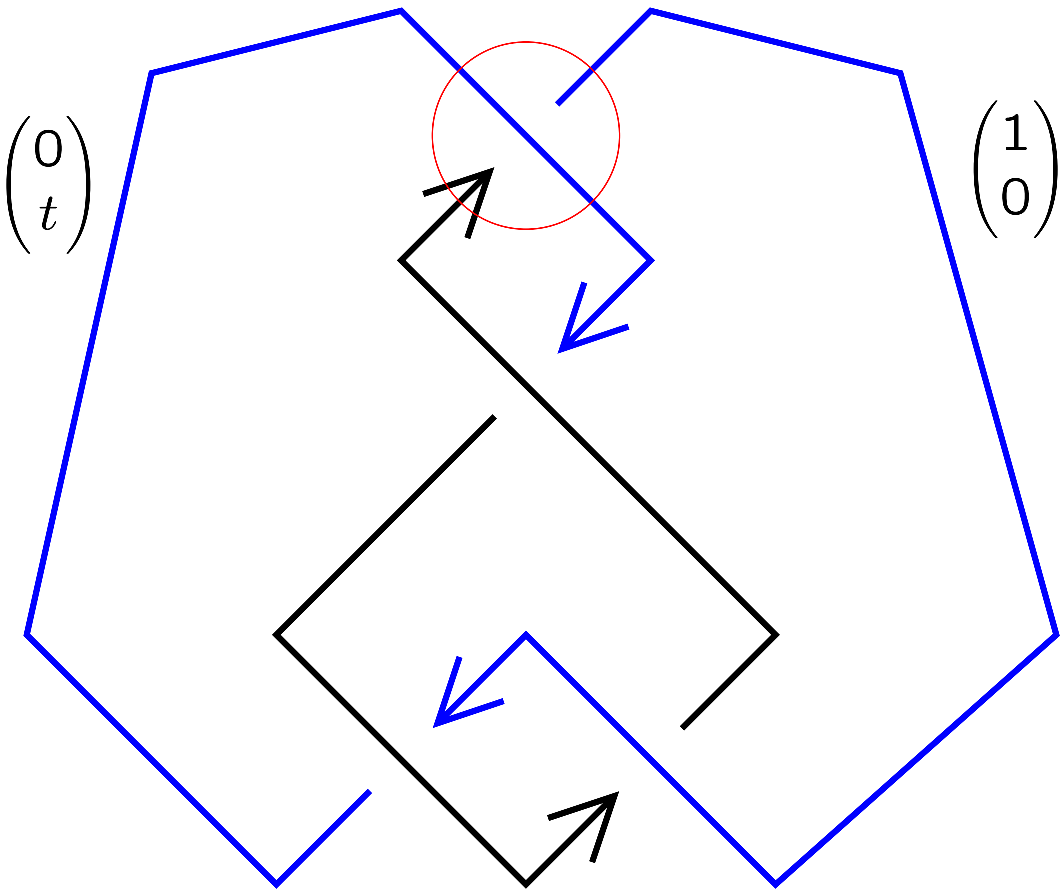
Arc coloring of the figure eight knot by $\mathcal{P}$



Color two arcs by

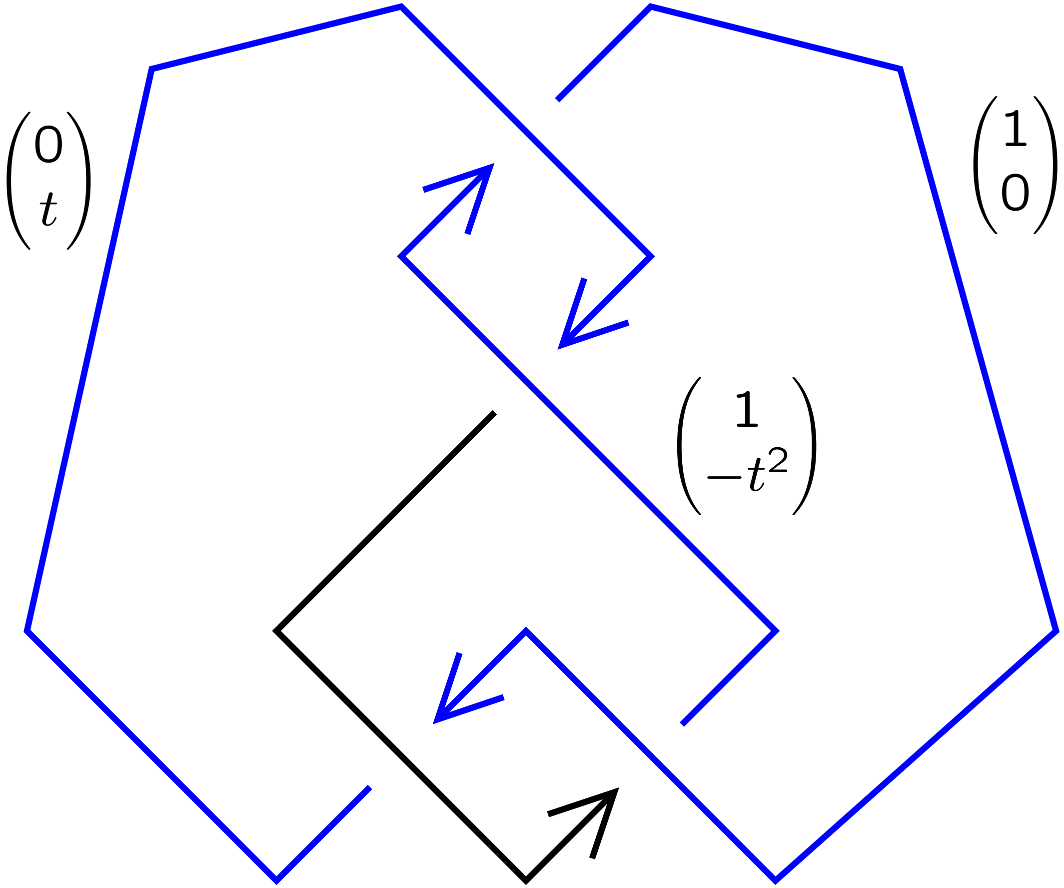
$(\mathbb{C}^2 \setminus \{0\})/\pm$.

Arc coloring of the figure eight knot by $\mathcal{P}$



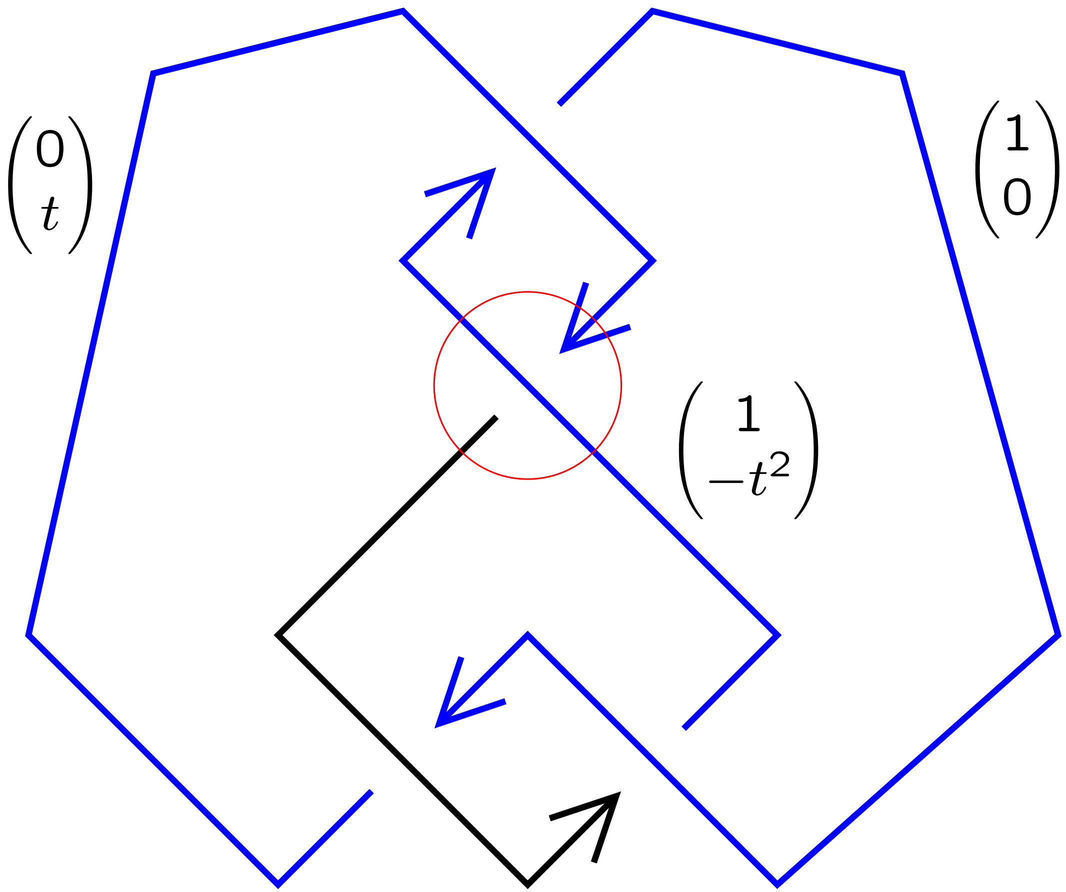
Consider the relation at this crossing.

Arc coloring of the figure eight knot by $\mathcal{P}$



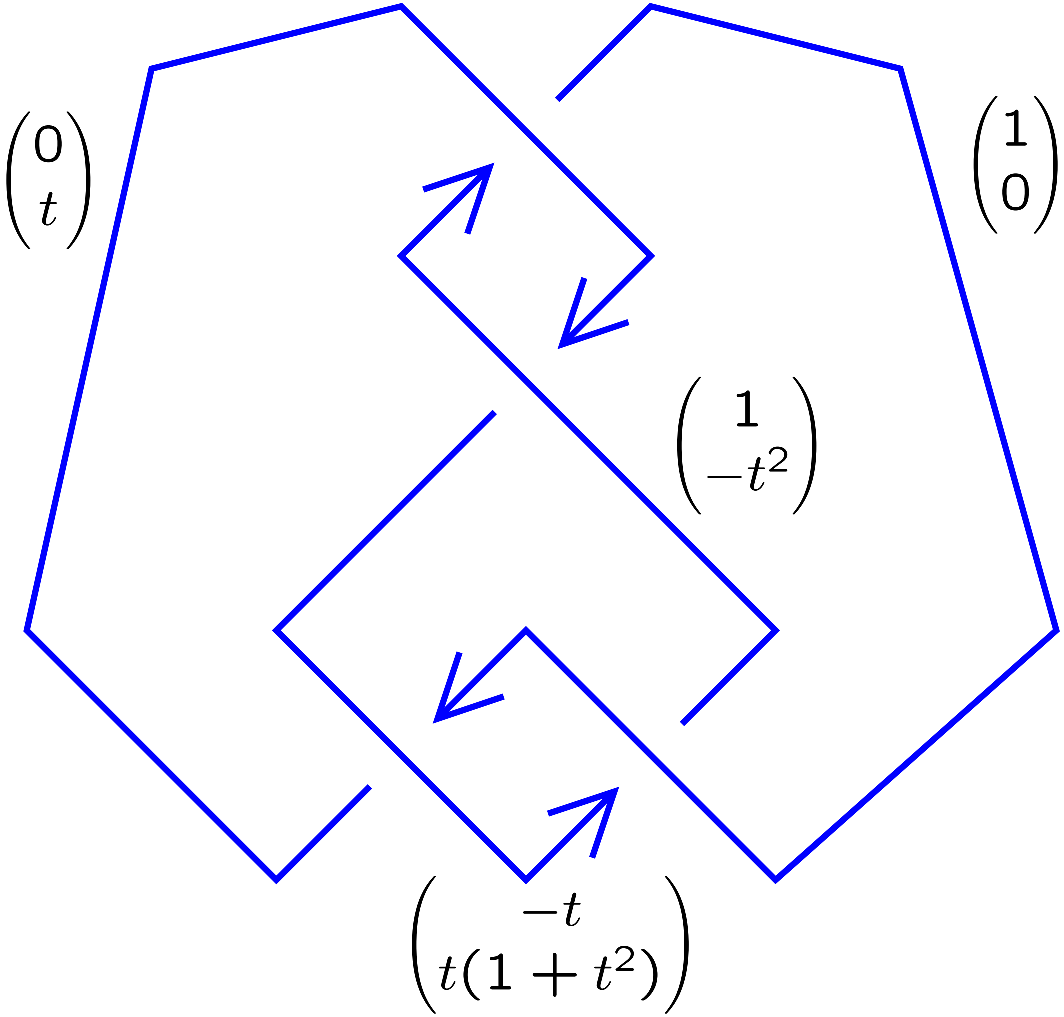
$$\begin{pmatrix}1\\0\end{pmatrix} *^{-1} \begin{pmatrix}0\\t\end{pmatrix} = \begin{pmatrix}1\\-t^2\end{pmatrix}$$

Arc coloring of the figure eight knot by $\mathcal{P}$



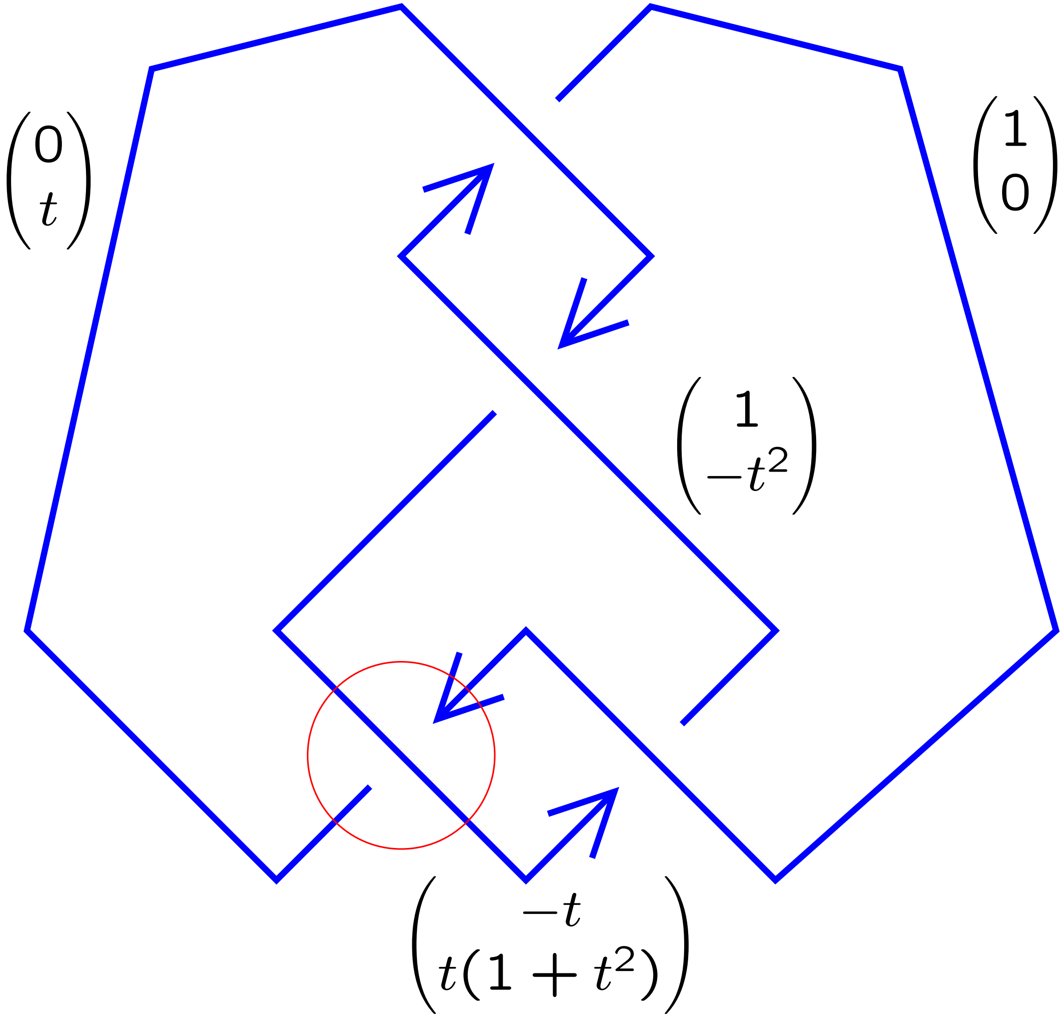
Consider the relation at this crossing.

Arc coloring of the figure eight knot by $\mathcal{P}$



$$\begin{pmatrix} 0 \\ t \end{pmatrix} * \begin{pmatrix} 1 \\ -t^2 \end{pmatrix} = \begin{pmatrix} -t \\ t(1+t^2) \end{pmatrix}$$

Arc coloring of the figure eight knot by $\mathcal{P}$



The relation at this crossing is

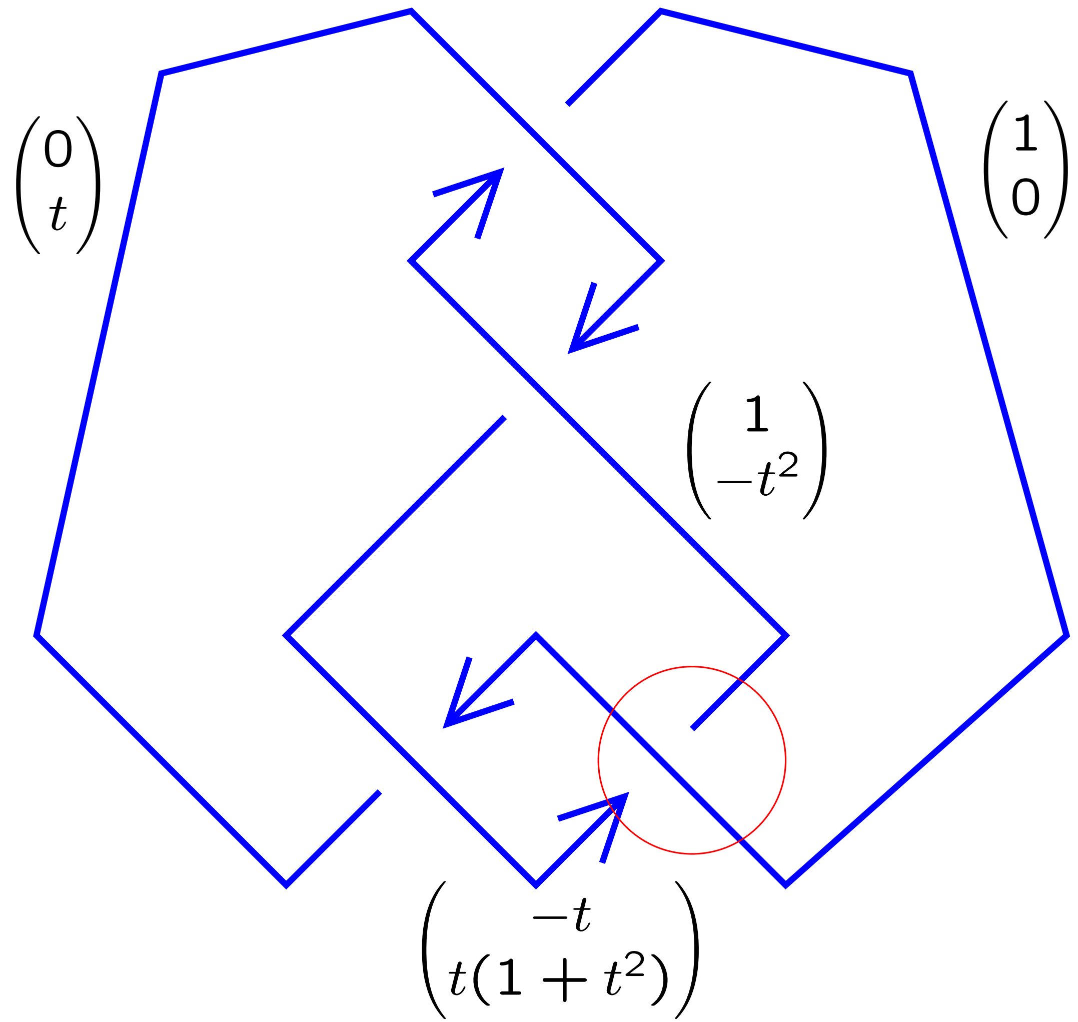
$$\left(\begin{pmatrix} 0 \\ t \end{pmatrix} * \begin{pmatrix} -t \\ t(1+t^2) \end{pmatrix} = \right)$$

$$\begin{pmatrix} -t^3 \\ t(1+t^2+t^4) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} (t+1)(t^2-t+1) = 0 \\ t(t^2+t+1)(t^2-t+1) = 0 \end{cases}$$

$$\therefore t^2 - t + 1 = 0$$

Arc coloring of the figure eight knot by $\mathcal{P}$



The relation at this crossing is

$$\left(\begin{pmatrix}1\\-t^2\end{pmatrix} * \begin{pmatrix}1\\0\end{pmatrix} =\right)$$

$$\begin{pmatrix}1+t^2\\-t^2\end{pmatrix} = \begin{pmatrix}-t\\t(1+t^2)\end{pmatrix}$$

$$\begin{cases} t^2+t+1=0 \\ t(t^2+t+1)=0 \end{cases}$$

$$\therefore t^2+t+1=0$$

Arc coloring of the figure eight knot by $\mathcal{P}$

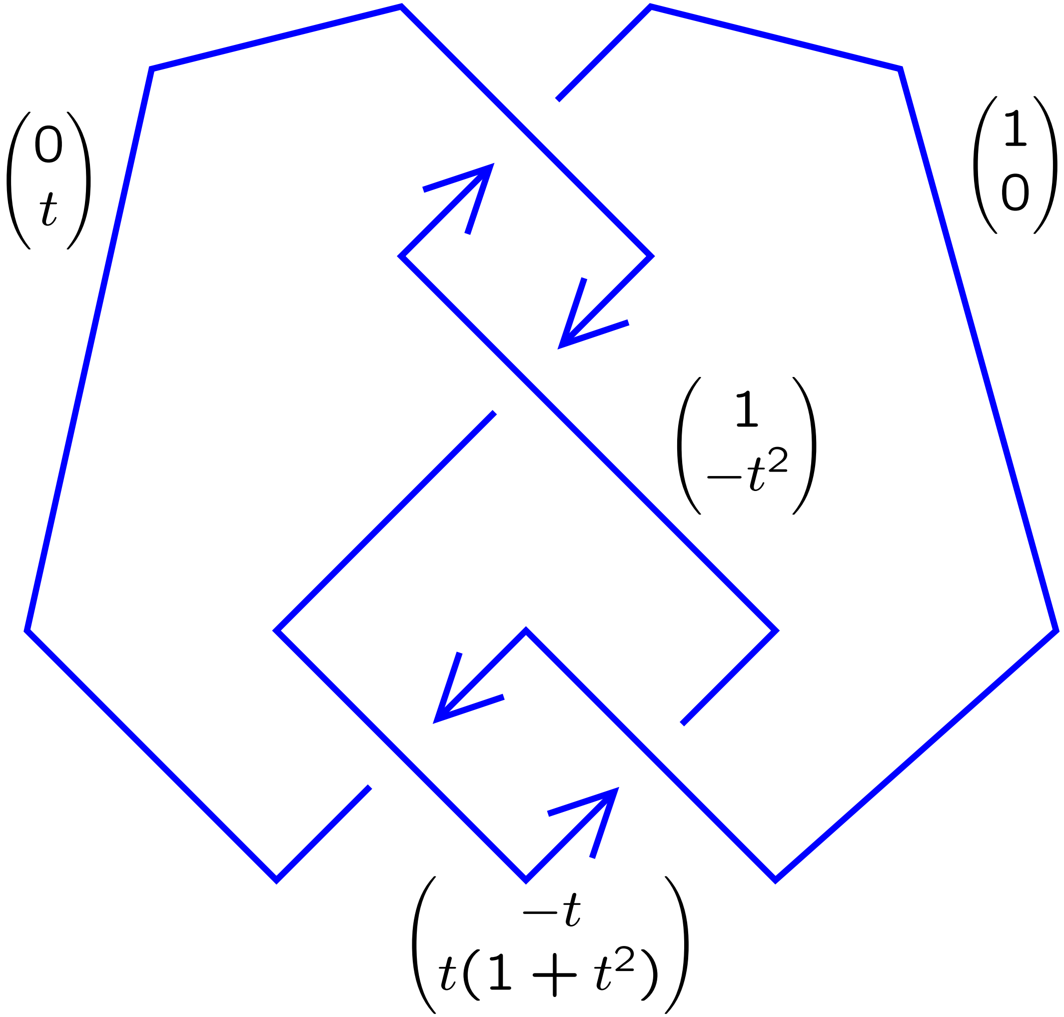
There are two relations

$$t^2 + t + 1 = 0, \quad t^2 - t + 1 = 0$$

which do not have any common solution. But we have a coloring by $(\mathbb{C}^2 \setminus \{0\})/\pm \cong \mathcal{P}$.

$$t = \pm\tfrac{1+\sqrt{3}i}{2} \text{ or } \pm\tfrac{1-\sqrt{3}i}{2}$$

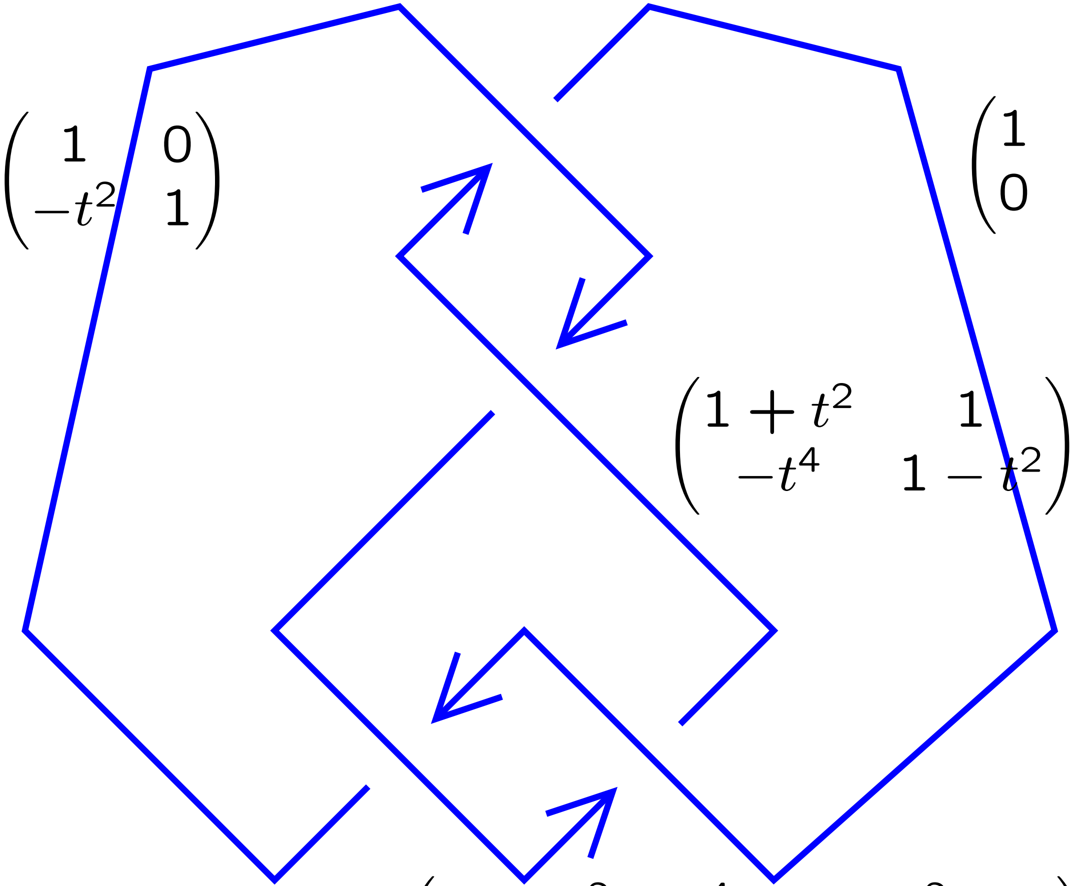
Arc coloring of the figure eight knot by $\mathcal{P}$



A parabolic representation can be obtained by the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - xy & x^2 \\ -y^2 & 1 + xy \end{pmatrix}$$

Arc coloring of the figure eight knot by $\mathcal{P}$



$$\begin{pmatrix} 1 & 0 \\ -t^2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

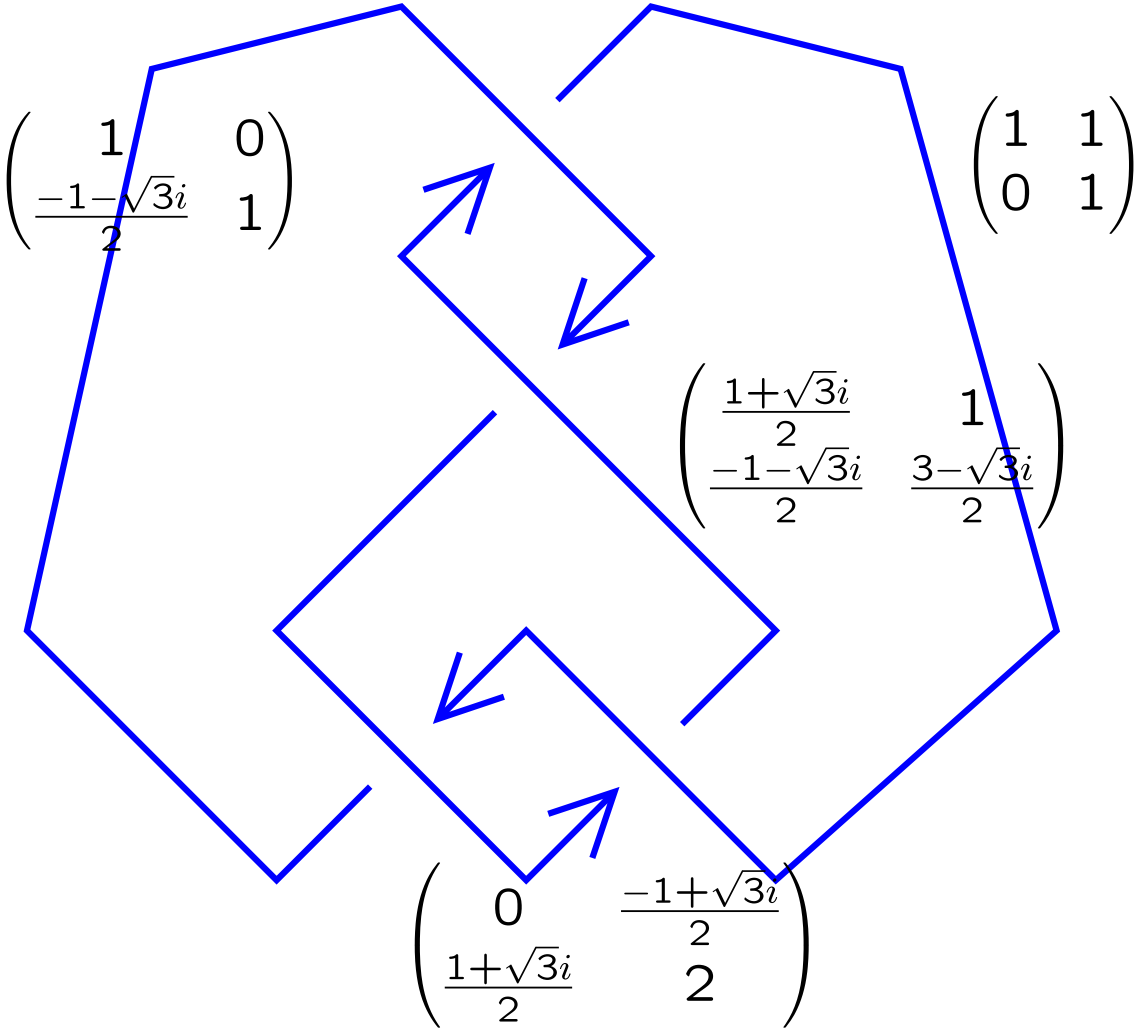
$$\begin{pmatrix} 1+t^2 & 1 \\ -t^4 & 1-t^2 \end{pmatrix}$$

$$\begin{pmatrix} 1+t^2+t^4 & t^2 \\ -t^2(1+t^2)^2 & 1-t^2-t^4 \end{pmatrix}$$

A parabolic representation can be obtained by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1-xy & x^2 \\ -y^2 & 1+xy \end{pmatrix}$$

Arc coloring of the figure eight knot by $\mathcal{P}$



Evaluate at $t^2 = \frac{-1+\sqrt{3}i}{2}$.

We obtain a discrete faithful representation of the figure eight knot complement.

As we have seen, an arc coloring by $\mathcal{P}$ gives a representation $\pi_1(S^3 \setminus K) \to \mathrm{PSL}(2,\mathbb{C})$ which sends each meridian to the corresponding parabolic element of $\mathrm{PSL}(2,\mathbb{C})$.

We call such a representation *parabolic representation*. E.g. a discrete faithful representation of a hyperbolic knot complement.

From now on, we construct an invariant for parabolic representations with values in *quandle homology*, then give a description of the volume and the Chern-Simons invariant.

Outline

1.

$\rho : \pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{C})$
parabolic representations

$\overset{1:1}{\longleftrightarrow}$

Arc colorings $\mathcal{A}$
by the quandle $\mathcal{P}$

2. Define a shadow coloring $\mathcal{S}$ and construct an invariant $[C(\mathcal{S})]$ with values in the *quandle homology* $H_2^Q(\mathcal{P}; \mathbb{Z}[\mathcal{P}])$.

3.

Quandle homology	general theory $\downarrow$	Simplicial quandle homology	Dupont -Zickert $\downarrow$	Extended Bloch group

$$
\begin{array}{ccccc}
H_2^Q(\mathcal{P}; \mathbb{Z}[\mathcal{P}]) & \xrightarrow{\varphi_*} & H_3^{\Delta}(\mathcal{P}) & \to & \widehat{\mathcal{B}}(\mathbb{C}) \\
\Cup & & & & R \downarrow \text{ Neumann} \\
[C(\mathcal{S})] & & & & \mathbb{C}/\pi^2\mathbb{Z} \\
& & & & i(\mathrm{Vol} + i\mathrm{CS})
\end{array}
$$

Quandle homology (Carter-Jelsovsky-Kamada-Langford-Saito, 2003)

Let $C_n^R(X) = \text{span}_{\mathbb{Z}[G_X]}\{(x_1, \ldots, x_n) | x_i \in X\}$. Define the boundary operator $\partial : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\begin{aligned}\partial(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i \{ &(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \\ &- x_i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\}\end{aligned}$$

Let M be a right $\mathbb{Z}[G_X]$-module. The homology group of $M \otimes_{\mathbb{Z}[G_X]} C_n^R(X)$ is called the *rack homology* $H_n^R(X; M)$.

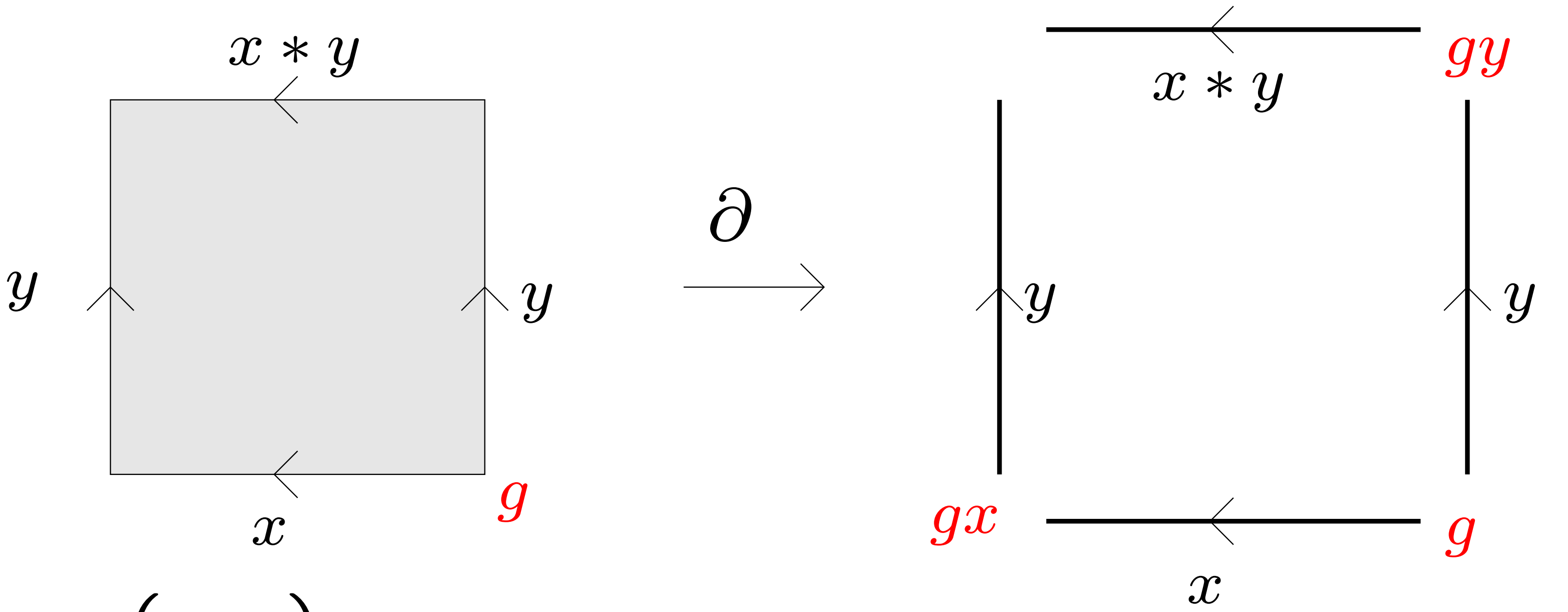
Factoring degenerate chains, we also define the quandle homology $H_n^Q(X;M)$.

Let

$$C_n^D(X) = \mathrm{span}_{\mathbb{Z}[G_X]}\{(x_1,\ldots,x_n) | x_i \in X,$$
$$x_i = x_{i+1} (\text{for some } i)\}.$$

This is a subcomplex of $C_n^R(X)$. Let $C_n^Q(X)$ be the quotient $C_n^R(X)/C_n^D(X)$. The homology of $M \otimes_{\mathbb{Z}[G_X]} C_n^Q(X)$ is called the *quandle homology* $H_n^Q(X;M)$

Geometric interpretation $\quad C_2^R(X) \to C_1^R(X)$

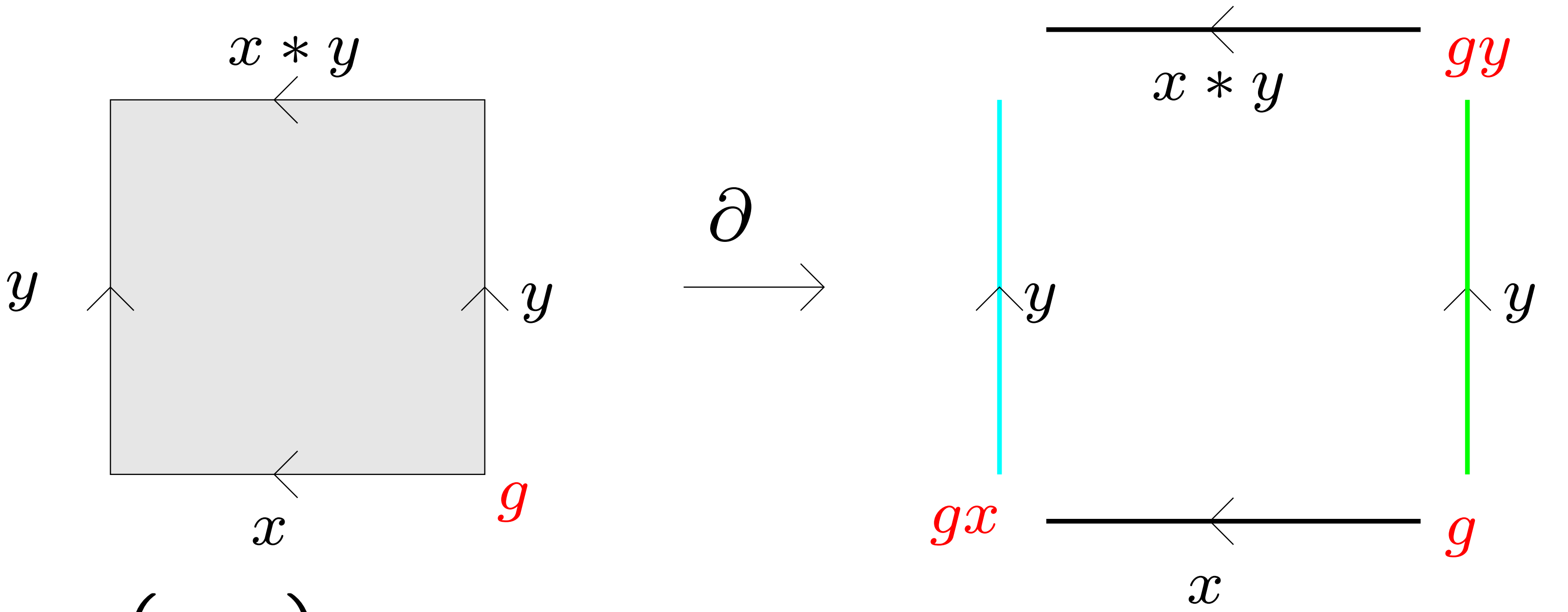


$$g(x, y)$$

$$-g(y) + gx(y)$$
$$+g(x) - gy(x * y)$$

$$\sum_{i=1}^{n} (-1)^i \{(x_1, \ldots, \widehat{x_i}, \ldots, x_n) - x_i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\}$$

Geometric interpretation $\quad C_2^R(X) \to C_1^R(X)$

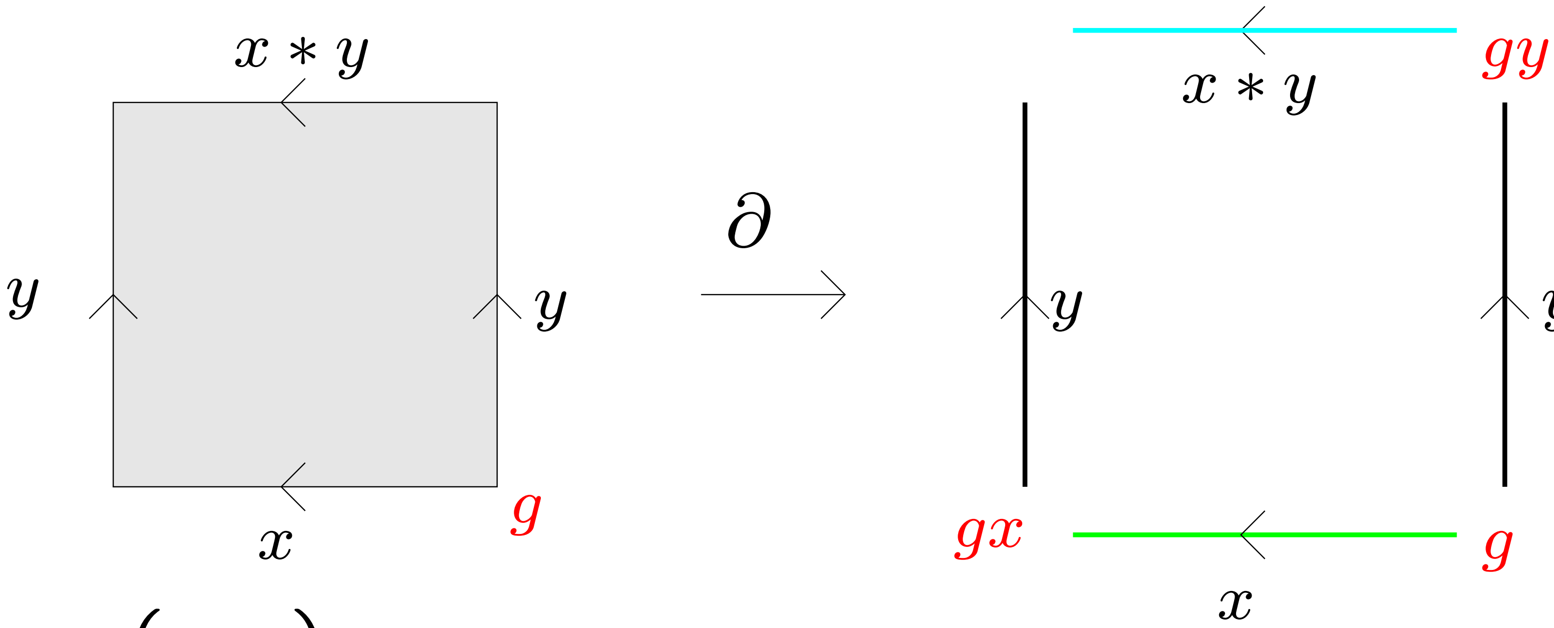


$r(x, y)$

$$-g(y) + gx(y)$$
$$+g(x) - gy(x * y)$$

$$\sum_{i=1}^{n} (-1)^i \{ (x_1, \ldots, \widehat{x_i}, \ldots, x_n) - x_i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n) \}$$

Geometric interpretation $\quad C_2^R(X) \to C_1^R(X)$

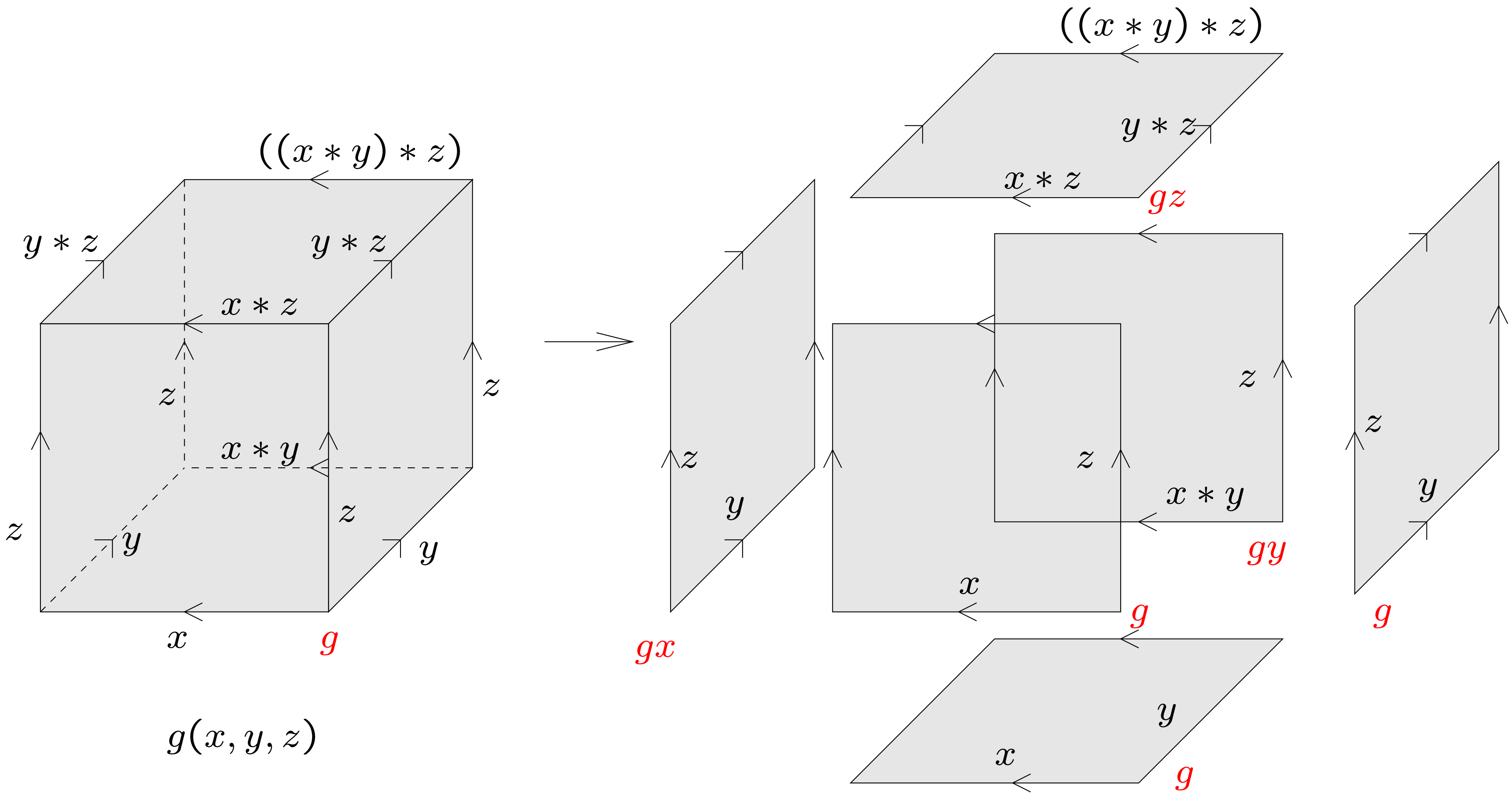


$r(x, y)$

$$-g(y) + gx(y)$$
$$+g(x) - gy(x * y)$$

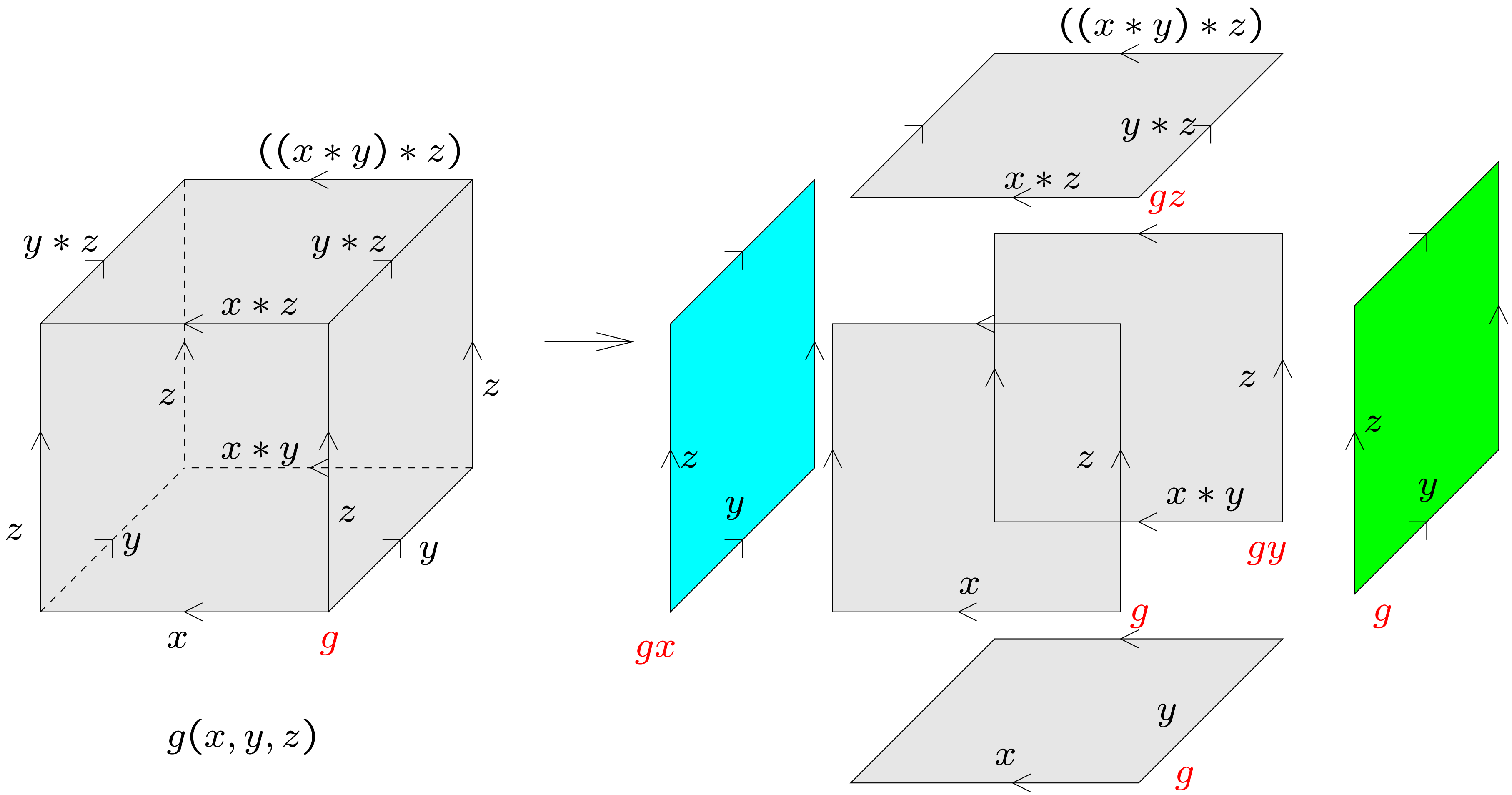
$$\sum_{i=1}^{n} (-1)^i \{(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \\ - x_i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\}$$

Geometric interpretation $C_3^R(X) \to C_2^R(X)$



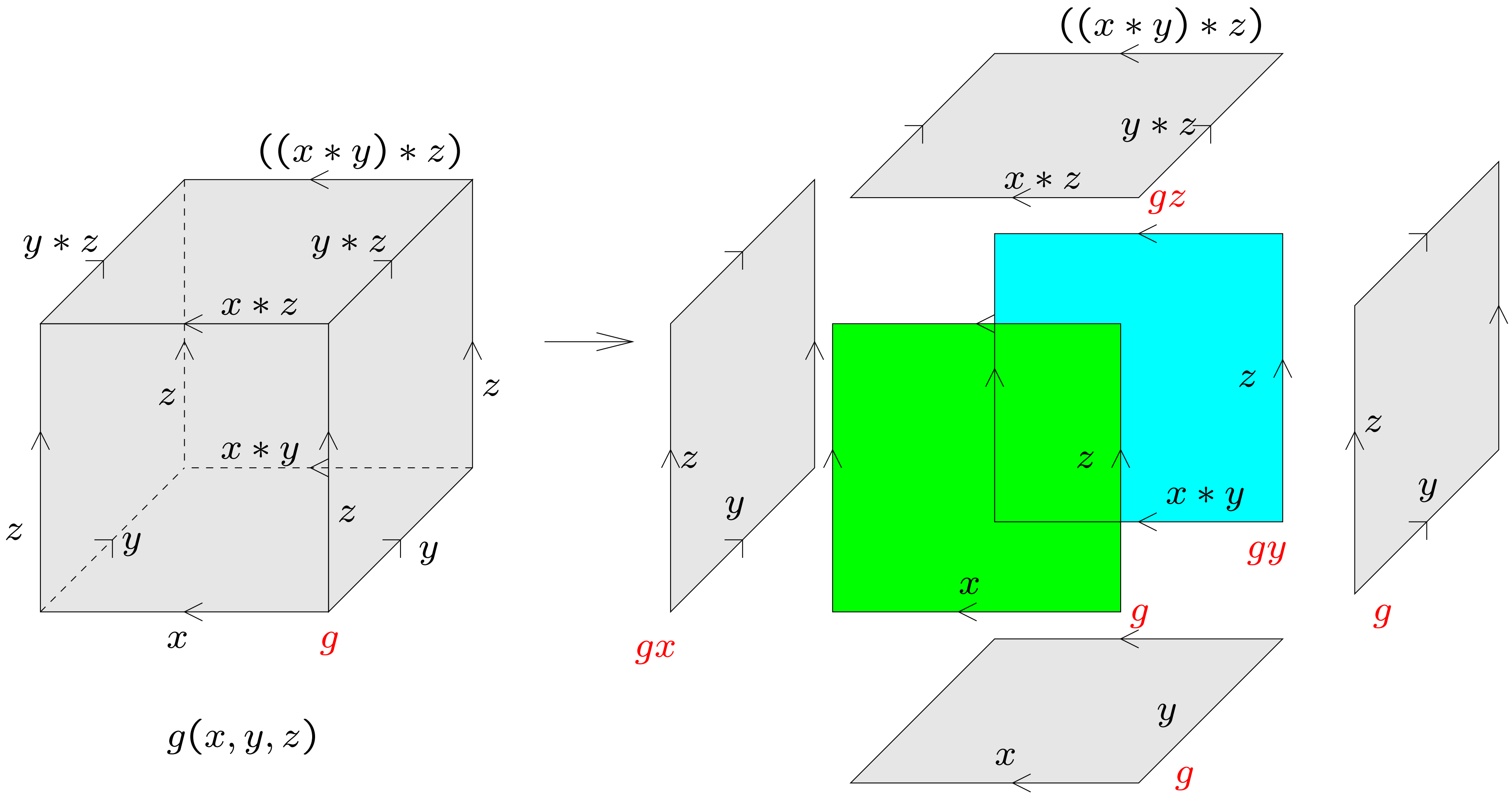
$$g(x,y,z) \mapsto -g(y,z) + gx(y,z) + g(x,z) - gy(x*y,z)$$
$$-g(x,y) + gz(x*z,y*z)$$

Geometric interpretation $C_3^R(X) \to C_2^R(X)$



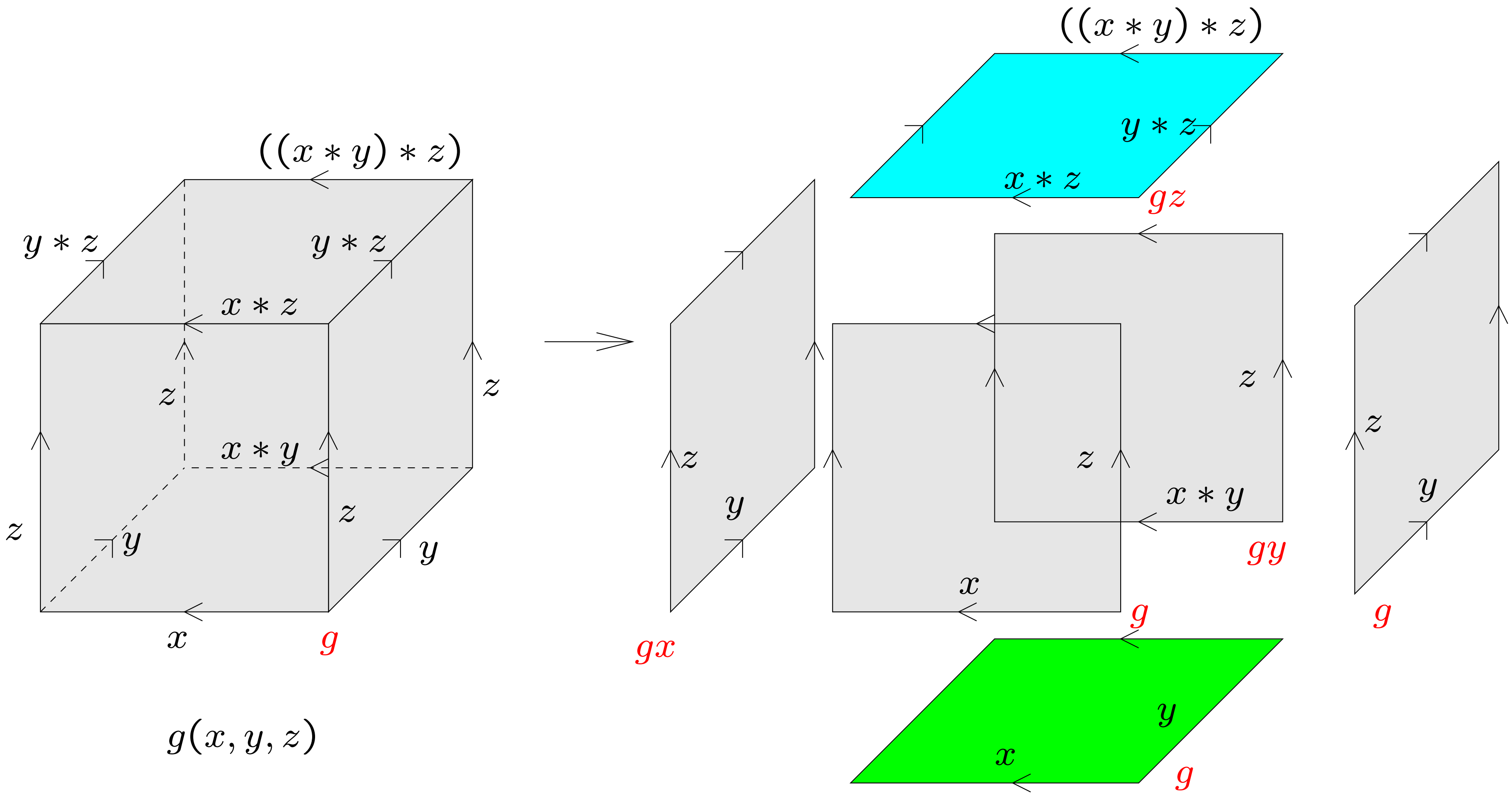
$$g(x,y,z) \mapsto -{\color{green}g(y,z)} + {\color{blue}gx(y,z)} + g(x,z) - gy(x*y,z)$$
$$-g(x,y) + gz(x*z, y*z)$$

Geometric interpretation $C_3^R(X) \to C_2^R(X)$



$$g(x,y,z) \mapsto -g(y,z) + gx(y,z) + g(x,z) - gy(x*y,z)$$
$$-g(x,y) + gz(x*z, y*z)$$

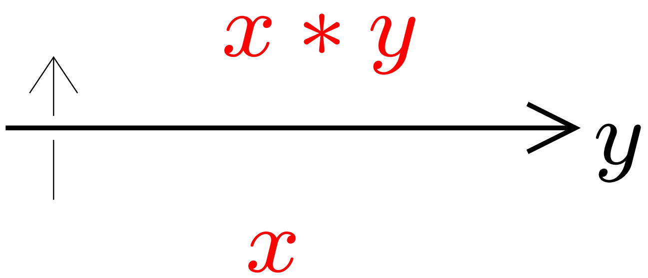
Geometric interpretation $\quad C_3^R(X) \to C_2^R(X)$



$$g(x,y,z) \mapsto -g(y,z) + gx(y,z) + g(x,z) - gy(x*y,z)$$
$$-g(x,y) + gz(x*z, y*z)$$

Region coloring

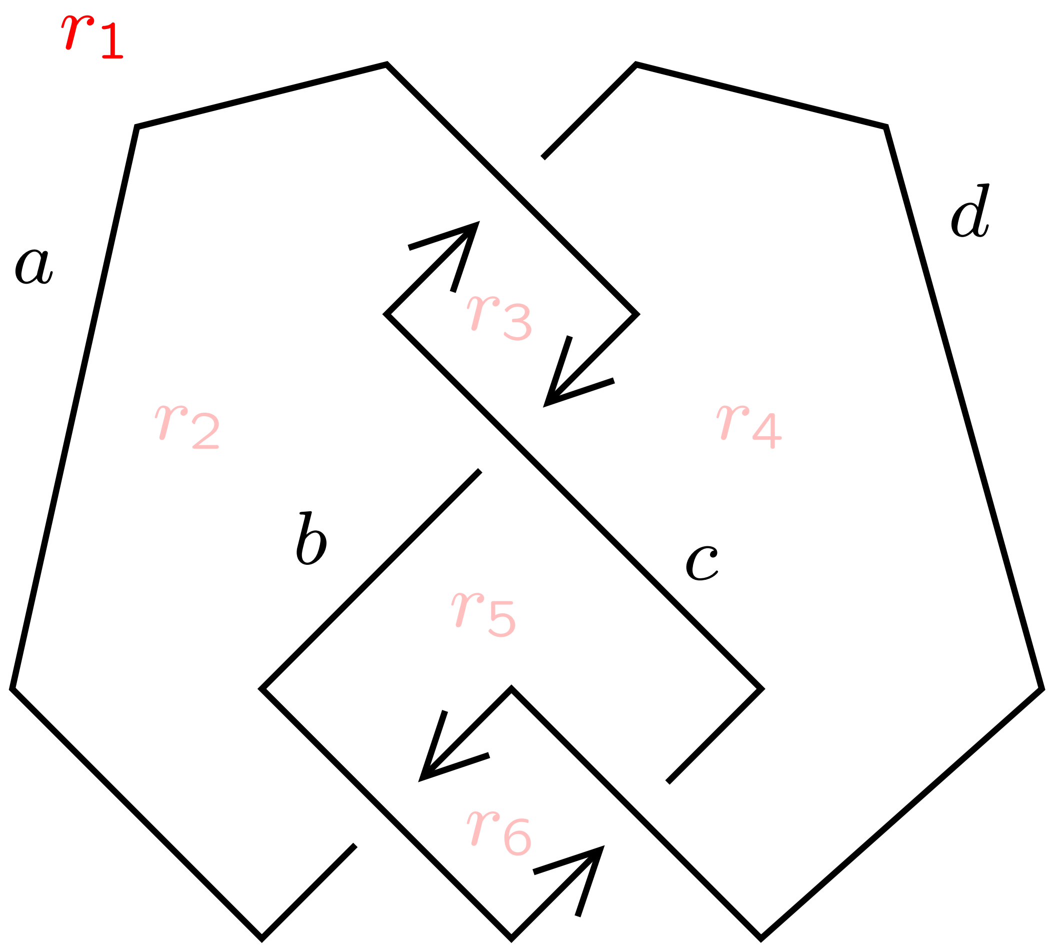
Let D be a diagram and $\mathcal{A}$ be an arc coloring by X. A map $\mathcal{D}$: $\{$regions of $D\} \to X$ is called an *region coloring* if it satisfies the following relation:



x, y and $x * y \in X$

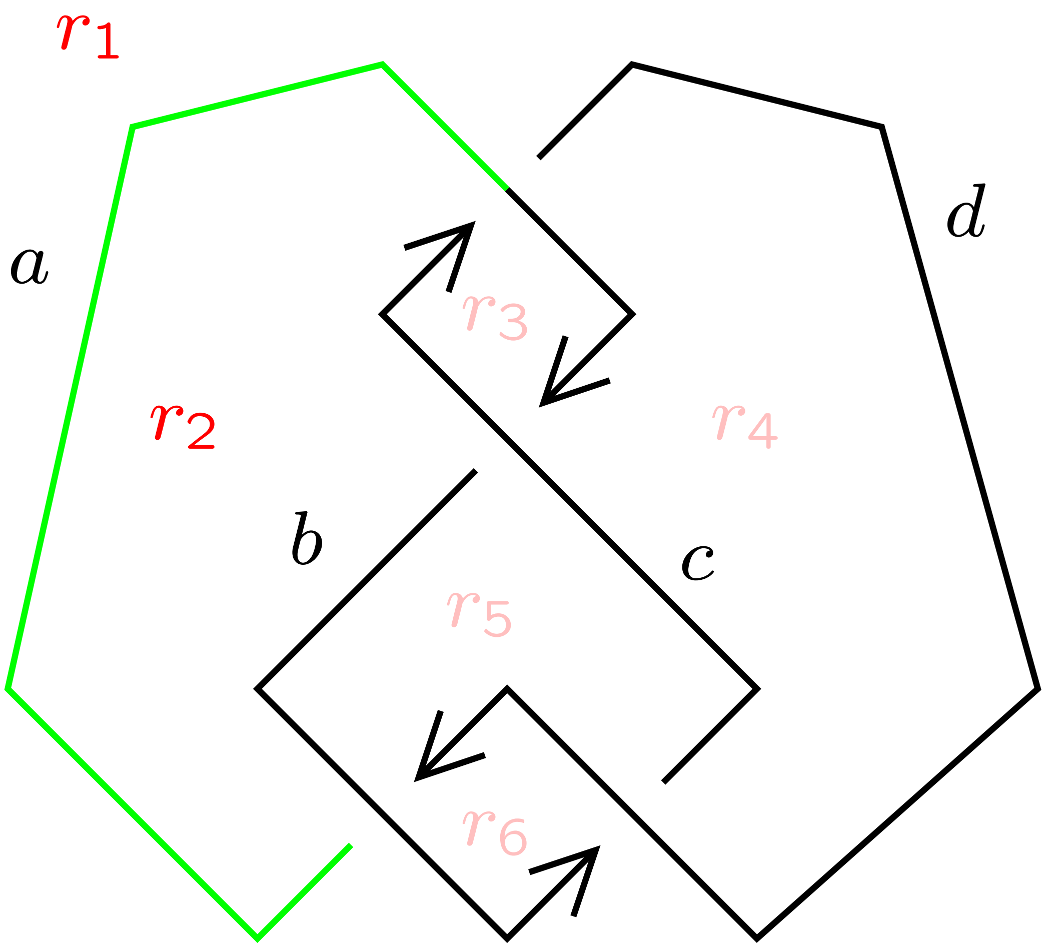
We call a pair $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ ($\mathcal{A}$: arc coloring, $\mathcal{R}$: region coloring) a *shadow coloring*.

Shadow coloring of the figure eight knot



$$r_2 * a = r_1, \quad r_3 * c = r_2,$$
$$r_3 * a = r_4, \quad r_2 * b = r_5,$$
$$r_5 * d = r_6,$$

Shadow coloring of the figure eight knot

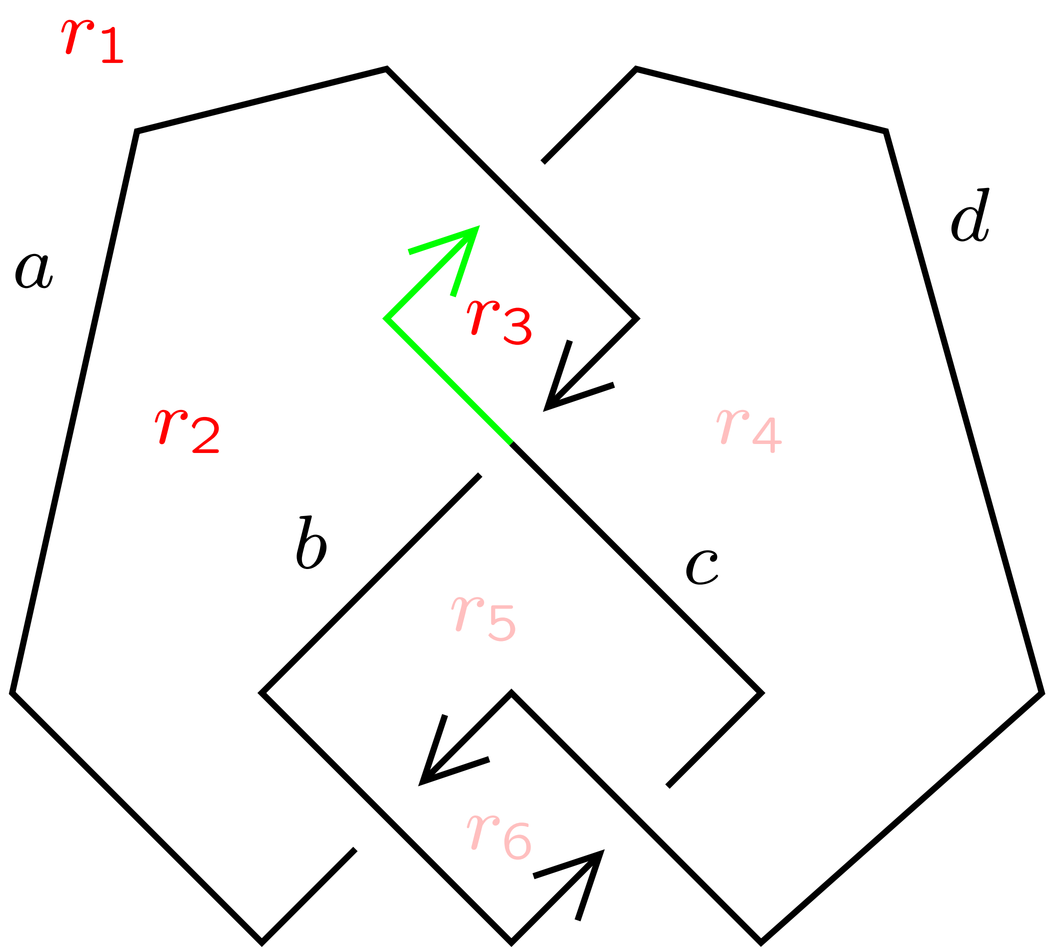


$r_2 * a = r_1, \quad r_3 * c = r_2,$

$r_3 * a = r_4, \quad r_2 * b = r_5,$

$r_5 * d = r_6,$

Shadow coloring of the figure eight knot



$r_2 * a = r_1, \quad r_3 * c = r_2,$

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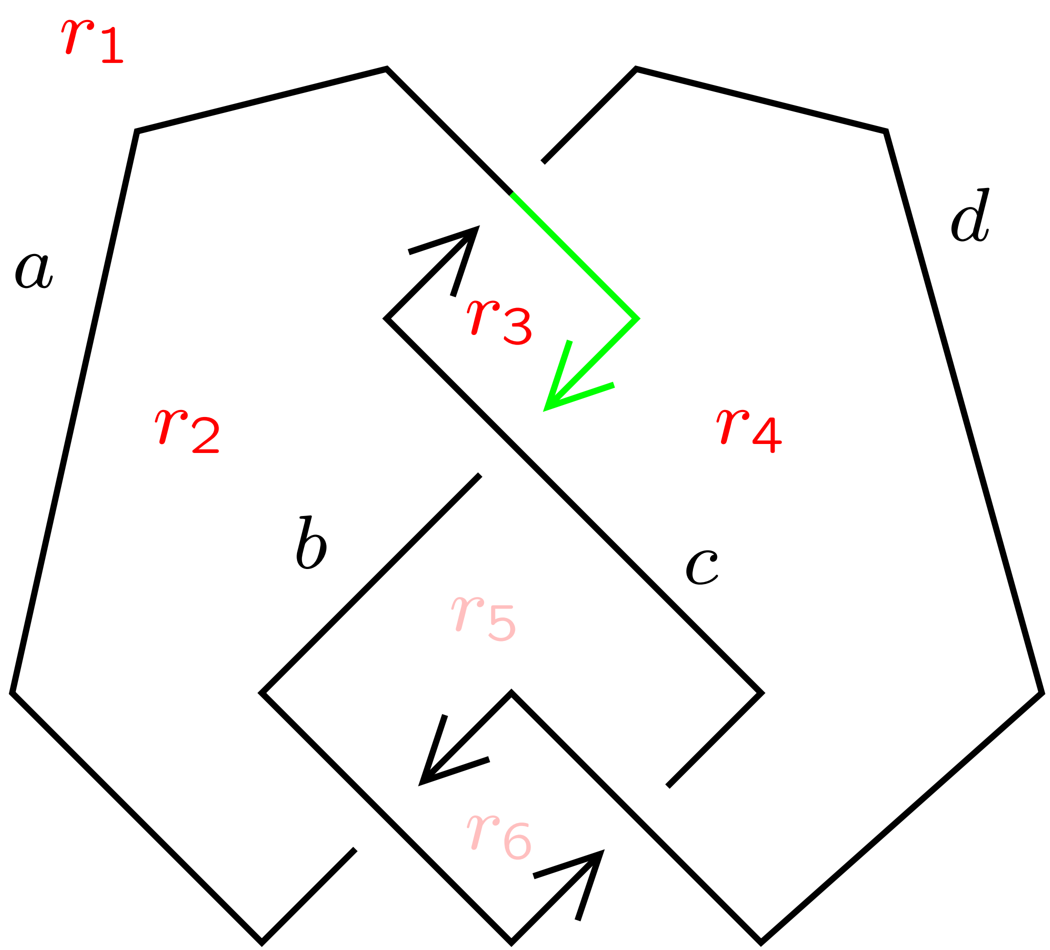
$r_5 * d = r_6,$

Shadow coloring of the figure eight knot

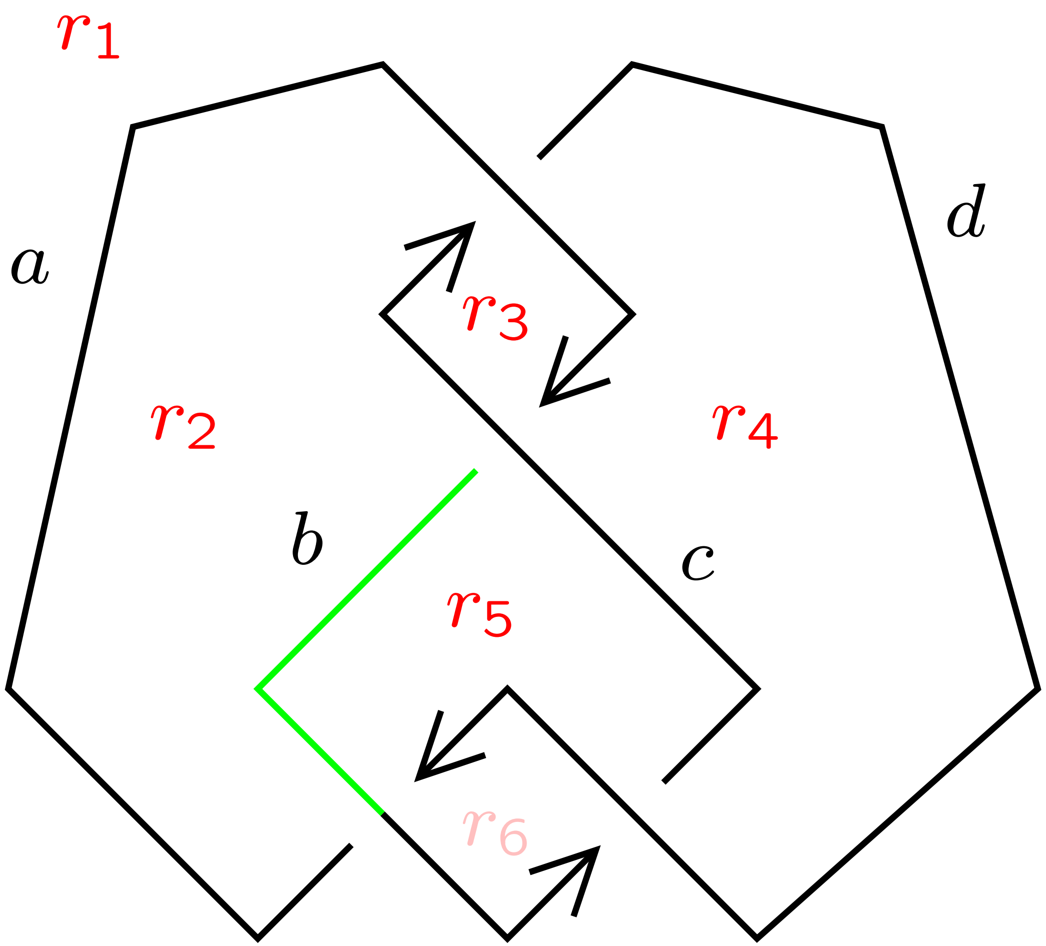


$r_2 * a = r_1, \quad r_3 * c = r_2,$

$r_3 * a = r_4, \quad r_2 * b = r_5,$

$r_5 * d = r_6,$

Shadow coloring of the figure eight knot

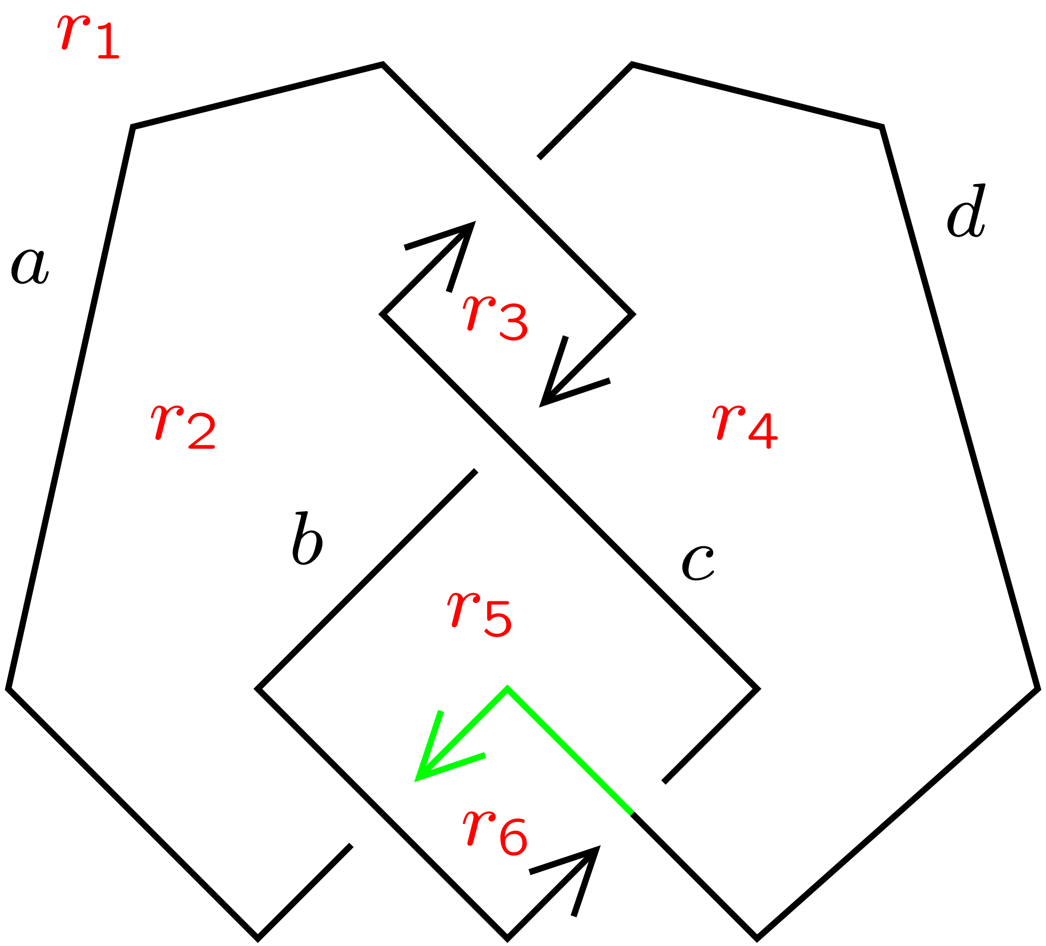


$r_2 * a = r_1$, $\quad r_3 * c = r_2$,

$r_3 * a = r_4$, $\quad r_2 * b = r_5$,

$r_5 * d = r_6$,

Shadow coloring of the figure eight knot



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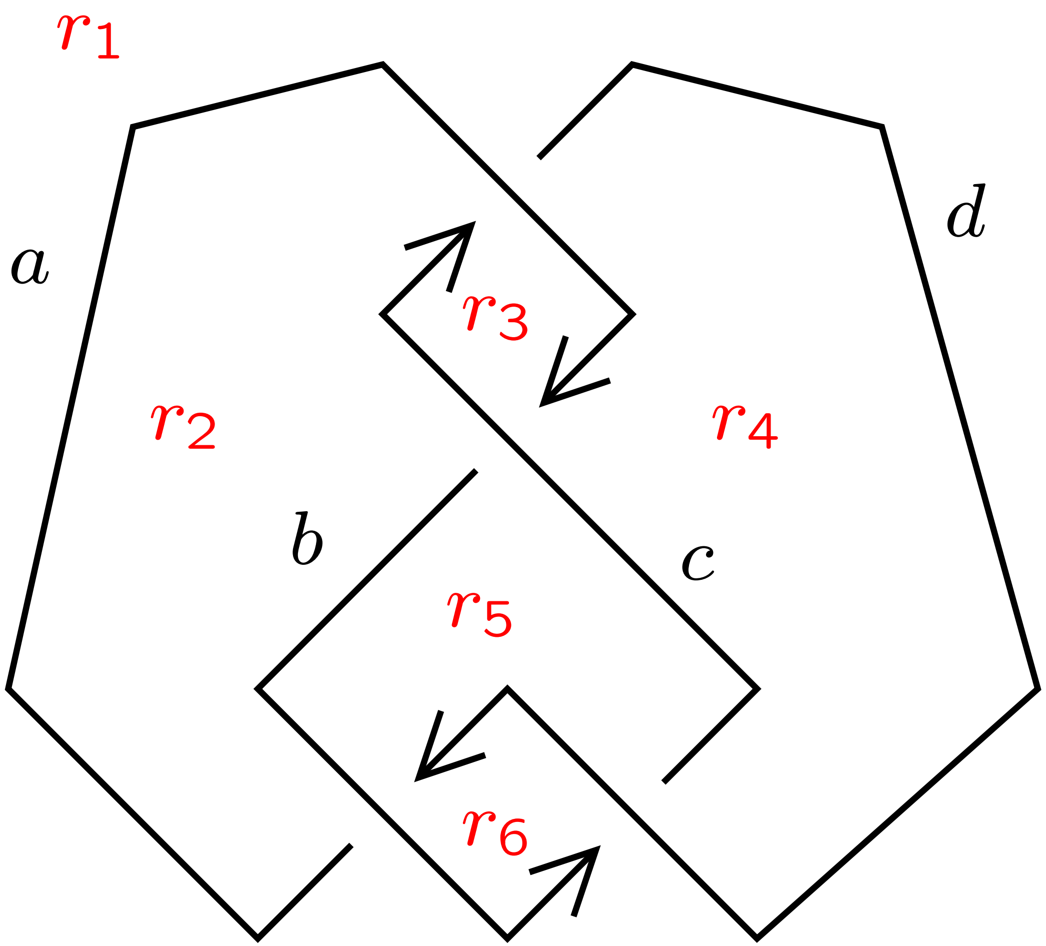


If we fix a color of one region, then the colors of other regions are uniquely determined.

Remark

Region colorings give no information on the representation of knot group, but it is useful to compute volume and Chern-Simons.

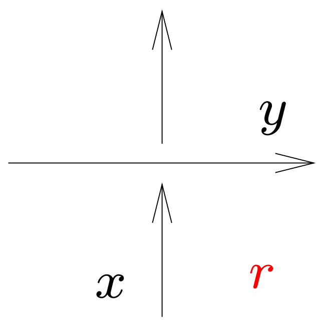
Cycle $[C(\mathcal{S})]$ associated with a shadow coloring

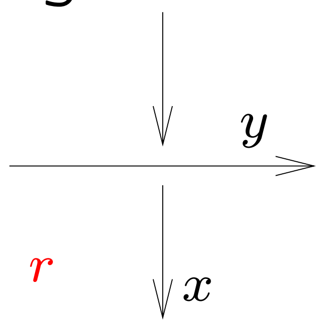
A quandle X itself has a right G_X-action defined by

$$x * (x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}) = (\dots((x *^{\varepsilon_1} x_1) *^{\varepsilon_2} x_2) \dots) *^{\varepsilon_n} x_n.$$

So the free abelian group $\mathbb{Z}[X]$ is a right $\mathbb{Z}[G_X]$-module.

Let $\mathcal{S}$ be a shadow coloring by a quandle X. Assign

$+r \otimes (x, y)$ for 

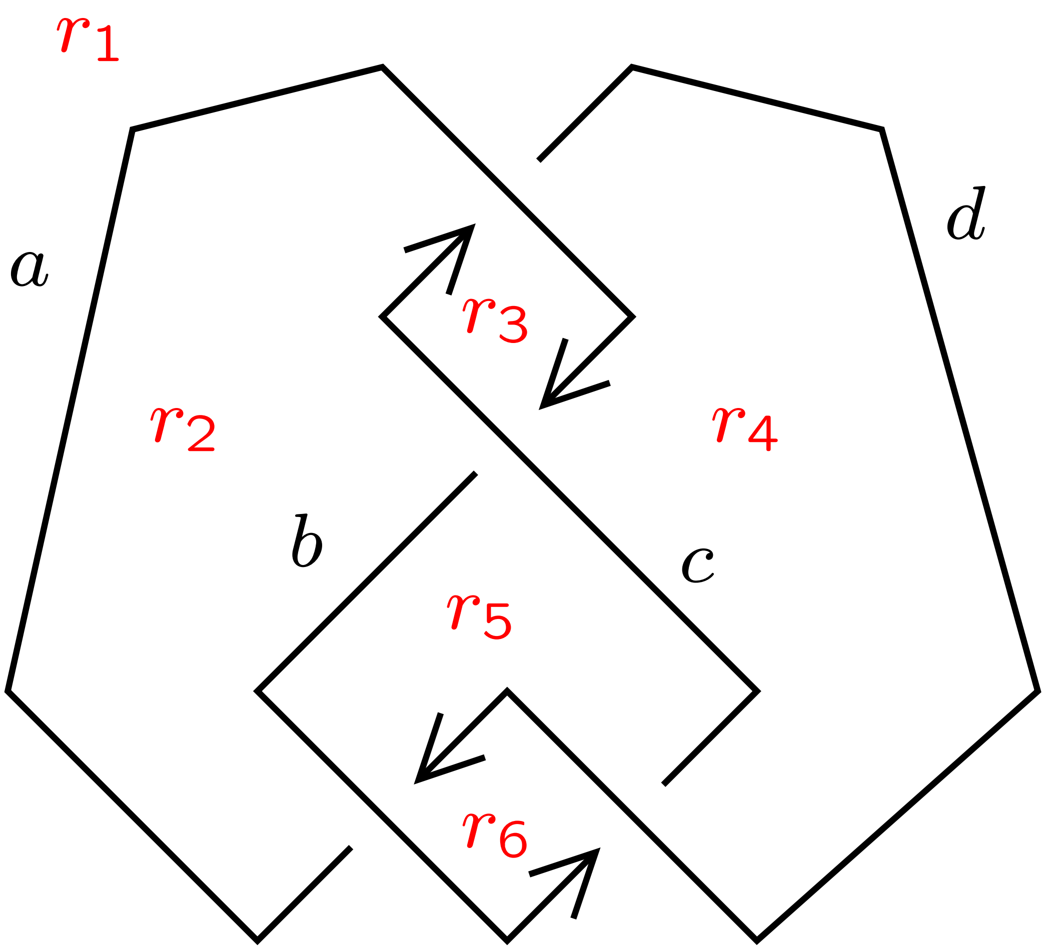
and $-r \otimes (x, y)$ for 

.

Let

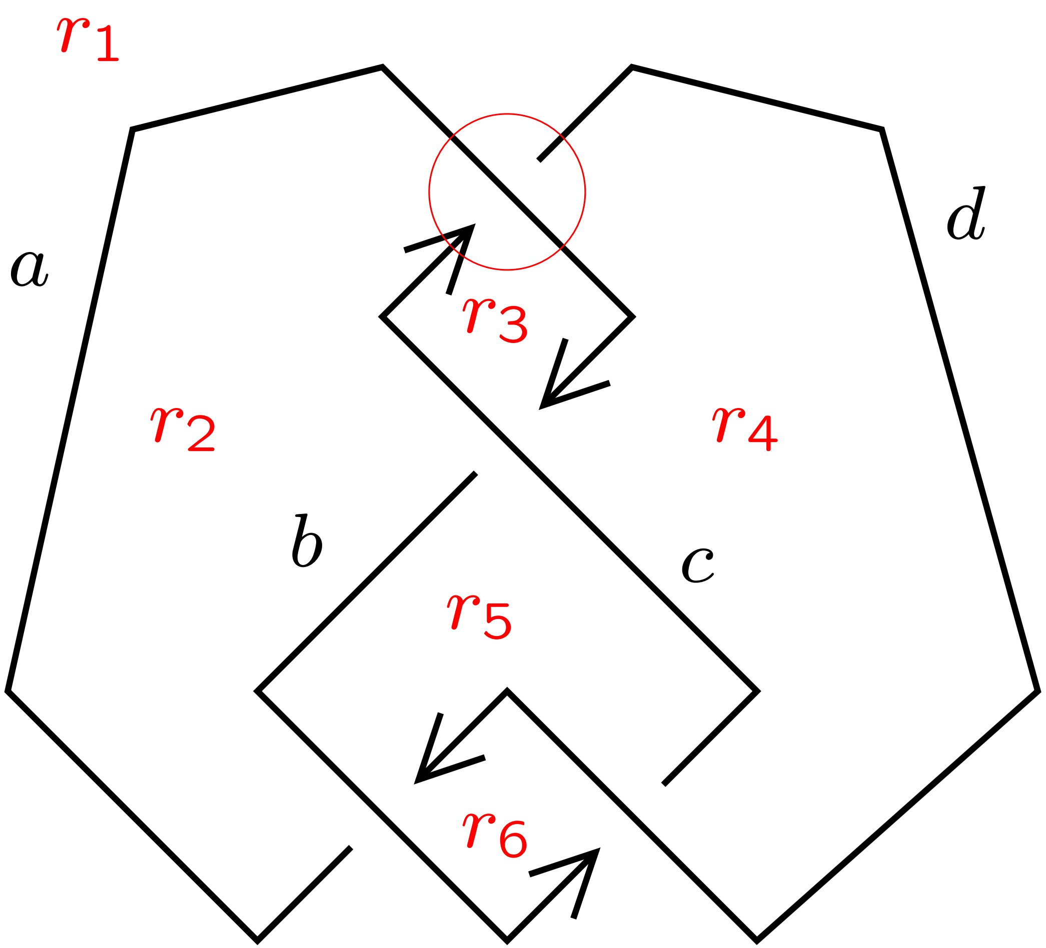
$$C(\mathcal{S}) = \sum_{c:\text{crossing}} \varepsilon_c r_c \otimes (x_c, y_c) \in C_2^Q(X; \mathbb{Z}[X]).$$

Example: $C(\mathcal{S})$ for the figure eight knot



$$C(\mathcal{S}) = r_3 \otimes (c,a) + r_3 \otimes (b,c) - r_2 \otimes (a,b) - r_4 \otimes (c,d)$$

Example: $C(\mathcal{S})$ for the figure eight knot

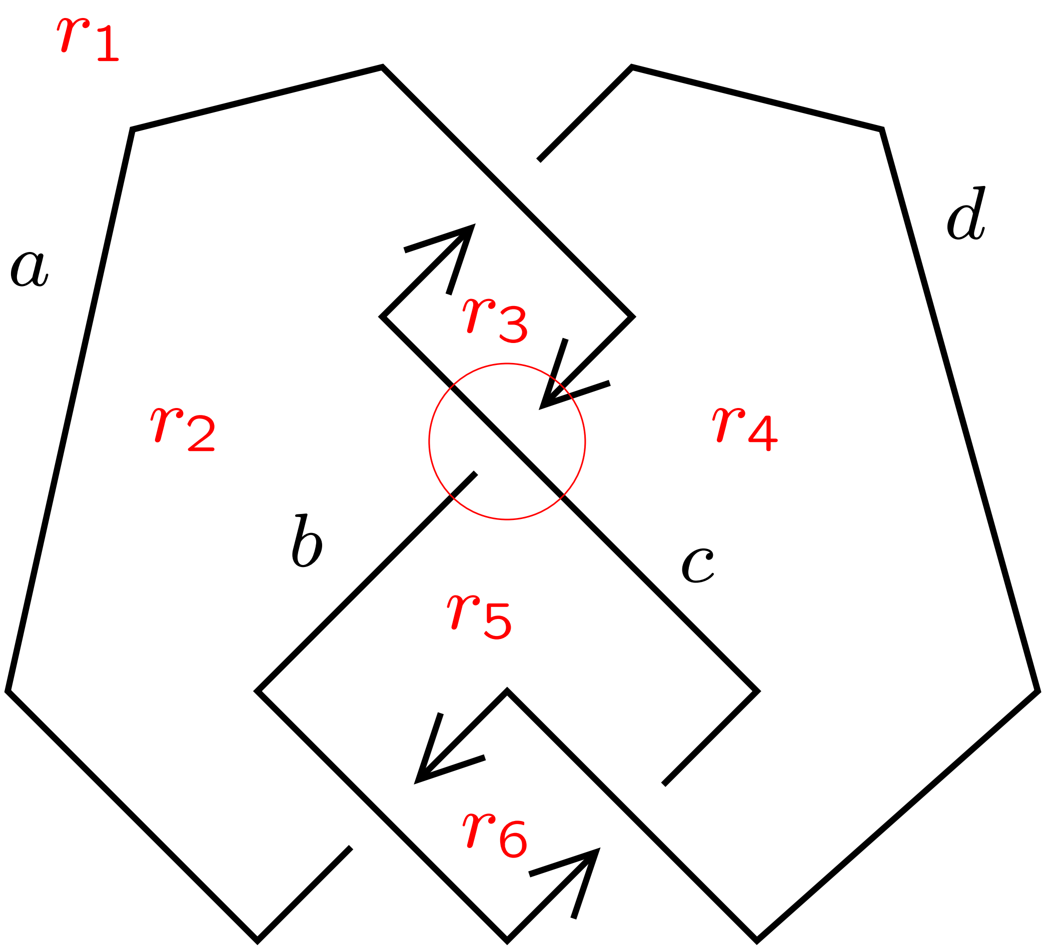


$$C(\mathcal{S}) =$$

$$r_3 \otimes (c, a) + r_3 \otimes (b, c)$$

$$- r_2 \otimes (a, b) - r_4 \otimes (c, d)$$

Example: $C(\mathcal{S})$ for the figure eight knot



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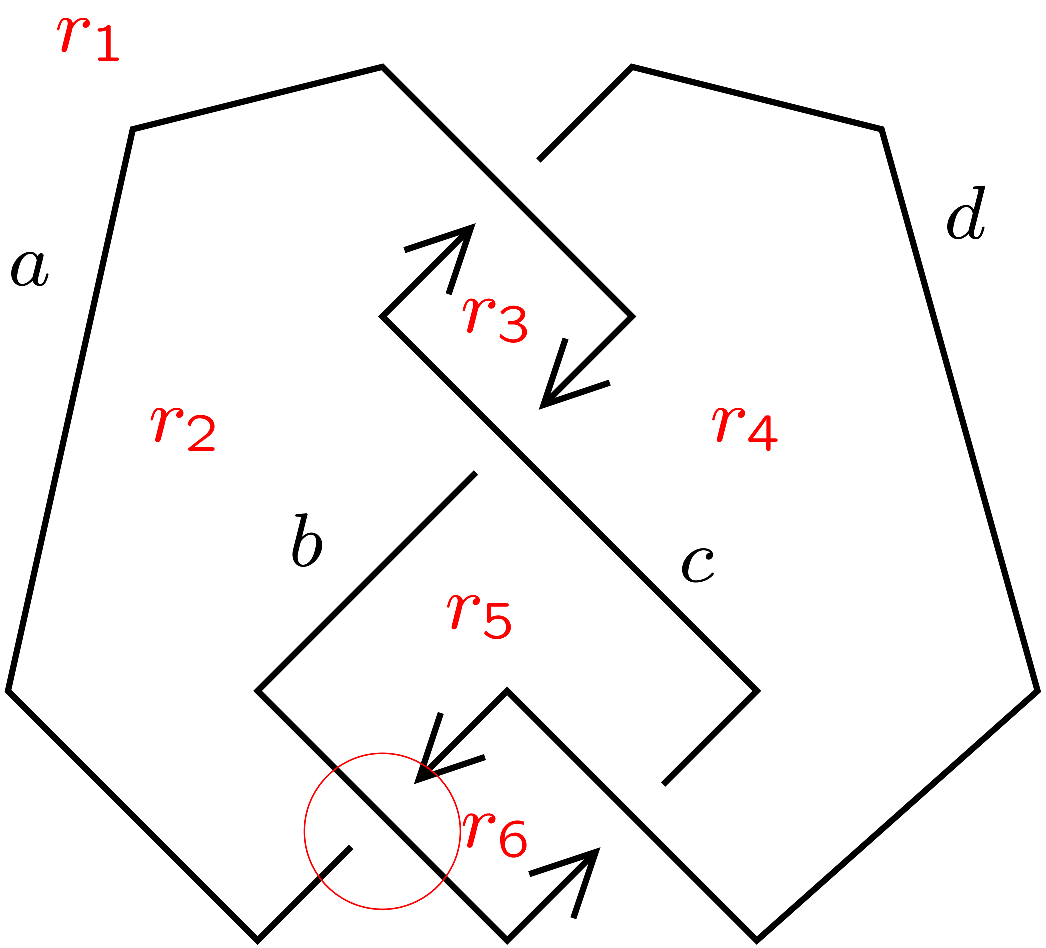
Example: $C(\mathcal{S})$ for the figure eight knot



$$C(\mathcal{S}) =$$

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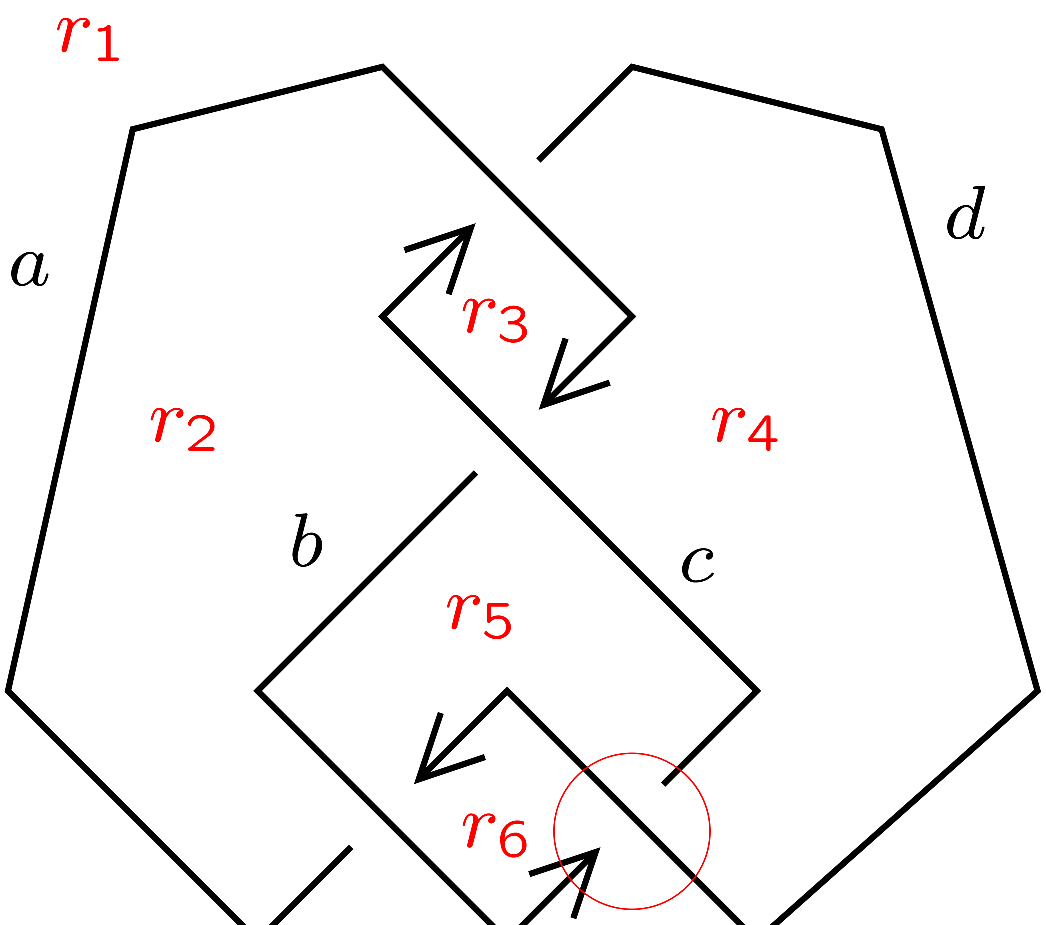
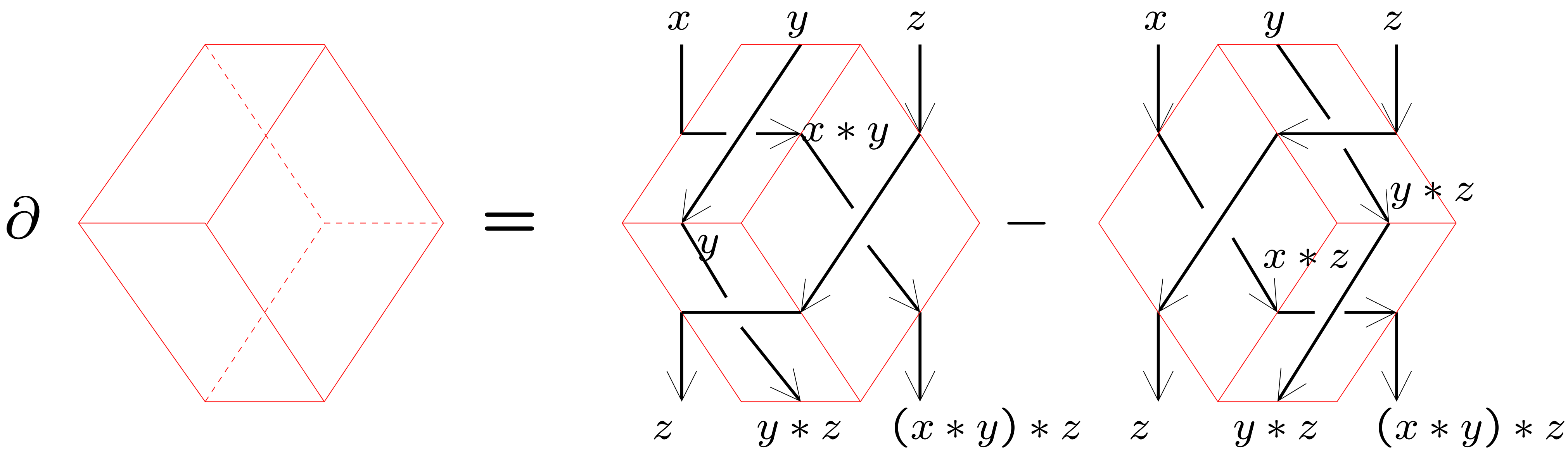
Example: $C(\mathcal{S})$ for the figure eight knot



$$C(\mathcal{S}) =$$
$$r_3 \otimes (c, a) + r_3 \otimes (b, c)$$
$$- r_2 \otimes (a, b) - r_4 \otimes (c, d)$$

$C(\mathcal{S})$ is a cycle. The homology class $[C(\mathcal{S})]$ in $H_2^Q(X;\mathbb{Z}[X])$ is invariant under the Reidemeister moves. The invariance under the Reidemeister III move is shown in the following figure.



$$\partial(r \otimes (x,y,z)) = (r \otimes (x,y) + r*y \otimes (x*y,z) + r \otimes (y,z))$$
$$- (r \otimes (x,z) + r*x \otimes (y,z) + r*z \otimes (x*z, y*z))$$

We can show that the homology class $[C(\mathcal{S})]$ does not depend on the region coloring. Moreover it only depends on the conjugacy class of the representation $\pi_1(S^3 \setminus K) \to G_X$ induced by the arc coloring. When $X = \mathcal{P}$ (quandle formed by parabolic elements of $\mathrm{PSL}(2, \mathbb{C})$),

Prop (Inoue - K.) *The homology class* $[C(\mathcal{S})]$ *in* $H_2^Q(\mathcal{P}, \mathbb{Z}[\mathcal{P}])$ *only depends on the conjugacy class of the parabolic representation* $\pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{C})$ *induced by the arc coloring* $\mathcal{A}$.

Simplicial quandle homology $H_n^{\Delta}(X)$

Let $C_n^{\Delta}(X) = \text{span}_{\mathbb{Z}}\{(x_0, \ldots, x_n) | x_i \in X\}$. Define the boundary operator $\partial : C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$ by

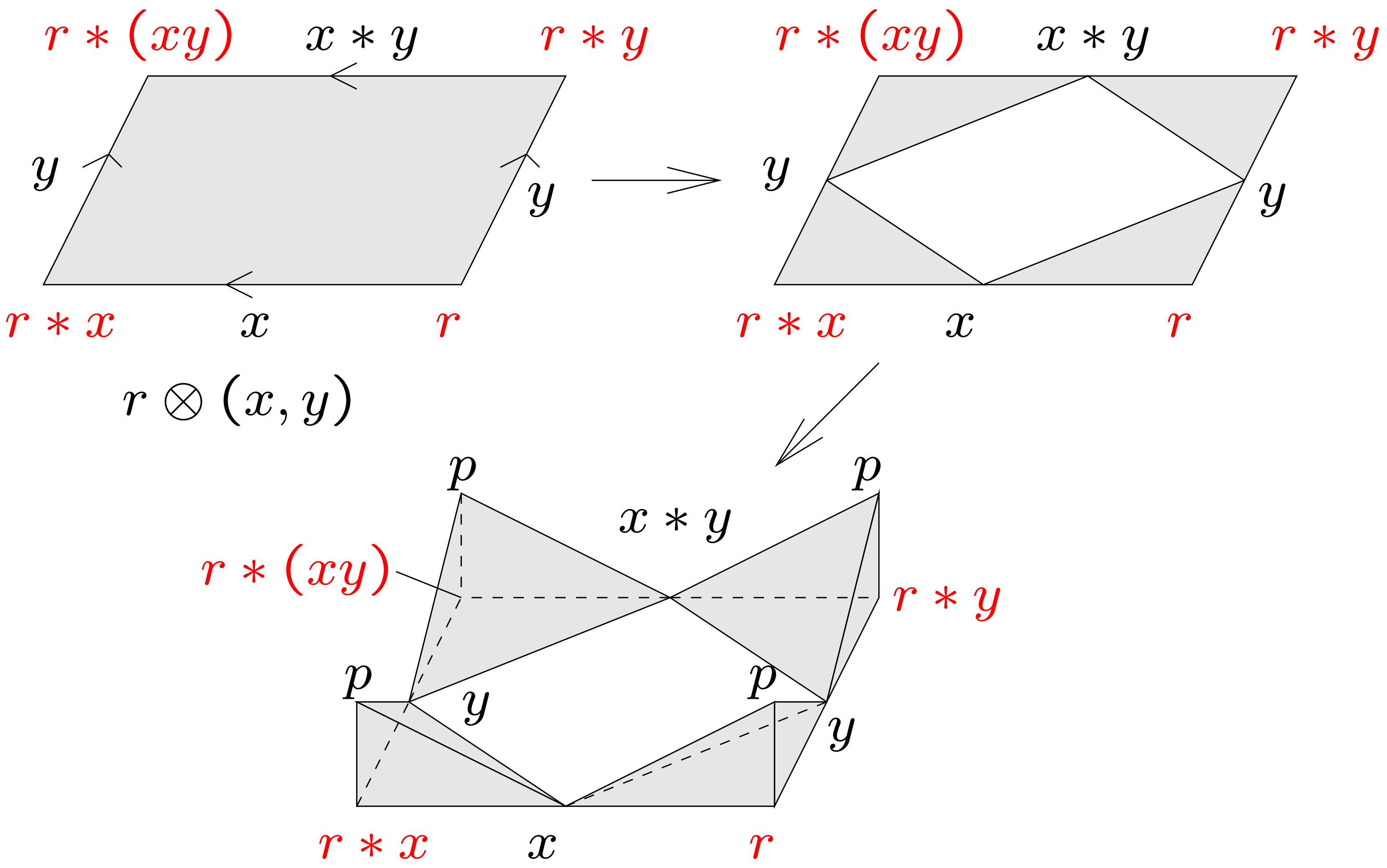
$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_n).$$

$C_n^{\Delta}(X)$ has a natural right action by $\mathbb{Z}[G_X]$. Denote the homology of $C_n^{\Delta}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ by $H_n^{\Delta}(X)$. We can construct a map

$$\varphi_* : H_n^R(X; \mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X)$$

in the following way:

$\underline{n=2}$ $\varphi : C_2^R(X;\mathbb{Z}[X]) \to C_3^{\Delta}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$



$$(p,r,x,y) - (p,r*x,x,y)$$
$$-(p,r*y,x*y,y) + (p,r*(xy),x*y,y)$$

For general case, let I_n be the set of maps $\iota : \{1, 2, \cdots, n\} \to \{0, 1\}$. Let $|\iota|$ denote the cardinality of the set $\{k \mid \iota(k) = 1,\ 1 \le k \le n\}$. For $r \otimes (x_1, x_2, \cdots, x_n) \in C_n^R(X; \mathbb{Z}[X])$ and $\iota \in I_n$, define

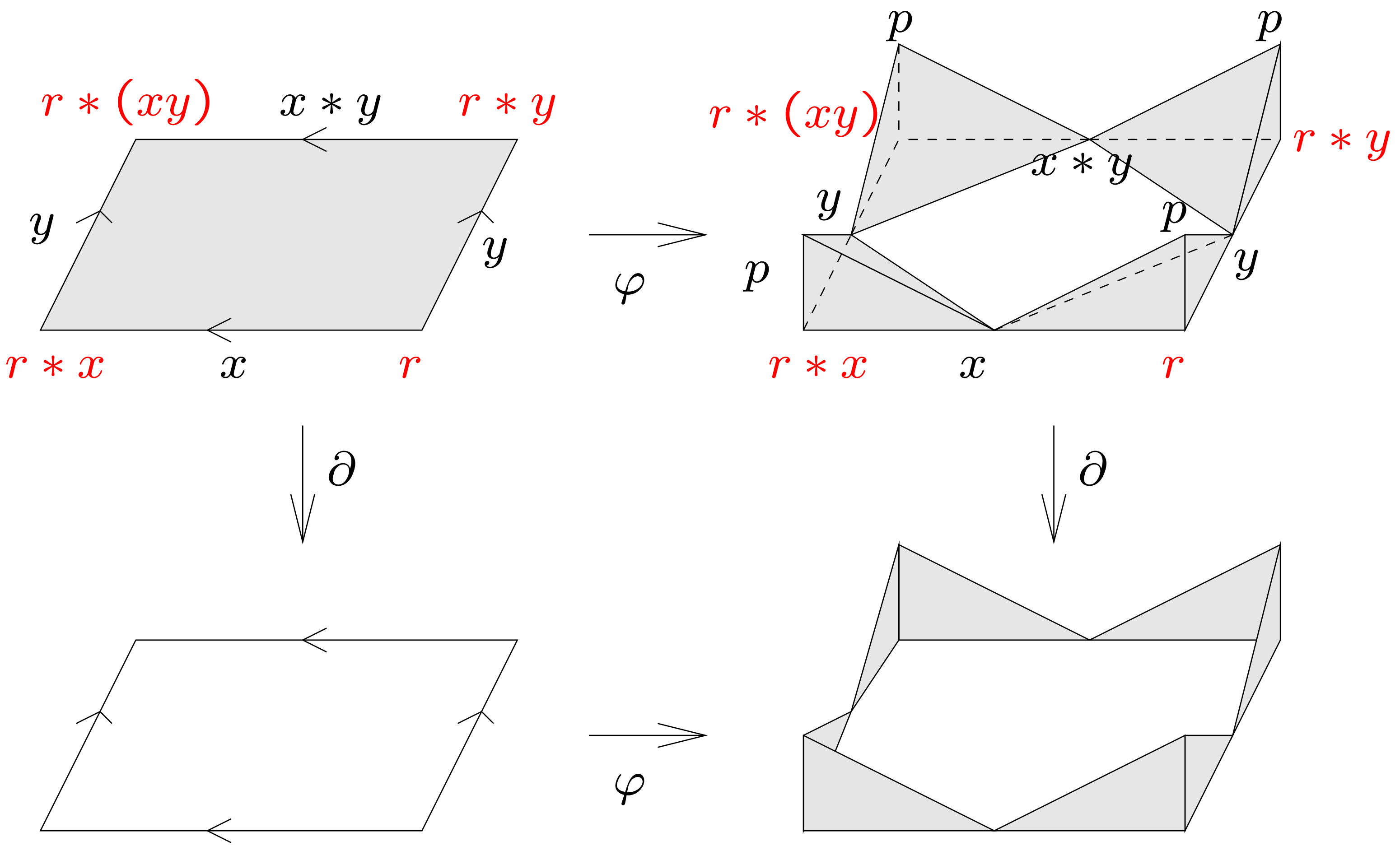
$$\begin{aligned} r(\iota) &= r * (x_1^{\iota(1)} x_2^{\iota(2)} \cdots x_n^{\iota(n)}) \\ x(\iota, i) &= x_i * (x_{i+1}^{\iota(i+1)} x_{i+2}^{\iota(i+2)} \cdots x_n^{\iota(n)}). \end{aligned}$$

Fix $p \in X$. Define $\varphi : C_n^R(X; \mathbb{Z}[X]) \longrightarrow C_{n+1}^{\Delta}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ by

$$\begin{aligned} &\varphi(r \otimes (x_1, x_2, \cdots, x_n)) \\ &\quad = \sum_{\iota \in I_n} (-1)^{|\iota|} (p, r(\iota), x(\iota, 1), x(\iota, 2), \cdots, x(\iota, n)). \end{aligned}$$

Thm $\varphi : C_n^R(X;\mathbb{Z}[X]) \longrightarrow C^{\Delta}_{n+1}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ *is a chain map.*

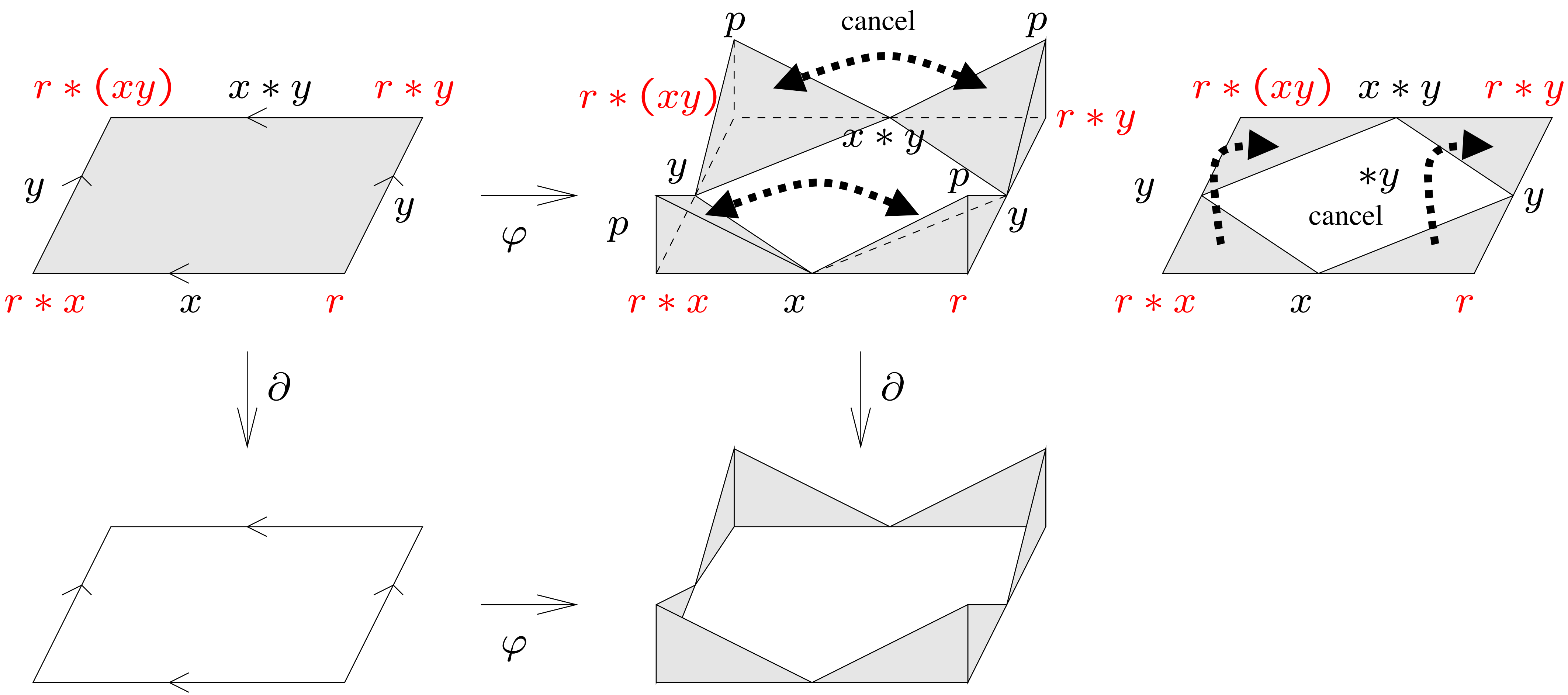
Proof.



∎

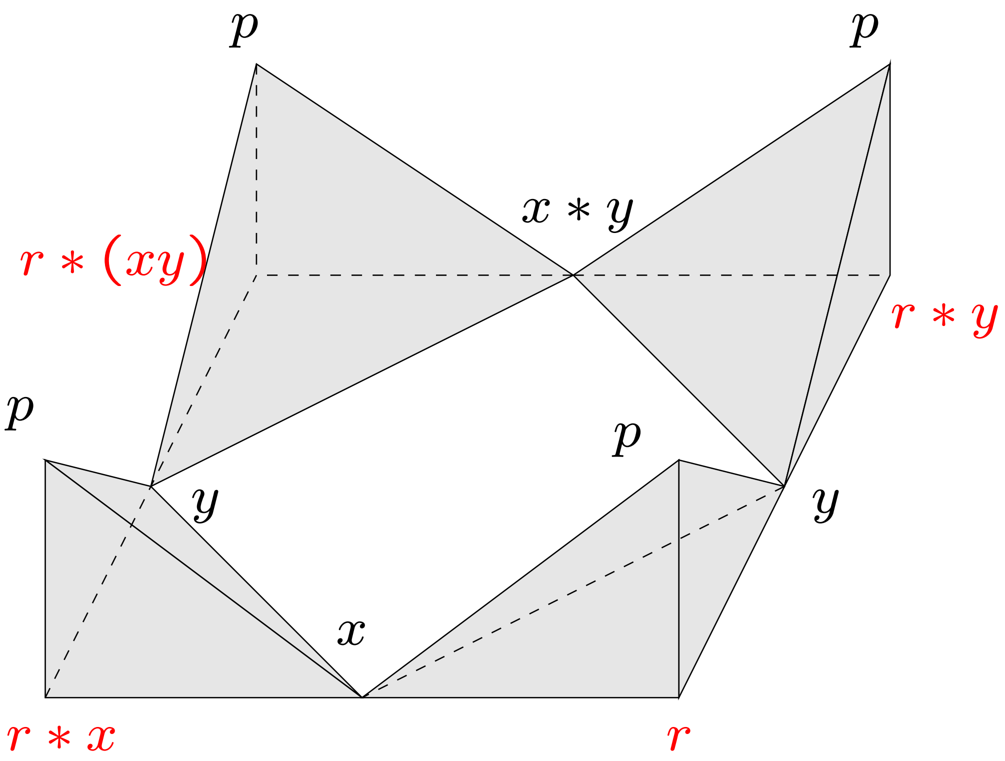
Thm $\varphi : C_n^R(X;\mathbb{Z}[X]) \longrightarrow C_{n+1}^{\Delta}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ *is a chain map.*

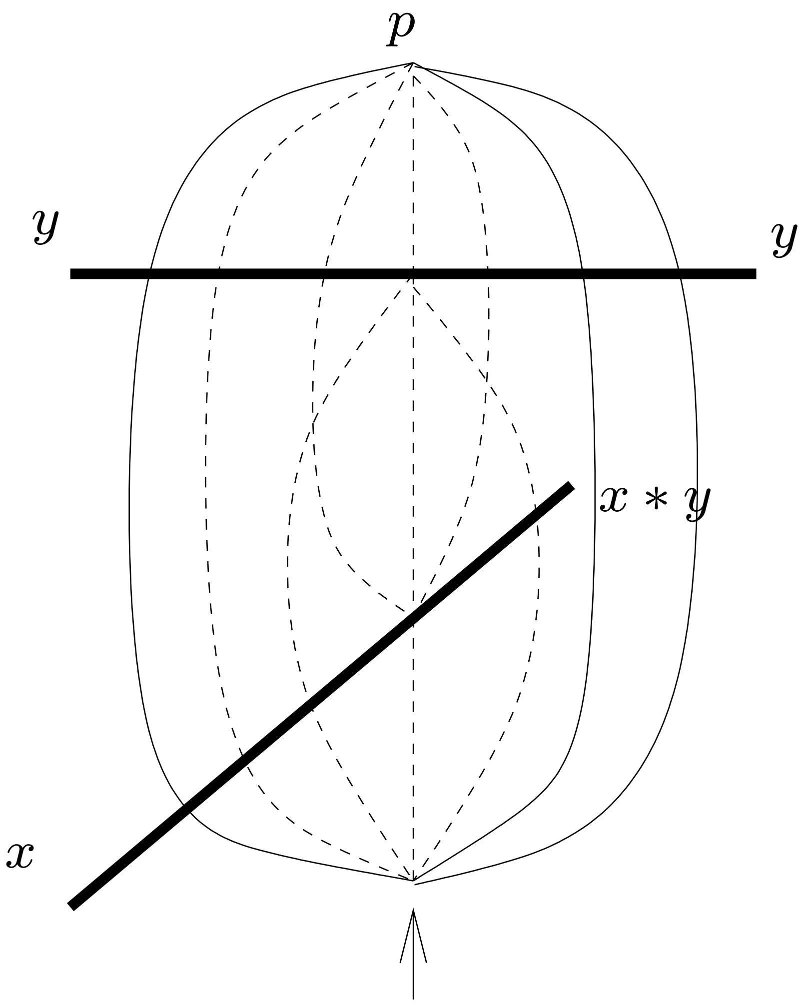
Proof.



■

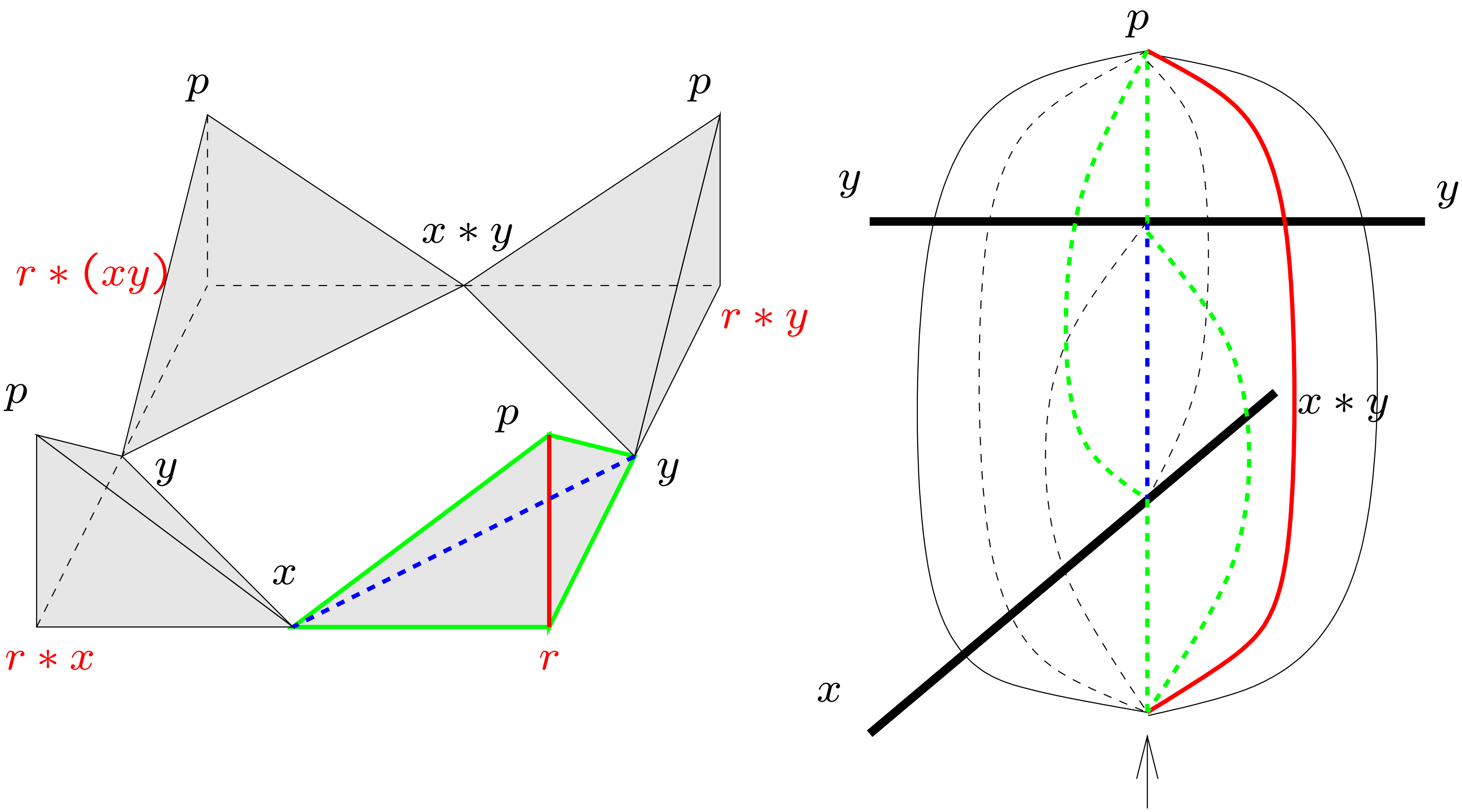
The result after gluing





$$r \sim r * x \sim r * y \sim r * (xy)$$

The result after gluing



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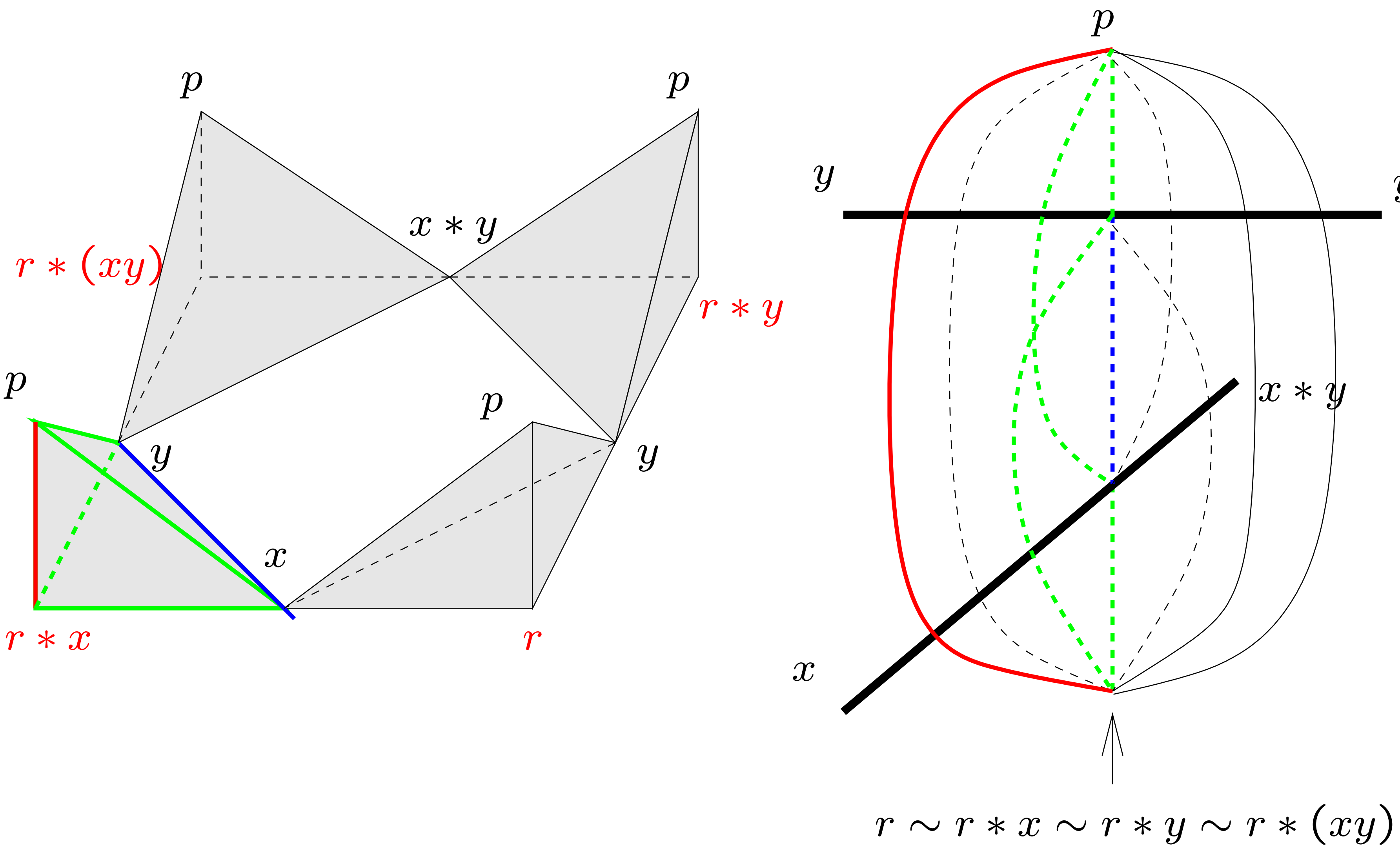

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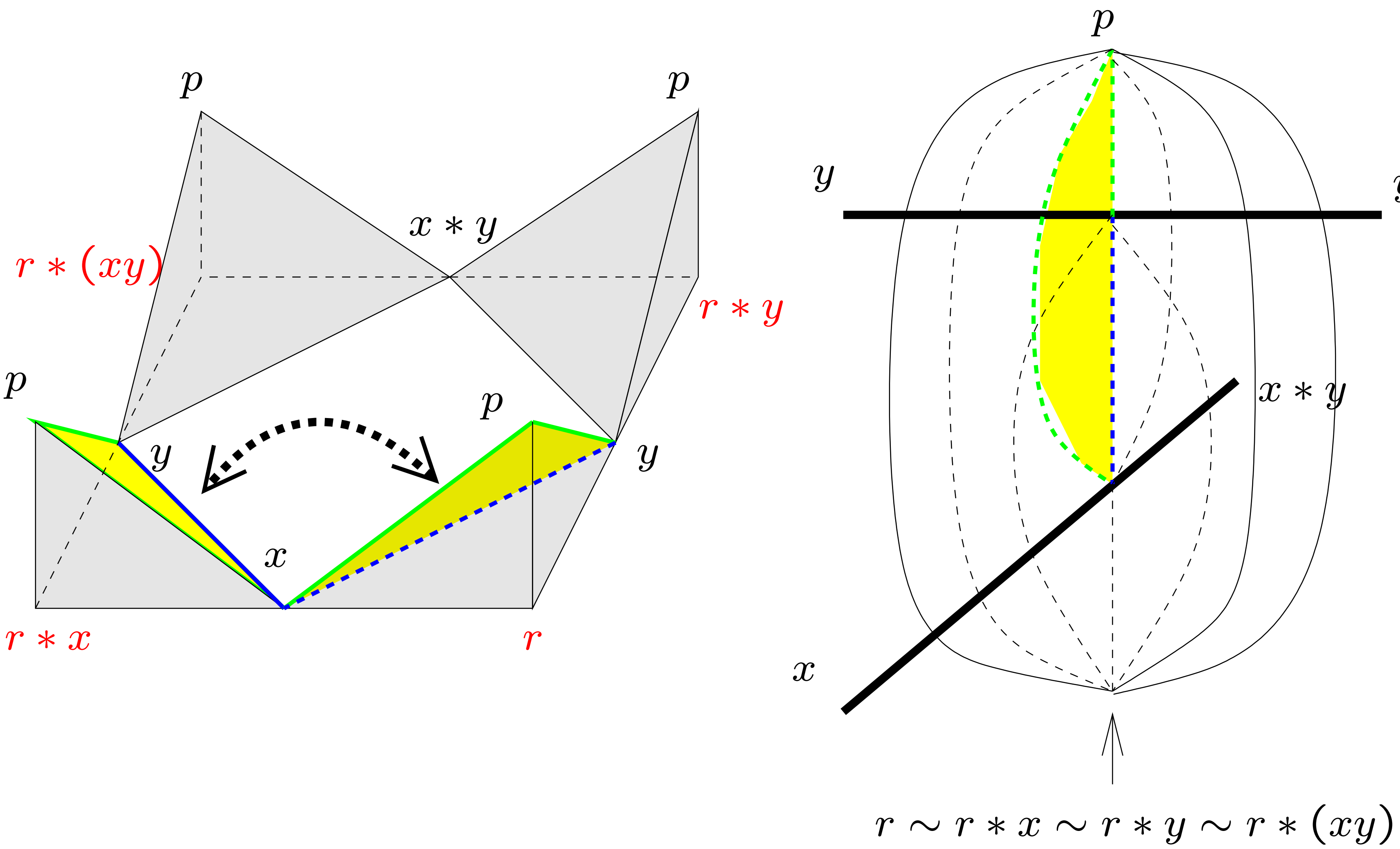

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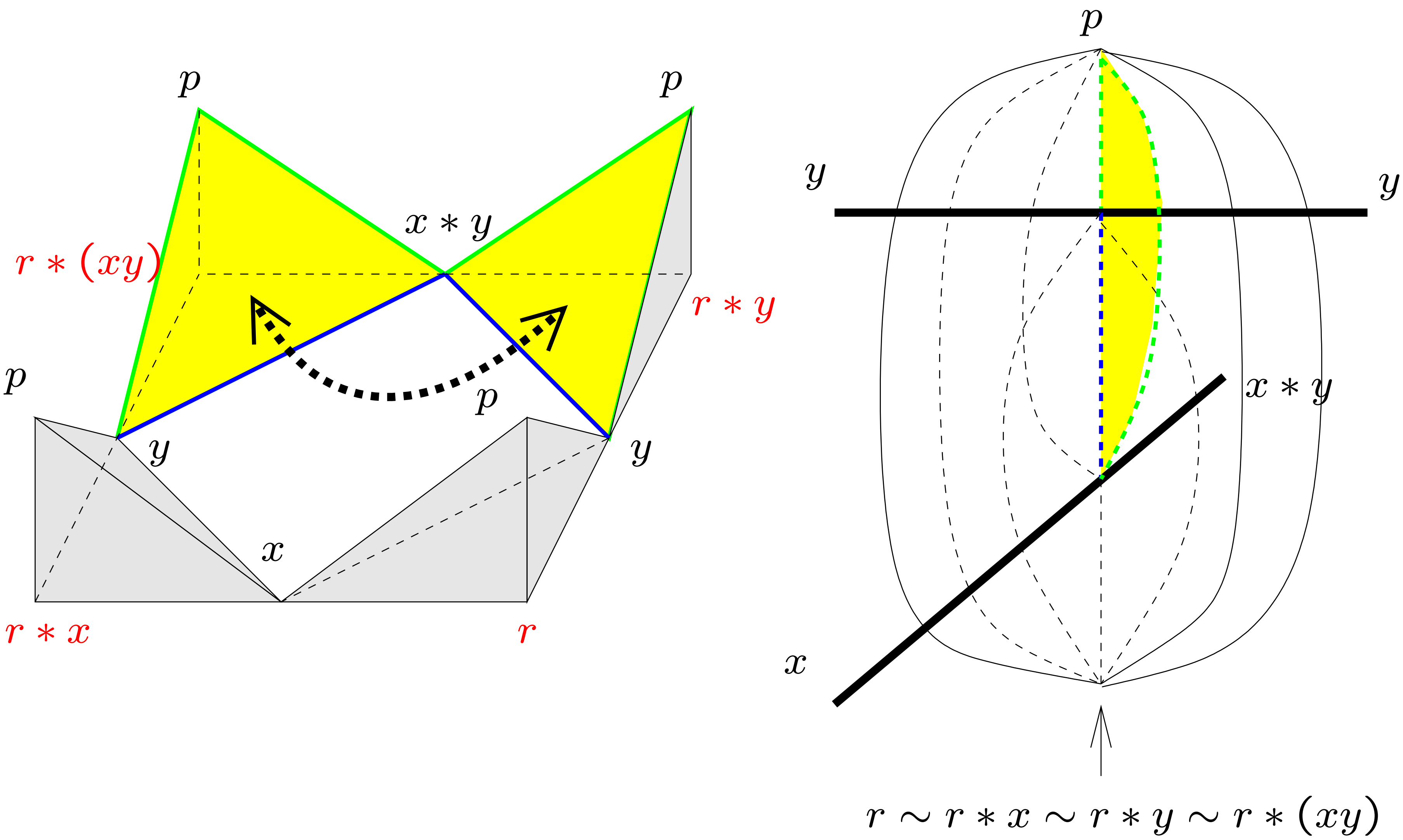

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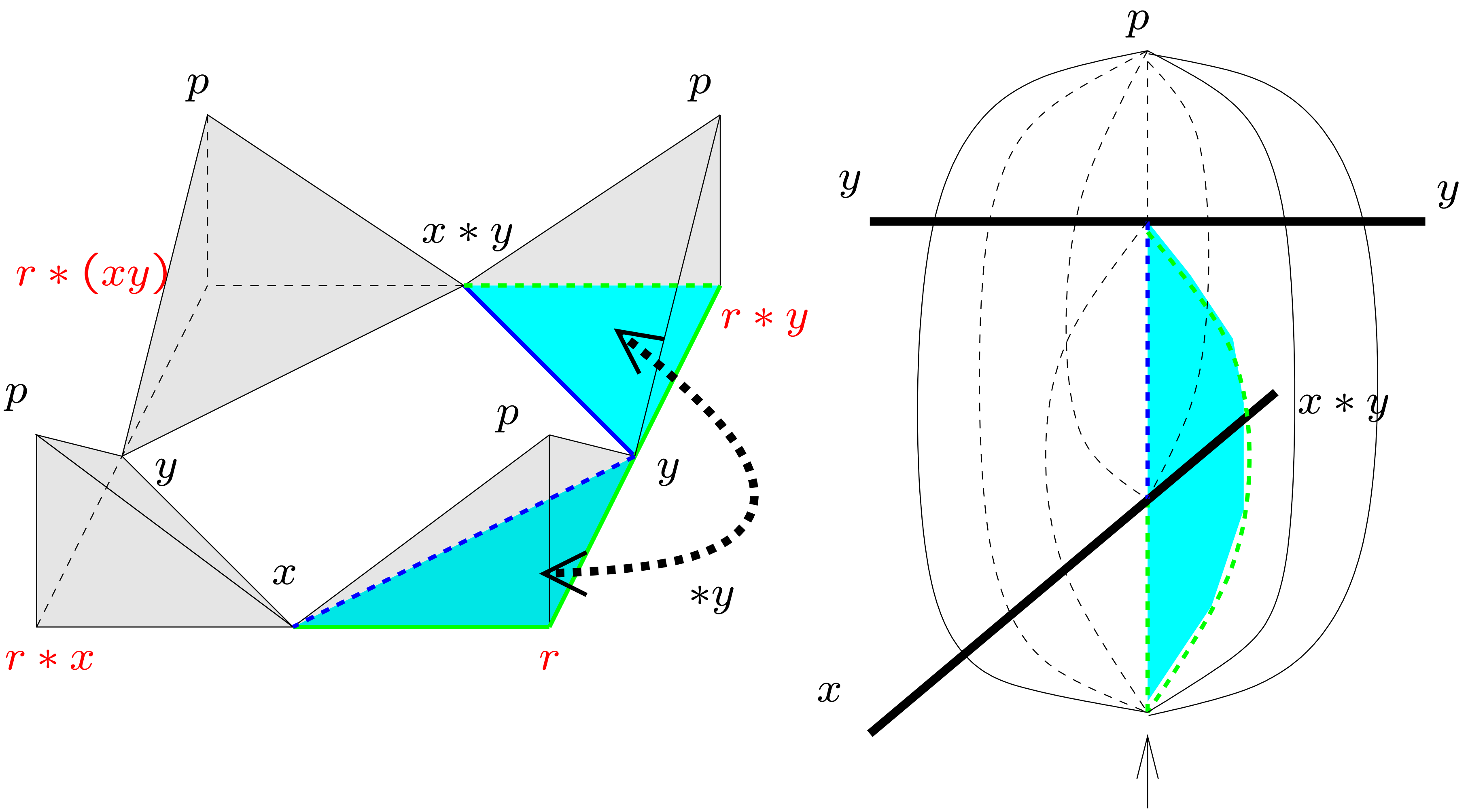

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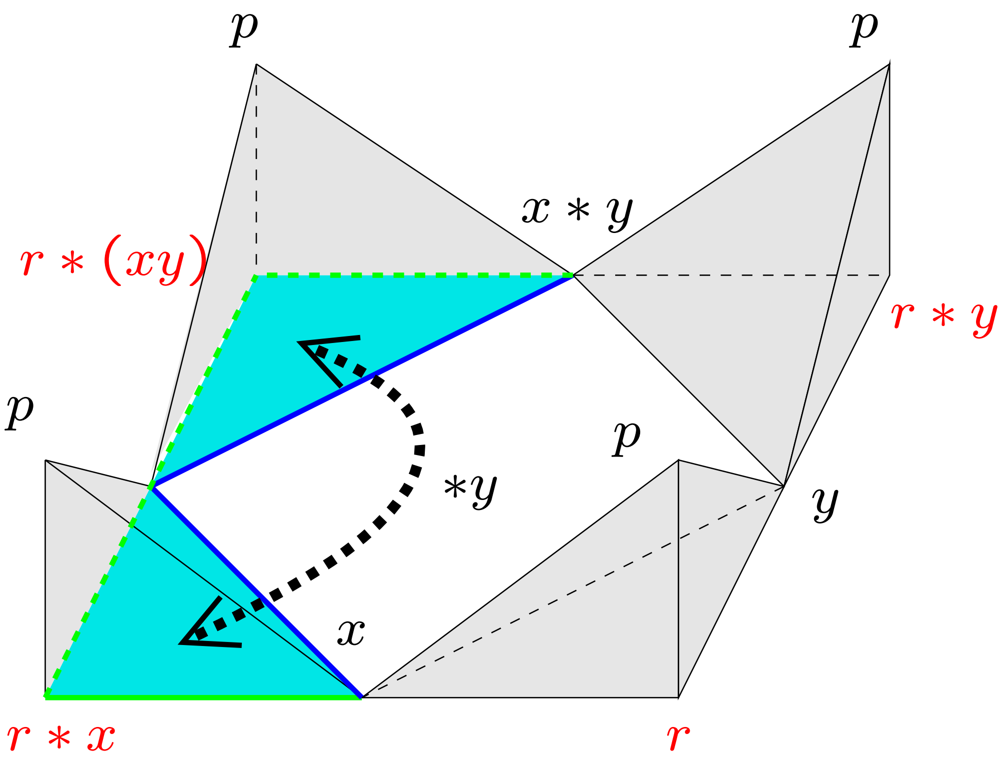

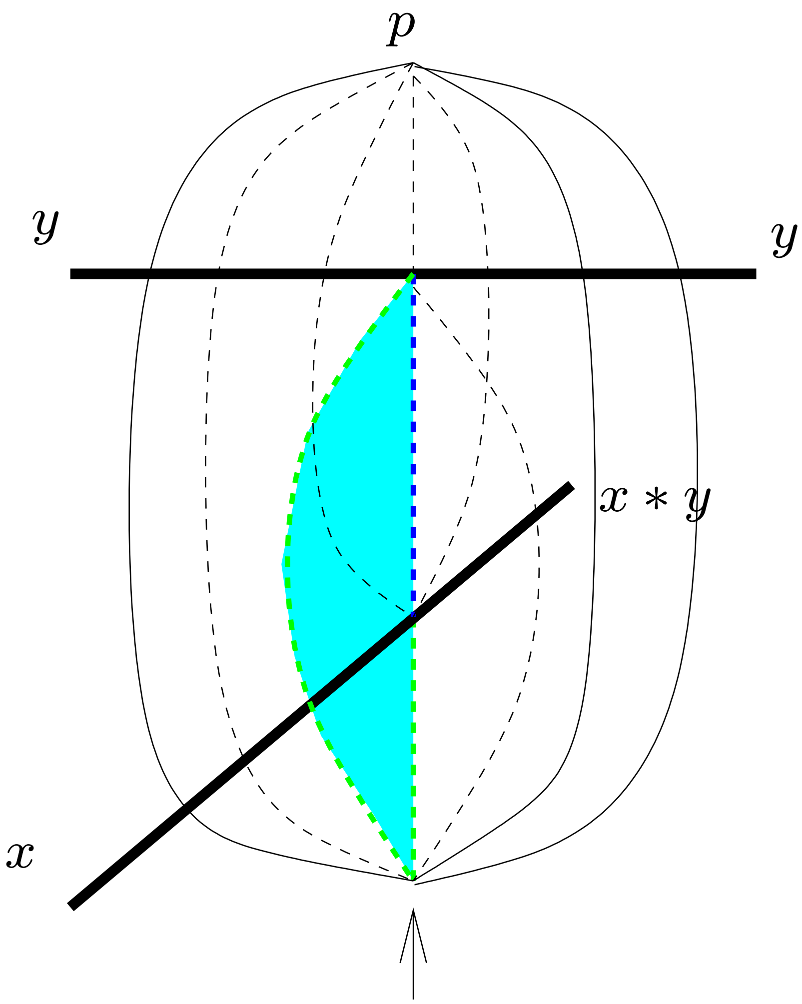

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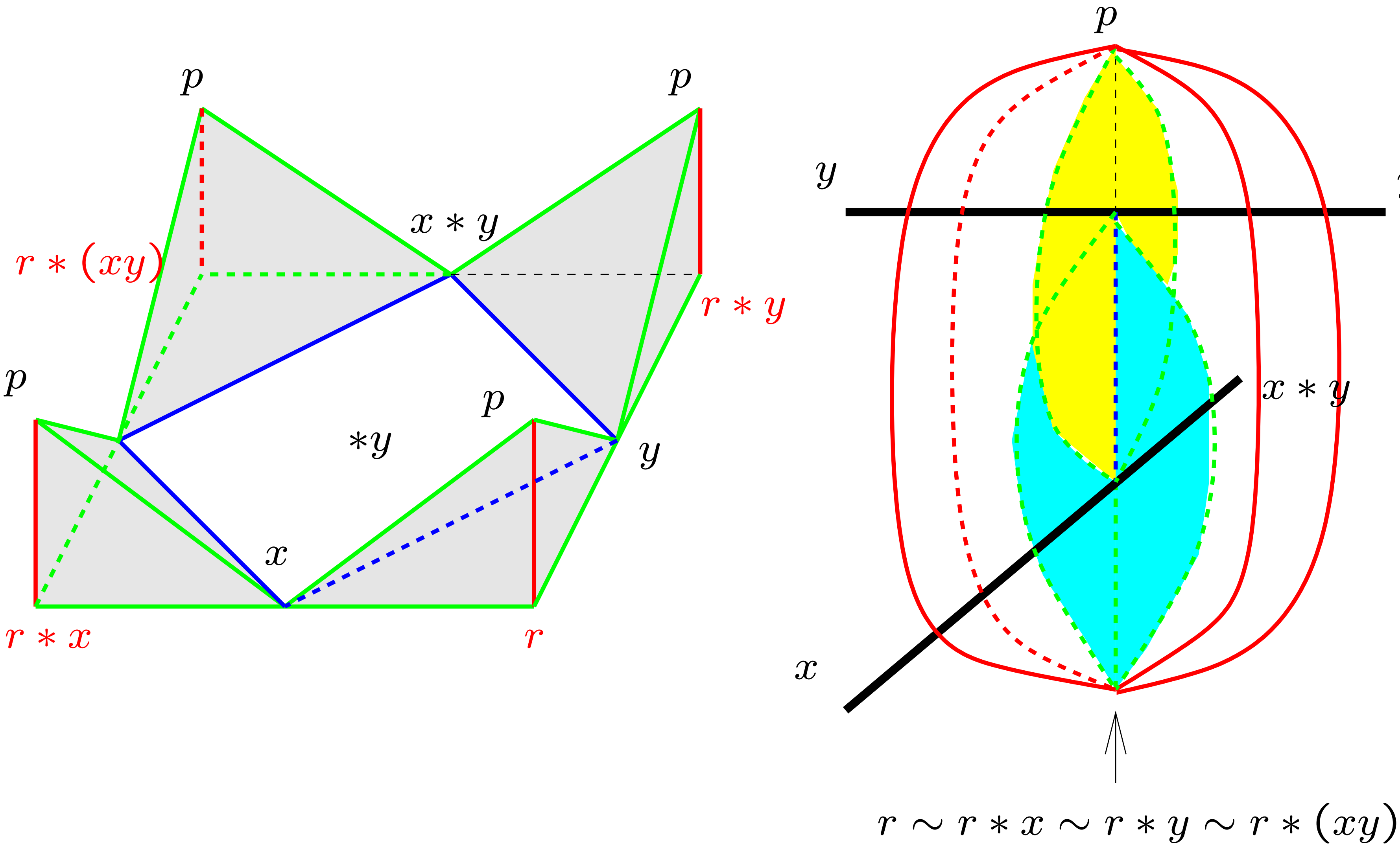


We obtain a triangulation of the knot complement.

The map φ induces a homomorphism

$$H_n^R(X;\mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X).$$

So we can construct a quandle cocycle from a cocycle of $H_{n+1}^{\Delta}(X)$. If we have a function f from X^{k+1} to some abelian group A satifying

1. $\sum_i (-1)^i f(x_0,\ldots,\widehat{x_i},\ldots,x_{k+1}) = 0$ and
2. $f(x_0 * y,\ldots,x_k * y) = f(x_0,\ldots,x_k)$ and
3. $f(x_0,\ldots,x_k) = 0$ if $x_i = x_{i+1}$ for some i,

then f gives a cocycle of $H_k^{\Delta}(X)$ and a cocycle of $H_{k-1}^{Q}(X;\mathbb{Z}[X])$.

If X has a 'geometric structure', we can construct a cocycle for $H_k^{\Delta}(X)$.

Let $\mathcal{P}_n$ be the quandle formed by parabolic elements of $\mathrm{Isom}^+(\mathbb{H}^n)$. For $x \in \mathcal{P}_n$, let $(x)_\infty$ be the unique fixed point at infinity $\partial\overline{\mathbb{H}^n}$ of x. The function $(\mathcal{P}_n)^{n+1} \to \mathbb{R}$ defined by

$$(x_0, x_1, \ldots, x_n) \mapsto \mathrm{Vol}(\mathrm{ConvHull}((x_0)_\infty, (x_1)_\infty, \ldots, (x_n)_\infty))$$

satisfies the previous three conditions.

Thm (Inoue-K.) *The n-dimensional hyperbolic volume is a quandle cocycle of $\mathcal{P}_n$.*

We further study three dimensional case. In this case, Chern-Simons invariant is also a quandle cocycle.

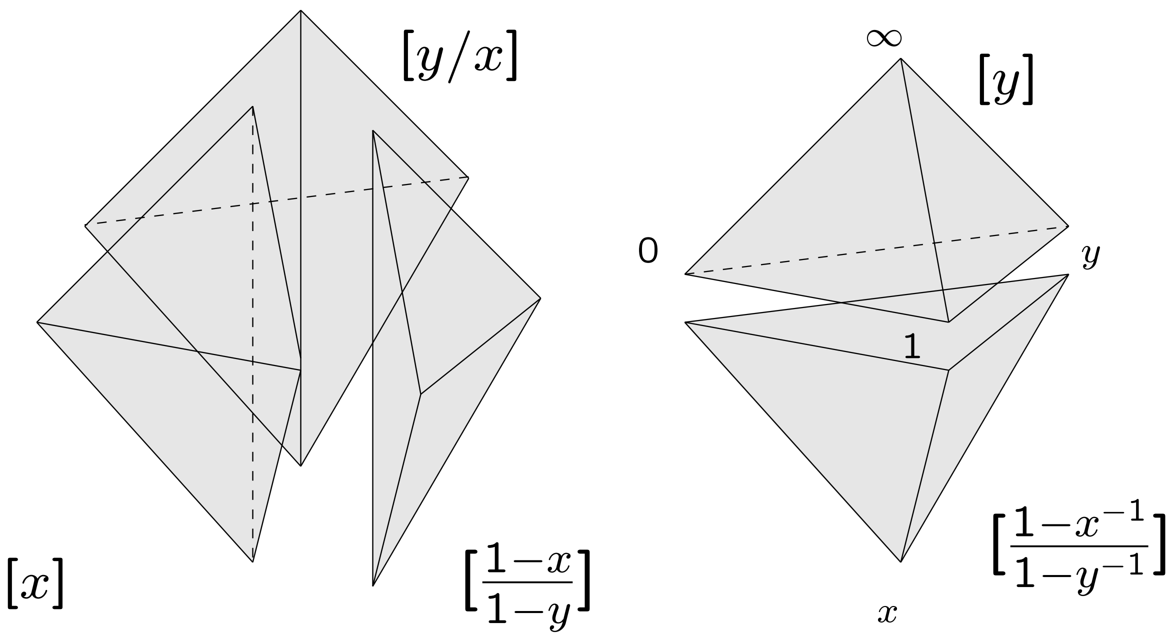
We will construct a map from $H_3^{\Delta}(\mathcal{P})$ to the extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$ along with the work of Dupont and Zickert.

Bloch group

Recall that an ideal tetrahedron in $\mathbb{H}^3$ is parametrized by $\mathbb{C} \setminus \{0, 1\}$. Let $\mathcal{P}(\mathbb{C})$ be the abelian group generated by $\mathbb{C} \setminus \{0, 1\}$ and factored by the following *five term relation*:

$$[x] - [y] + [y/z] - [\frac{1 - x^{-1}}{1 - y^{-1}}] + [\frac{1 - x}{1 - y}] = 0$$

The Bloch group $\mathcal{B}(\mathbb{C})$ is the kernel of the map $\mathcal{P}(\mathbb{C}) \to \mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^*$: $[z] \mapsto z \wedge_{\mathbb{Z}} (1 - z)$.



Extended Bloch group

The extended pre-Bloch group $\widehat{\mathcal{P}}(\mathbb{C})$ is, in some sense, a universal abelian cover of $\mathcal{P}(\mathbb{C})$. $\widehat{\mathcal{P}}(\mathbb{C})$ is generated by the element $[z; p, q]$ with $z \in \mathbb{C} \setminus \{0, 1\}$ and $p, q \in \mathbb{Z}$. The integers p, q represents branches at 0 and 1 respectively. $\widehat{\mathcal{P}}(\mathbb{C})$ is the quotient by *lifted five term relation*.

We can define a map $\widehat{\mathcal{P}}(\mathbb{C}) \to \mathbb{C} \wedge_{\mathbb{Z}} \mathbb{C}$. The kernel of this map is the *extended Bloch group* $\hat{\mathcal{B}}(\mathbb{C})$.

Neumann defined the extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$ and showed that $\widehat{\mathcal{B}}(\mathbb{C}) \cong H_3(\mathrm{BPSL}(2,\mathbb{C})^\delta;\mathbb{Z})$. He also defined the Rogers' dilogarithmic function $R : \widehat{\mathcal{B}}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Z}$.

$$R(z;p,q) = \mathcal{R}(z) + \frac{\pi i}{2}\left(q\mathrm{Log}(z) - p\mathrm{Log}\left(\frac{1}{1-z}\right)\right) - \frac{\pi^2}{6},$$
$$\mathcal{R}(z) = -\int_0^z \frac{\mathrm{Log}(1-t)}{t}dt + \frac{1}{2}\mathrm{Log}(z)\mathrm{Log}(1-z)$$

When a closed hyperbolic 3-manifold M is given, the fundamental class $[M]$ defines an element of $H_3(\mathrm{BPSL}(2,\mathbb{C})^\delta;\mathbb{Z})$. Under the isomorphism, we obtained an element of $\widehat{\mathcal{B}}(\mathbb{C})$. Neumann showed that the image of this element by R is equal to $i(\mathrm{Vol} + i\mathrm{CS})$.

Dupont and Zickert's work

Let $C_n(\mathbb{C}^2) = \text{span}_{\mathbb{Z}}\{(v_0, \ldots, v_n) | v_i \in \mathbb{C}^2 \setminus \{0\}\}$ and define the boundary operator of $C_n(\mathbb{C}^2)$ by

$$\partial(v_0, \ldots v_n) = \sum_{i=0}^{n} (-1)^i (v_0, \ldots, \widehat{v_i}, \ldots, v_n).$$

Thm (Dupont-Zickert) *There is an explicit map $C_3(\mathbb{C}^2) \to \widehat{\mathcal{P}}(\mathbb{C})$ which induces*

$$H_3(C_*(\mathbb{C}^2)_{\mathrm{PSL}(2,\mathbb{C})}) \to \widehat{\mathcal{B}}(\mathbb{C})$$

Remark *In their paper, they studied for* $\mathrm{SL}(2,\mathbb{C})$ *not* $\mathrm{PSL}(2,\mathbb{C})$.

Since $\mathcal{P} \cong (\mathbb{C}^2 \setminus \{0\})/\pm$, $C_*^{\Delta}(\mathcal{P})$ is nearly equal to $C_*(\mathbb{C}^2)$. So we can "construct" a map from $H_3^{\Delta}(\mathcal{P}) \to \widehat{\mathcal{B}}(\mathbb{C})$.

Thm (Inoue-K.) *There is a homomorphism*

$$H_2^Q(\mathcal{P}; \mathbb{Z}[\mathcal{P}]) \to \widehat{\mathcal{B}}(\mathbb{C}).$$

The image of $[C(\mathcal{S})]$ *by this map gives the extended Bloch invariant of the parabolic representation.*

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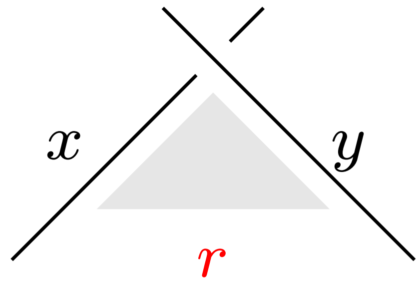
The image of $[C(\mathcal{S})]$ *by this map gives the extended Bloch invariant of the parabolic representation.*

Our work is based on the quandle homology theory, but we do not have to use it for actual calculation.

Fix an element p_0 of $\mathbb{C}^2 \setminus \{0\}$.

At a corner colored by

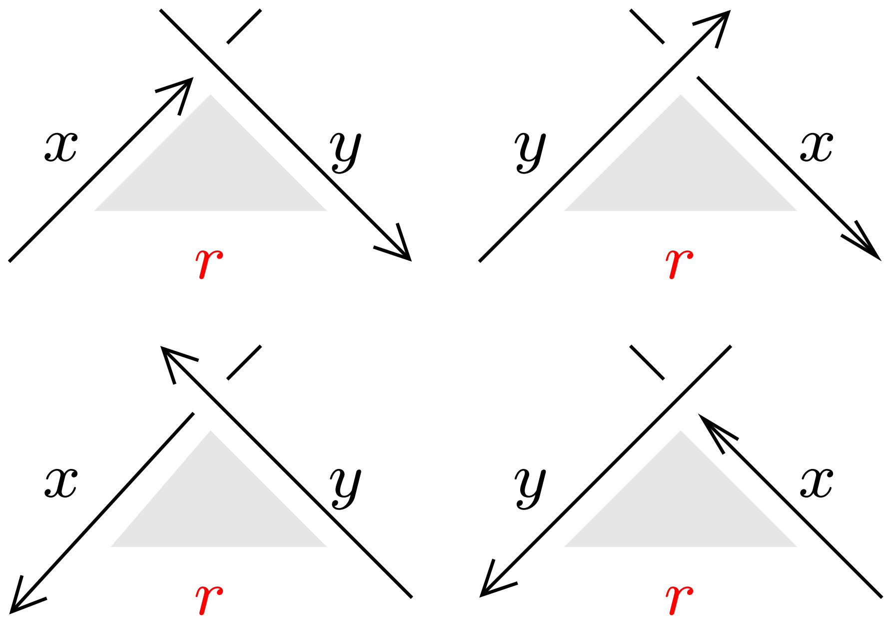


($x \leftrightarrow$ under arc, $y \leftrightarrow$ over arc), we let

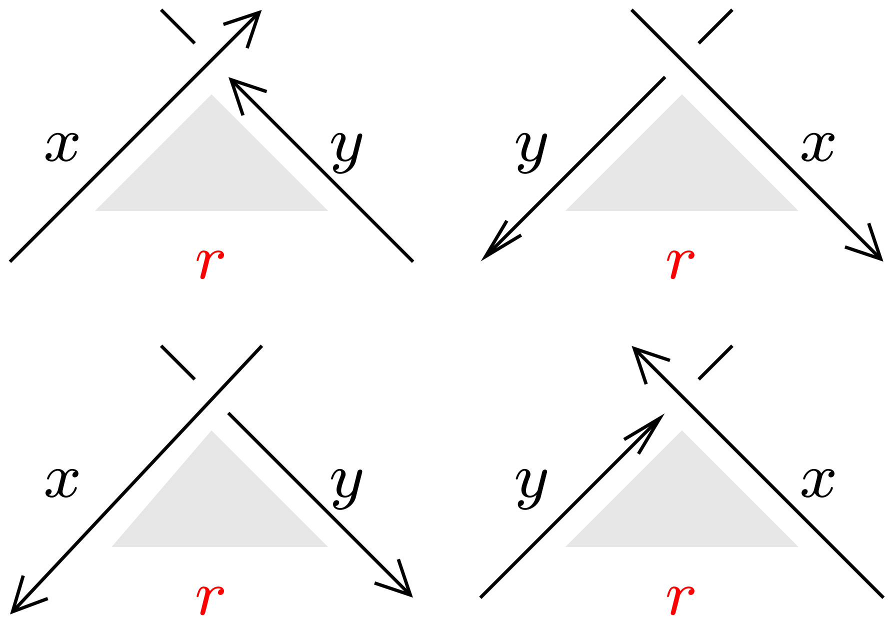
$$
\begin{aligned}
z =& \frac{\det(p_0, y)\det(r, x)}{\det(r, y)\det(p_0, x)} \\
p\pi i =& \mathrm{Log}(\det(p_0, y)) + \mathrm{Log}(\det(r, x)) \\
& - \mathrm{Log}(\det(r, y)) - \mathrm{Log}(\det(p_0, x)) - \mathrm{Log}(z) \\
q\pi i =& \mathrm{Log}(\det(p_0, x)) + \mathrm{Log}(\det(r, y)) \\
& - \mathrm{Log}(\det(p_0, r)) - \mathrm{Log}(\det(x, y)) - \mathrm{Log}(\frac{1}{1-z})
\end{aligned}
$$

where $\mathrm{Log}(z) = \log|z| + i\arg(z)$ $(-\pi < \arg(z) \leq \pi)$

Then define the sign in the following rule:



and



$+[z; p, q]$

(in-out or out-in)

$-[z; p, q]$

(in-in or out-out)

Thm (Inoue-K.)

$$\sum_{c:\text{corners}} \varepsilon_c [z_c; p_c, q_c] \in \widehat{\mathcal{B}}(\mathbb{C})$$

is the extended Bloch invariant.

Let $R : \widehat{\mathcal{B}}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Z}$ be the Rogers dilogarithmic function defined by Neumann. When the arc coloring corresponding to the faithful discrete representation of a hyperbolic knot K, then we have

$$\sum_{c:\text{corners}} \varepsilon_c R(z_c; p_c, q_c) = i(\text{Vol}(S^3 \setminus K) + i\text{CS}(S^3 \setminus K)).$$

Application to dihedral quandles

Let $R_p = \{0, 1, \ldots, p-1\} (= \mathbb{F}_p)$ and $x * y = 2y - x \mod p$ for $x, y \in R_p$. This is called the *dihedral quandle*.

Let f be a group 3-cocycle of $\mathbb{Z}/p$ defined by

$$f : [a|b|c] \mapsto \overline{a}(\overline{b+c} - \overline{b} - \overline{c}) \quad \text{mod } p$$

where $\overline{a}$ is a lift to $\mathbb{Z}$. In homogeneous notation, we have

$$\tilde{f} : (w, x, y, z) \mapsto \overline{x-w}(\overline{y-x} + \overline{z-y} - \overline{\overline{y-x} + \overline{z-y}}).$$

Let $g(w, x, y, z) = \tilde{f}(w, x, y, z) + \tilde{f}(-w, -x, -y, -z)$ for $w, x, y, z \in R_p$.

The function g satisfies the following properties:

1. $\sum_i (-1)^i g(x_0, \ldots, \widehat{x_i}, \ldots, x_4) = 0$,
2. $g(x_0 * y, \ldots, x_3 * y) = g(x_0, \ldots, x_3)$,
3. $g(x_0, \ldots, x_3) = 0$ if $x_i = x_{i+1}$.

By our construction, this gives a cocycle on $H_2^Q(R_p; \mathbb{Z}[R_p])$. Since there exists a map $H_2^Q(R_p; \mathbb{Z}[R_p]) \to H_3^Q(R_p; \mathbb{Z})$, g gives a quandle 3-cocycle in $H_Q^3(R_p; \mathbb{Z}/p)$.

On the other hand, there is a non-trivial quandle 3-cocycle of R_p given by

$$(x, y, z) \mapsto (x-y)((2z-y)^p + y^p - 2z^p)/p \mod p$$

This is called the *Mochizuki's 3-cocycle*. Our cocycle g must be a constant multiple of the Mochizuki's 3-cocycle up to coboundary, because $\dim_{\mathbb{F}_p} H^3_Q(R_p; \mathbb{Z}/p) = 1$. By computer calculation, we have:

p	(Our cocycle) $= c \cdot$ (Mochizuki's cocycle)
3	1
5	4
7	4
11	4
$\vdots$	$\vdots$

Thank you