

# On generic 4-fold covers of algebraic surfaces

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## Notation

In this poster, all varieties are defined over the field of complex numbers  $\mathbf{C}$ .

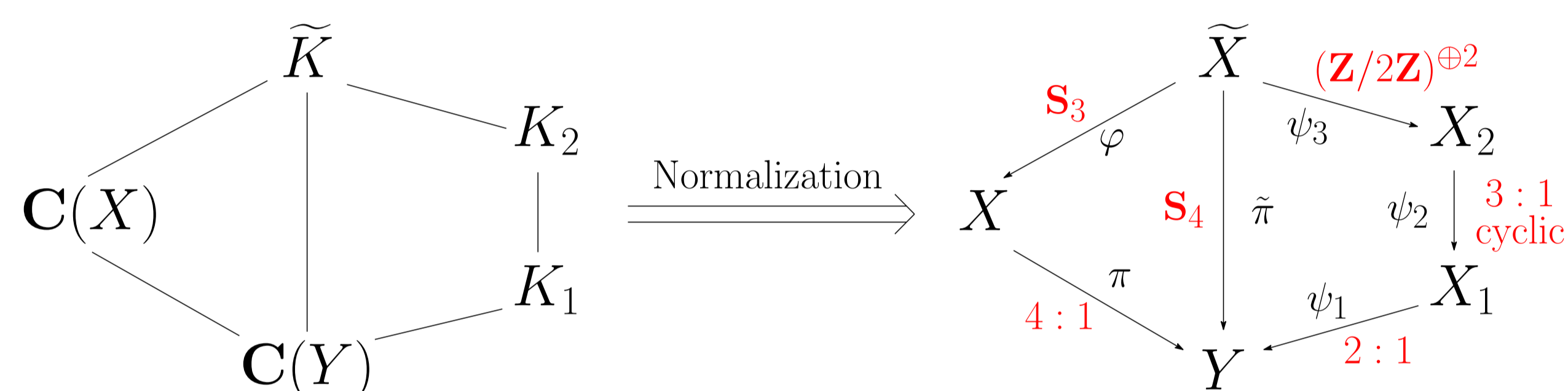
- $Y$  : a **smooth** projective surface.
- $X$  : a normal projective surface.
- $\pi : X \rightarrow Y$  : a 4-fold cover (i.e. a finite surjective morphism with  $\deg \pi = 4$ ).
- $\Delta(\pi)$  : the branch locus of  $\pi$ . ( $\Delta(\pi) \neq \emptyset$ , in general.)
- $\mathbf{C}(X), \mathbf{C}(Y)$  : the rational function fields of  $X$  and  $Y$ , respectively.
- $K_Y, e(Y)$  : a canonical divisor, topologically Euler number of  $Y$ , respectively.
- $S_i$  : the symmetric group of degree  $i$ .

## 4-fold covers and $S_4$ -covers

$\mathbf{C}(X)$  is a finite extension of  $\mathbf{C}(Y)$  with  $[\mathbf{C}(X) : \mathbf{C}(Y)] = 4$ . Hence there is  $z \in \mathbf{C}(X)$  such that its minimal polynomial over  $\mathbf{C}(Y)$  is

$$f := z^4 + g_1 z^2 + g_2 z + g_3.$$

Based on Lagrange's method of solving  $f = 0$ , we canonically obtain the following diagram:



$$K_1 := \mathbf{C}(Y)(\theta_1), \quad K_2 := K_1(\theta_{2,1}) = K_1(\theta_{2,2}),$$

$$\tilde{K} := K_2(\theta_{3,1}, \theta_{3,2}) = K_2(\theta_{3,2}, \theta_{3,3}) = K_2(\theta_{3,3}, \theta_{3,1}),$$

where  $\theta_1, \theta_{2,i}$  and  $\theta_{3,j}$  are elements such that

$$\theta_1^2 = \delta_f, \quad \theta_{2,1}^3 = -\frac{h_2}{2} + \theta_1, \quad \theta_{2,2}^3 = -\frac{h_2}{2} - \theta_1,$$

$$\theta_{3,1}^2 = \theta_{2,1} + \theta_{2,2}, \quad \theta_{3,2}^2 = \zeta_3 \theta_{2,1} + \zeta_3^2 \theta_{2,2}, \quad \theta_{3,3}^2 = \zeta_3^2 \theta_{2,1} + \zeta_3 \theta_{2,2}.$$

Here  $h_1, h_2 \in \mathbf{C}(Y)$  and  $\zeta_3 = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$ .

We define four distinct algebraic subsets of  $\Delta(\tilde{\pi})$  as follows:

$$A := \overline{\{P \in Y \mid \#\psi_1^{-1}(P) = 1, \#(\psi_1 \circ \psi_2)^{-1}(P) = 3, \#\tilde{\pi}^{-1}(P) = 12\}},$$

$$B := \overline{\{P \in Y \mid \#\psi_1^{-1}(P) = 1, \#(\psi_1 \circ \psi_2)^{-1}(P) = 3, \#\tilde{\pi}^{-1}(P) = 6, P \notin A\}},$$

$$C := \overline{\{P \in Y \mid \#\psi_1^{-1}(P) = 2, \#(\psi_1 \circ \psi_2)^{-1}(P) = 2, \#\tilde{\pi}^{-1}(P) = 8\}},$$

$$D := \overline{\{P \in Y \mid \#\psi_1^{-1}(P) = 2, \#(\psi_1 \circ \psi_2)^{-1}(P) = 6, \#\tilde{\pi}^{-1}(P) = 12\}}.$$

Note that  $\Delta(\pi) = A + B + C + D$ .

## Generic 4-fold covers

**Def.** Let  $\pi : X \rightarrow Y$  be a 4-fold cover.

$$\pi : \text{generic} \stackrel{\text{def}}{\iff} \Delta(\pi) = A \quad (\text{i.e. } B = C = D = 0)$$

Let  $\pi : X \rightarrow Y$  be a generic 4-fold cover, which  $\Delta(\pi) \subset Y$  is a reduced curve with at worst **simple singularities**. Let

- $\gamma : Z_1 \rightarrow X_1$  : the minimal resolution of  $X_1$ ,
- $\hat{\psi}_2 : Z_2 \rightarrow Z_1$  : the  $K_2$ -normalization of  $Z_1$ ,
- $\hat{\psi}_3 : \tilde{Z} \rightarrow Z_2$  : the  $\tilde{K}$ -normalization of  $Z_2$ .

We put

$\delta_1$  : the number of connected components of  $\Delta(\hat{\psi}_2) \subset Z_1$ ,

$\delta_2$  : the number of connected components of  $\hat{\psi}_2(\Delta(\hat{\psi}_3)) \subset Z_1$ .

Then our **main** theorem is the following:

**Main Thm.** Let  $\pi : X \rightarrow Y$  be a generic 4-fold cover as above, and  $\bar{X}$  the minimal resolution of  $X$ . Then

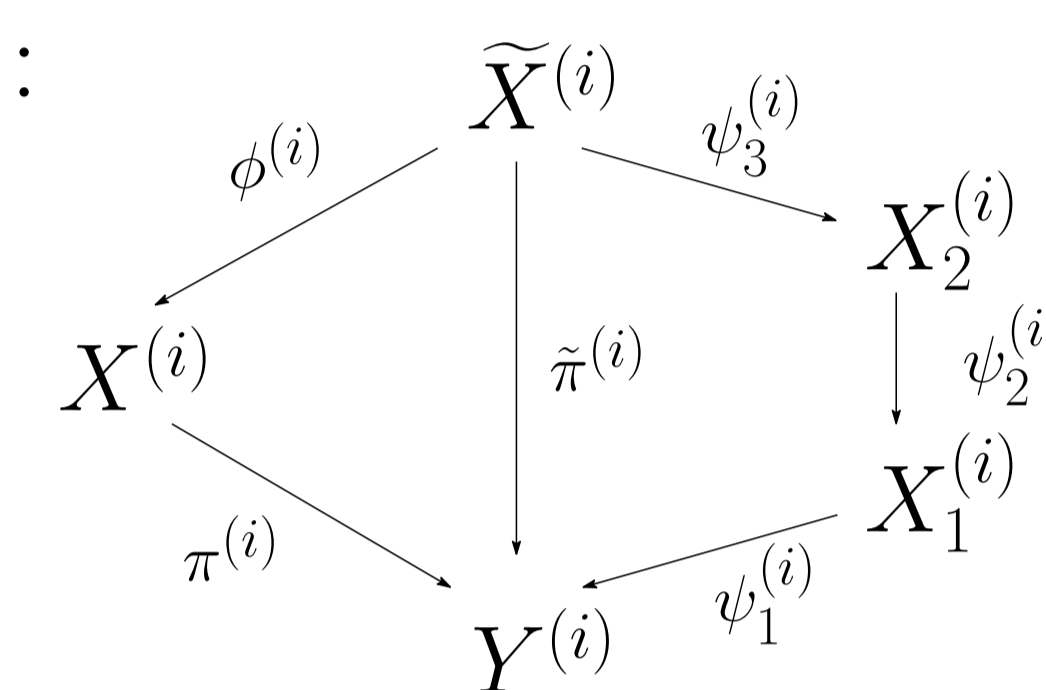
$$K_{\bar{X}}^2 = 4K_Y^2 + 2A.K_Y + \frac{1}{2}A^2 - \delta_1 - \delta_2,$$

$$e(\bar{X}) = 4e(Y) + A.K_Y + A^2 - 3\delta_1 - 2\delta_2.$$

## Resolution of 4-fold covers

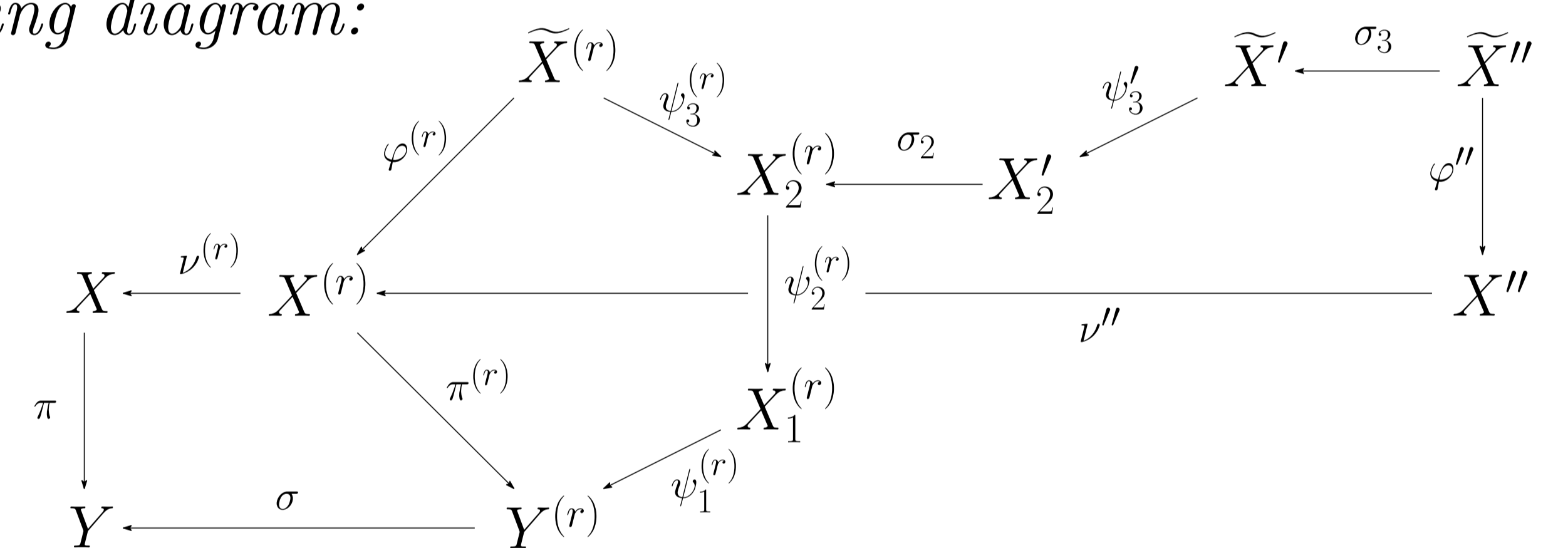
We summarise resolution of any 4-fold covers, which is used in our proof of Main Theorem.

For any integer  $i > 0$ , let  $\sigma^{(i)} : Y^{(i)} \rightarrow Y^{(i-1)}$  be a blowing-up of  $Y^{(i-1)}$  ( $Y^{(0)} := Y$ ). Then we obtain the following diagram by normalizing  $Y^{(i)}$ :



Let  $A^{(i)}, B^{(i)}, C^{(i)}$  and  $D^{(i)}$  be the reduced divisors as  $A, B, C$  and  $D$ .

**Thm.** Let  $\pi : X \rightarrow Y$  be a 4-fold cover. Then there is a resolution  $X''$  of  $X$  such that it is constructed by the following diagram:



where  $\sigma := \sigma^{(1)} \circ \dots \circ \sigma^{(r)}$  is blowing-ups, and  $\pi^{(r)}$  is a certain 4-fold cover. If  $\pi$  is generic, then

$$K_{X''}^2 = 4K_Y^2 + 2A.K_Y + \frac{1}{2}A^2 - \sum_{i=0}^{r-1} \frac{a_i(a_i - 4)}{2} - \sum_{i=0}^{r-1} \frac{3b_i(3b_i - 8)}{4}$$

$$- \sum_{i=0}^{r-1} \frac{4c_i(c_i - 3)}{3} - \sum_{i=0}^{r-1} d_i(d_i - 4) - \sum_{i=0}^{r-1} d_i(a_i + 3b_i) - 4r,$$

$$e(X'') = 4e(Y) + A.K_Y + A^2 - \sum_{i=0}^{r-1} a_i(a_i - 1) - \sum_{i=0}^{r-1} 3b_i(b_i - 1)$$

$$- \sum_{i=0}^{r-1} 2c_i(c_i - 1) - \sum_{i=0}^{r-1} 2d_i(d_i - 1) - \sum_{i=0}^{r-1} d_i(2a_i + 3b_i) + 4r - 3s,$$

where  $a_i, b_i, c_i, d_i$  and  $s$  are as follows:

For any reduced divisor  $H$  and  $P \in Y$ , we denote the multiplicity of  $H$  at  $P$  by  $\mu_P(H)$ . Let  $P^{(i)}$  be the center of the blowing-up  $\sigma^{(i+1)}$  for  $i \geq 0$ .

Then we put  $a_i, b_i, c_i$  and  $d_i$  as follows;

$$a_i := \begin{cases} \mu_{P^{(i-1)}}(A^{(i-1)}) - 1 & (\text{if } E^{(i)} \subset A^{(i)}) \\ \mu_{P^{(i-1)}}(A^{(i-1)}) & (\text{otherwise}) \end{cases}$$

$$b_i := \begin{cases} \mu_{P^{(i-1)}}(B^{(i-1)}) - 1 & (\text{if } E^{(i)} \subset B^{(i)}) \\ \mu_{P^{(i-1)}}(B^{(i-1)}) & (\text{otherwise}) \end{cases}$$

$$c_i := \begin{cases} \mu_{P^{(i-1)}}(C^{(i-1)}) - 1 & (\text{if } E^{(i)} \subset C^{(i)}) \\ \mu_{P^{(i-1)}}(C^{(i-1)}) & (\text{otherwise}) \end{cases}$$

$$d_i := \begin{cases} \mu_{P^{(i-1)}}(D^{(i-1)}) - 1 & (\text{if } E^{(i)} \subset D^{(i)}) \\ \mu_{P^{(i-1)}}(D^{(i-1)}) & (\text{otherwise}) \end{cases}$$

Let  $s$  be the number of singular points of  $D^{(r)}$ .