On generic 4-fold covers of algebraic surfaces

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Notation

In this poster, all varieties are defined over the field of complex numbers $\mathbf{C}$.

Y	: a **smooth** projective surface.
X	: a normal projective surface.
$\pi : X \to Y$	: a 4-fold cover (i.e. a finite surjective morphism with $\deg \pi = 4$).
$\Delta(\pi)$	: the branch locus of π. ($\Delta(\pi) \neq \emptyset$, in general.)
$\mathbf{C}(X)$, $\mathbf{C}(Y)$	: the rational function fields of X and Y, respectively.
K_Y, $e(Y)$	: a canonical divisor, topologically Euler number of Y, respectively.
$\mathbf{S}_i$	: the symmetric group of degree i.

4-fold covers and $\mathbf{S}_4$-covers

$\mathbf{C}(X)$ is a finite extension of $\mathbf{C}(Y)$ with $[\mathbf{C}(X) : \mathbf{C}(Y)] = 4$. Hence there is $z \in \mathbf{C}(X)$ such that its minimal polynomial over $\mathbf{C}(Y)$ is

$$f := z^4 + g_1 z^2 + g_2 z + g_3.$$

Based on Lagrange's method of solving $f = 0$, we canonically obtain the following diagram:

$$\begin{tikzcd}
 & \widetilde{K} \arrow[dl, dash] \arrow[r, dash] \arrow[dd, dash] & K_2 \arrow[d, dash] \\
\mathbf{C}(X) \arrow[dr, dash] & & K_1 \arrow[dl, dash] \\
 & \mathbf{C}(Y) &
\end{tikzcd}
\quad \xRightarrow{\text{Normalization}} \quad
\begin{tikzcd}
 & \widetilde{X} \arrow[dl, "\mathbf{S}_3"', "\varphi"] \arrow[r, "(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}", "\psi_3"'] \arrow[dd, "\mathbf{S}_4"', "\tilde{\pi}"] & X_2 \arrow[d, "\psi_2"', "3:1\ \text{cyclic}"] \\
X \arrow[dr, "4:1"', "\pi"] & & X_1 \arrow[dl, "\psi_1"', "2:1"] \\
 & Y &
\end{tikzcd}$$

$$K_1 := \mathbf{C}(Y)(\theta_1), \qquad K_2 := K_1(\theta_{2,1}) = K_1(\theta_{2,2}),$$
$$\widetilde{K} := K_2(\theta_{3,1}, \theta_{3,2}) = K_2(\theta_{3,2}, \theta_{3,3}) = K_2(\theta_{3,3}, \theta_{3,1}),$$

where θ_1, $\theta_{2,i}$ and $\theta_{3,j}$ are elements such that

$$\theta_1^2 = \delta_f, \qquad \theta_{2,1}^3 = -\frac{h_2}{2} + \theta_1, \qquad \theta_{2,2}^3 = -\frac{h_2}{2} - \theta_1,$$
$$\theta_{3,1}^2 = \theta_{2,1} + \theta_{2,2}, \qquad \theta_{3,2}^2 = \zeta_3\theta_{2,1} + \zeta_3^2\theta_{2,2}, \qquad \theta_{3,3}^2 = \zeta_3^2\theta_{2,1} + \zeta_3\theta_{2,2}.$$

Here $h_1,\ h_2 \in \mathbf{C}(Y)$ and $\zeta_3 = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$.

We define four distinct algebraic subsets of $\Delta(\tilde{\pi})$ as follows:

$$A := \overline{\{P \in Y \mid \sharp\psi_1^{-1}(P) = 1,\ \sharp(\psi_1 \circ \psi_2)^{-1}(P) = 3,\ \sharp\tilde{\pi}^{-1}(P) = 12\}},$$
$$B := \overline{\{P \in Y \mid \sharp\psi_1^{-1}(P) = 1,\ \sharp(\psi_1 \circ \psi_2)^{-1}(P) = 3,\ \sharp\tilde{\pi}^{-1}(P) = 6,\ P \notin A\}},$$
$$C := \overline{\{P \in Y \mid \sharp\psi_1^{-1}(P) = 2,\ \sharp(\psi_1 \circ \psi_2)^{-1}(P) = 2,\ \sharp\tilde{\pi}^{-1}(P) = 8\}},$$
$$D := \overline{\{P \in Y \mid \sharp\psi_1^{-1}(P) = 2,\ \sharp(\psi_1 \circ \psi_2)^{-1}(P) = 6,\ \sharp\tilde{\pi}^{-1}(P) = 12\}}.$$

Note that $\Delta(\pi) = A + B + C + D$.

Generic 4-fold covers

Def. Let $\pi : X \to Y$ be a 4-fold cover.

$$\pi : \textit{generic} \overset{\text{def}}{\iff} \Delta(\pi) = A \quad (\text{i.e. } B = C = D = 0)$$

Let $\pi : X \to Y$ be a generic 4-fold cover, which $\Delta(\pi) \subset Y$ is a reduced curve with at worst **simple singularities**. Let

$$\gamma : Z_1 \to X_1 \ : \text{the minimal resolution of } X_1,$$
$$\hat{\psi}_2 : Z_2 \to Z_1 \ : \text{the } K_2\text{-normalization of } Z_1,$$
$$\hat{\psi}_3 : \widetilde{Z} \to Z_2 \ : \text{the } \widetilde{K}\text{-normalization of } Z_2.$$

We put

δ_1 : the number of connected components of $\Delta(\hat{\psi}_2) \subset Z_1$,

δ_2 : the number of connected components of $\hat{\psi}_2(\Delta(\hat{\psi}_3)) \subset Z_1$.

Then our **main** theorem is the following:

Main Thm. *Let $\pi : X \to Y$ be a generic 4-fold cover as above, and $\overline{X}$ the minimal resolution of X. Then*

$$K_{\overline{X}}^2 = 4K_Y^2 + 2A.K_Y + \frac{1}{2}A^2 - \delta_1 - \delta_2,$$
$$e(\overline{X}) = 4e(Y) + A.K_Y + A^2 - 3\delta_1 - 2\delta_2.$$

Resolution of 4-fold covers

We summatise resolution of any 4-fold covers, which is used in our proof of Main Theorem.

For any integer $i > 0$, let $\sigma^{(i)} : Y^{(i)} \to Y^{(i-1)}$ be a blowing-up of $Y^{(i-1)}$ ($Y^{(0)} := Y$). Then we obtain the following diagram by normalizing $Y^{(i)}$:

$$\begin{tikzcd}
 & \widetilde{X}^{(i)} \arrow[dl, "\phi^{(i)}"'] \arrow[r, "\psi_3^{(i)}"] \arrow[dd, "\tilde{\pi}^{(i)}"] & X_2^{(i)} \arrow[d, "\psi_2^{(i)}"] \\
X^{(i)} \arrow[dr, "\pi^{(i)}"'] & & X_1^{(i)} \arrow[dl, "\psi_1^{(i)}"] \\
 & Y^{(i)} &
\end{tikzcd}$$

Let $A^{(i)}$, $B^{(i)}$, $C^{(i)}$ and $D^{(i)}$ be the reduced divisors as A, B, C and D.

Thm. *Let $\pi : X \to Y$ be a 4-fold cover. Then there is a resolution X'' of X such that it is constructed by the following diagram:*

$$\begin{tikzcd}
 & & \widetilde{X}^{(r)} \arrow[ddl, "\varphi^{(r)}"'] \arrow[dr, "\psi_3^{(r)}"] & & & \widetilde{X}' \arrow[dl, "\psi_3'"'] & \widetilde{X}'' \arrow[l, "\sigma_3"'] \arrow[dd, "\varphi''"] \\
 & & & X_2^{(r)} \arrow[dd, "\psi_2^{(r)}"] & X_2' \arrow[l, "\sigma_2"'] & & \\
X & X^{(r)} \arrow[l, "\nu^{(r)}"'] \arrow[ddr, "\pi^{(r)}"] & & & & & X'' \arrow[lllll, "\nu''"] \\
 & & & X_1^{(r)} \arrow[dl, "\psi_1^{(r)}"] & & & \\
Y \arrow[from=uu, "\pi"'] & & Y^{(r)} \arrow[ll, "\sigma"'] & & & &
\end{tikzcd}$$

where $\sigma := \sigma^{(1)} \circ \cdots \circ \sigma^{(r)}$ is blowing-ups, and $\pi^{(r)}$ is a certain 4-fold cover. If π is generic, then

$$K_{X''}^2 = 4K_Y^2 + 2A.K_Y + \frac{1}{2}A^2 - \sum_{i=0}^{r-1}\frac{a_i(a_i-4)}{2} - \sum_{i=0}^{r-1}\frac{3b_i(3b_i-8)}{4}$$
$$- \sum_{i=0}^{r-1}\frac{4c_i(c_i-3)}{3} - \sum_{i=0}^{r-1} d_i(d_i-4) - \sum_{i=0}^{r-1} d_i(a_i+3b_i) - 4r,$$
$$e(X'') = 4e(Y) + A.K_Y + A^2 - \sum_{i=0}^{r-1} a_i(a_i-1) - \sum_{i=0}^{r-1} 3b_i(b_i-1)$$
$$- \sum_{i=0}^{r-1} 2c_i(c_i-1) - \sum_{i=0}^{r-1} 2d_i(d_i-1) - \sum_{i=0}^{r-1} d_i(2a_i+3b_i) + 4r - 3s,$$

where a_i, b_i, c_i, d_i and s are as follows:

For any reduced divisor H and $P \in Y$, we denote the multiplicity of H at P by $\mu_P(H)$. Let $P^{(i)}$ be the center of the blowing-up $\sigma^{(i+1)}$ for $i \geq 0$.

Then we put a_i, b_i, c_i and d_i as follows;

$$a_i := \begin{cases} \mu_{P^{(i-1)}}(A^{(i-1)}) - 1 & (\textit{if } E^{(i)} \subset A^{(i)}) \\ \mu_{P^{(i-1)}}(A^{(i-1)}) & (\textit{otherwise}) \end{cases}$$
$$b_i := \begin{cases} \mu_{P^{(i-1)}}(B^{(i-1)}) - 1 & (\textit{if } E^{(i)} \subset B^{(i)}) \\ \mu_{P^{(i-1)}}(B^{(i-1)}) & (\textit{otherwise}) \end{cases}$$
$$c_i := \begin{cases} \mu_{P^{(i-1)}}(C^{(i-1)}) - 1 & (\textit{if } E^{(i)} \subset C^{(i)}) \\ \mu_{P^{(i-1)}}(C^{(i-1)}) & (\textit{otherwise}) \end{cases}$$
$$d_i := \begin{cases} \mu_{P^{(i-1)}}(D^{(i-1)}) - 1 & (\textit{if } E^{(i)} \subset D^{(i)}) \\ \mu_{P^{(i-1)}}(D^{(i-1)}) & (\textit{otherwise}) \end{cases}$$

Let s be the number of singular points of $D^{(r)}$.