Recent progress on topology of plane curves: A quick trip

Part V:
Orbifolds and Quasi-projective Groups

Enrique ARTAL BARTOLO

Departamento de Matemáticas, IUMA
Universidad de Zaragoza

Branched Coverings in Tokyo - March 7-10, 2011
1. Statements

2. Orbifolds
## Contents

1. Statements
2. Orbifolds
3. Characteristic varieties of orbifolds
Contents

1 Statements

2 Orbifolds

3 Characteristic varieties of orbifolds

4 Main result and applications
Definition

A quasiprojective group is the fundamental group of a quasiprojective smooth variety.
Definition

A *quasiprojective* group is the fundamental group of a quasiprojective smooth variety.

Theorem (Arapura)

Let $\Sigma$ be an irreducible component of $\Sigma_{G,1}$, $G = \pi_1(X)$, $X$ quasi-projective surface. Then,

In particular, positive dimensional irreducible components are subtori translated by torsion elements.
A *quasiprojective* group is the fundamental group of a quasiprojective smooth variety.

**Theorem (Arapura)**

Let $\Sigma$ be an irreducible component of $\Sigma_{G,1}$, $G = \pi_1(X)$, $X$ quasi-projective surface. Then,

1. If $\dim \Sigma > 0$ then there exists a primitive surjective morphism $\rho : X \to C$, $C$ algebraic curve, and a torsion element $\sigma$ such that $\Sigma = \sigma \rho^*(H^1(C; \mathbb{C}^*))$.

In particular, positive dimensional irreducible components are subtori translated by torsion elements.
Starting point

Definition

A *quasiprojective* group is the fundamental group of a quasiprojective smooth variety.

Theorem (Arapura)

Let $\Sigma$ be an irreducible component of $\Sigma_{G,1}$, $G = \pi_1(X)$, $X$ quasi-projective surface. Then,

1. If $\dim \Sigma > 0$ then there exists a primitive surjective morphism $\rho : X \to C$, $C$ algebraic curve, and a torsion element $\sigma$ such that $\Sigma = \sigma \rho^* (H^1(C; \mathbb{C}^*))$.
2. If $\dim \Sigma = 0$ then $\Sigma$ is unitary.

In particular, positive dimensional irreducible components are subtori translated by torsion elements.
Statements
Orbifolds
Characteristic varieties of orbifolds
Main result and applications

Orbifolds

Example

C a sextic with six cusps on a conic: $C = \{ f^3 - f^2 = 0 \}$. Note that $TH = C^*$ and $\Sigma C, 1 = \{ \zeta_6, \zeta_6^{-1} \}, \Sigma C, 2 = \emptyset$.

Consider the primitive map $\rho: P^2 \setminus C \to P^1 \{ [1:1] \}$ given by $[x:y:z] \mapsto [f^2(x,y,z):f^3(x,y,z)]$. The map is trivial on $\pi_{orb}$.
Example

- $C$ a sextic with six cusps on a conic: $C = \{f_2^3 - f_3^2 = 0\}$. Note that $T_H = \mathbb{C}^*$ and $\Sigma_{C,1} = \{\zeta_6, \zeta_6^{-1}\}$, $\Sigma_{C,2} = \emptyset$. 
Example

- $C$ a sextic with six cusps on a conic: $C = \{f_2^3 - f_3^2 = 0\}$. Note that $\mathbb{T}_H = \mathbb{C}^*$ and $\Sigma_{C,1} = \{\zeta_6, \zeta_6^{-1}\}$, $\Sigma_{C,2} = \emptyset$.

- Consider the primitive map $\rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\}$ given by $[x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2]$. The map is trivial on $\pi_1$. 
Example

- $C$ a sextic with six cusps on a conic: $C = \{f_2^3 - f_3^2 = 0\}$. Note that $T_H = \mathbb{C}^*$ and $\Sigma_{C,1} = \{\zeta_6, \zeta_6^{-1}\}$, $\Sigma_{C,2} = \emptyset$.
- Consider the primitive map $\rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\}$ given by $[x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2]$. The map is trivial on $\pi_1$.

Definition

An orbifold $X_\varphi$ is a quasiprojective Riemann surface $X$ with a function $\varphi : X \to \mathbb{N}$ such that $\text{Sing}(X_\varphi) := \{x \in X \mid \varphi(x) > 1\}$ is a finite set. Assume the following interpretation: the angle of a disk centered at $x$ equals $\frac{2\pi}{\varphi(x)}$. 

Statements
Orbifolds
Characteristic varieties of orbifolds
Main result and applications

Orbifolds

Example

- $C$ a sextic with six cusps on a conic: $C = \{f_2^3 - f_3^2 = 0\}$. Note that $T_H = \mathbb{C}^*$ and $\Sigma_{C,1} = \{\zeta_6, \zeta_6^{-1}\}$, $\Sigma_{C,2} = \emptyset$.
- Consider the primitive map $\rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\}$ given by $[x : y : z] \mapsto [f_2(x,y,z)^3 : f_3(x,y,z)^2]$. The map is trivial on $\pi_1$

Definition

An orbifold $X_\varphi$ is a quasiprojective Riemann surface $X$ with a function $\varphi : X \to \mathbb{N}$ such that $\text{Sing}(X_\varphi) := \{x \in X \mid \varphi(x) > 1\}$ is a finite set. Assume the following interpretation: the angle of a disk centered at $x$ equals $\frac{2\pi}{\varphi(x)}$.

Definition

For an orbifold $X_\varphi = X_{\varphi(x), x \in \text{Sing}(X_\varphi)}$ we define:

$\pi_1^{\text{orb}}(X_\varphi) := \pi_1(X \setminus \text{Sing}(X_\varphi)) / \langle \mu_x^{\varphi(x)} \rangle = 1$, $\forall x \in \text{Sing}(X_\varphi)$, $\mu_x$ a meridian of $x$. 

E. Artal
Fundamental Group and Braid Monodromy
Example

\( G_{p,q} := \pi_1^{\text{orb}}(\mathbb{C}_{p,q}) = \mathbb{Z}/p \ast \mathbb{Z}/q \). If \( \gcd(p, q) = 1 \) then \( H = \mathbb{Z}/pq \) and it is not hard to check that \( \Sigma_{G_{p,q},1} \) is composed by the primitive \( pq \)-roots of unity.
Orbifold morphism

Example

\[ G_{p,q} := \pi_1^{\text{orb}}(\mathbb{C}_{p,q}) = \mathbb{Z}/p \ast \mathbb{Z}/q. \]
If \( \gcd(p, q) = 1 \) then \( H = \mathbb{Z}/pq \) and it is not hard to check that \( \Sigma_{G_{p,q},1} \) is composed by the primitive \( pq \)-roots of unity.

Definition

Let \( X_\varphi \) be an orbifold and \( Y \) a smooth algebraic variety. A dominant algebraic morphism \( \rho : Y \to X \) defines an \textit{orbifold morphism} \( Y \to X_\varphi \) if for all \( x \in X \), \( \frac{1}{\varphi(x)} \rho^*(x) \) is a divisor.
**Orbifold morphism**

**Example**

\[
G_{p,q} := \pi_1^{\text{orb}}(\mathbb{C}_{p,q}) = \mathbb{Z}/p \ast \mathbb{Z}/q.
\]

If \(\gcd(p, q) = 1\) then \(H = \mathbb{Z}/pq\) and it is not hard to check that \(\Sigma_{G_{p,q},1}\) is composed by the primitive \(pq\)-roots of unity.

**Definition**

Let \(X_\varphi\) be an orbifold and \(Y\) a smooth algebraic variety. A dominant algebraic morphism \(\rho : Y \to X\) defines an **orbifold morphism** \(Y \to X_\varphi\) if for all \(x \in X\),

\[
\frac{1}{\varphi(x)} \rho^*(x)
\]

is a divisor.

**Remark**

Such a morphism induces a mapping \(\pi_1(Y) \to \pi_1^{\text{orb}}(X_\varphi)\) if it is primitive. Note that if we choose a tranversal disk to a smooth point of the regular part of \(\rho^*(x)\) then for suitable local coordinates, this map is of the form \(t \mapsto t^n\) for \(n\) a multiple of \(\varphi(x)\).
Orbifolds and characteristic varieties

Example

The map $\rho : \mathbb{P}^2 \mathbb{C} \to \mathbb{P}^1 \{ [1 : 1] \}$, $[x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2]$, is an orbifold map. We obtain an epimorphism $G \mathbb{C} \twoheadrightarrow \pi_{\text{orb}}^1(C^2, 3) = G^{2, 3}$. In fact, it is an isomorphism and the irreducible components of $\Sigma_{C, 1}$ are obtained by pull-back by $\rho$ of $\Sigma_{G^{2, 3}, 1}$.

Example $G^* n := \pi_{\text{orb}}^1(C^* n) = \mathbb{Z}^* \mathbb{Z} / n$. The torus $T_H$ has equation $t_n^2 = 1$ in $(C^* n)^2$ and $\Sigma_{G^* n, 1} = (C^* \times \{\zeta | \zeta \neq 1, \zeta^n = 1\}) \cup \{(1, 1)\}$.

Let $C$ be the curve of equation $(y^2 z - x^3) x z = 0$. It is not hard to see that $G_C = \langle a, b | [a, b]^2 = 1 \rangle$, $T_H = (C^* n)^2$ and $\Sigma_{C, 1} = (C^* \times \{-1\}) \cup \{(1, 1)\}$.

The mapping $[x : y : z] \mapsto [y^2 z : x^3]$ from $\mathbb{P}^2 \mathbb{C}$ misses only $\{ [1 : 0], [1 : 1] \}$ and the pull-back of $[0 : 1]$ is a double divisor. Then, it defines an orbifold morphism onto $C^* n$. This map pulls back the 1-dimensional component of $\Sigma_{G^* n, 1}$ into $\Sigma_{C, 1}$. 

E. Artal

Fundamental Group and Braid Monodromy
The map $\rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\}$, 
$[x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2]$, is an orbifold map.
Example

- The map $\rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\}$,
  $[x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2]$, is an orbifold map.
- We obtain an epimorphism $G_C \twoheadrightarrow \pi_1^{\text{orb}}(\mathbb{C}_{2,3}) = G_{2,3}$. 
Example

- The map $\rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\}$,
  $[x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2]$, is an orbifold map.
- We obtain an epimorphism $G_C \to \pi_1^{\text{orb}}(\mathbb{C}, 1) = G_{2,3}$.
- In fact, it is an isomorphism and the irreducible components of $\Sigma_{C,1}$ are obtained by pull-back by $\rho$ of $\Sigma_{G_{2,3},1}$. 
Example

- The map $\rho : \mathbb{P}^2 \setminus C \rightarrow \mathbb{P}^1 \setminus \{[1 : 1]\}$, 
  $[x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2]$, is an orbifold map.
- We obtain an epimorphism $G_C \twoheadrightarrow \pi_1^{\text{orb}}(\mathbb{C}_{2,3}) = G_{2,3}$.
- In fact, it is an isomorphism and the irreducible components of $\Sigma_{C,1}$ are obtained by pull-back by $\rho$ of $\Sigma_{G_{2,3},1}$.

Example
**Example**

- The map \( \rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\}, \)
  \([x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2] \), is an orbifold map.

- We obtain an epimorphism \( G_C \to \pi_{1,\text{orb}}(\mathbb{C}, 3) = G_{2,3} \).

- In fact, it is an isomorphism and the irreducible components of \( \Sigma_{C,1} \) are obtained by pull-back by \( \rho \) of \( \Sigma_{G_{2,3},1} \).

**Example**

- \( G_n^* := \pi_{1,\text{orb}}(\mathbb{C}^*_n) = \mathbb{Z} \ast \mathbb{Z}/n \). The torus \( \mathbb{T}_H \) has equation \( t_2^n = 1 \) in \( (\mathbb{C}^*)^2 \) and \( \Sigma_{G_n^*,1} = (\mathbb{C}^* \times \{\zeta \mid \zeta \neq 1, \zeta^n = 1\}) \cup \{(1,1)\} \).
Orbifolds and characteristic varieties

Example

1. The map \( \rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\} \), 
\([x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2] \), is an orbifold map.
2. We obtain an epimorphism \( G_C \to \pi_1^{\text{orb}}(\mathbb{C}_{2,3}) = G_{2,3} \).
3. In fact, it is an isomorphism and the irreducible components of \( \Sigma_{C,1} \) are obtained by pull-back by \( \rho \) of \( \Sigma_{G_{2,3},1} \).

Example

1. \( G_n^* := \pi_1^{\text{orb}}(\mathbb{C}_n^*) = \mathbb{Z} \ast \mathbb{Z}/n \). The torus \( T_H \) has equation \( t_2^n = 1 \) in \( (\mathbb{C}^*)^2 \) and \( \Sigma_{G_n^*,1} = (\mathbb{C}^* \times \{ \zeta \mid \zeta \neq 1, \zeta^n = 1 \}) \cup \{(1, 1)\} \).
2. Let \( C \) be the curve of equation \( (y^2 z - x^3)xz = 0 \). It is not hard to see that \( G_C = \langle a, b \mid [a, b^2] = 1 \rangle \), \( T_H = (\mathbb{C}^*)^2 \) and \( \Sigma_{C,1} = (\mathbb{C}^* \times \{-1\}) \cup \{(1, 1)\} \).
Orbifolds and characteristic varieties

Example

- The map $\rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\}$, 
  $[x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2]$, is an orbifold map.
- We obtain an epimorphism $G_C \twoheadrightarrow \pi_1^{\text{orb}}(\mathbb{C}^*, 3) = G_{2,3}$.
- In fact, it is an isomorphism and the irreducible components of $\Sigma_{C,1}$ are obtained by pull-back by $\rho$ of $\Sigma_{G_{2,3},1}$.

Example

- $G_n^* := \pi_1^{\text{orb}}(\mathbb{C}^*_n) = \mathbb{Z} \ast \mathbb{Z}/n$. The torus $\mathbb{T}_H$ has equation $t_2^n = 1$ in $(\mathbb{C}^*)^2$ and $\Sigma_{G_n^*,1} = (\mathbb{C}^* \times \{\zeta \mid \zeta \neq 1, \zeta^n = 1\}) \cup \{(1, 1)\}$.
- Let $C$ be the curve of equation $(y^2 z - x^3)xz = 0$. It is not hard to see that $G_C = \langle a, b \mid [a, b^2] = 1 \rangle$, $\mathbb{T}_H = (\mathbb{C}^*)^2$ and $\Sigma_{C,1} = (\mathbb{C}^* \times \{-1\}) \cup \{(1, 1)\}$.
- The mapping $[x : y : z] \mapsto [y^2 z : x^3]$ from $\mathbb{P}^2 \setminus C$ misses only $\{[1 : 0], [1 : 1]\}$ and the pull-back of $[0 : 1]$ is a double divisor. Then, it defines an orbifold morphism onto $\mathbb{C}^*_2$. 
Orbifolds and characteristic varieties

Example

- The map \( \rho : \mathbb{P}^2 \setminus C \rightarrow \mathbb{P}^1 \setminus \{[1 : 1]\}, \) \( [x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2], \) is an orbifold map.
- We obtain an epimorphism \( G_c \rightarrow \pi_1^{\text{orb}}(\mathbb{C}_{2,3}) = G_{2,3}. \)
- In fact, it is an isomorphism and the irreducible components of \( \Sigma_{G_{2,3},1} \) are obtained by pull-back by \( \rho \) of \( \Sigma_{G_{2,3},1}. \)

Example

- \( G_n := \pi_1^{\text{orb}}(\mathbb{C}^*_n) = \mathbb{Z} \ast \mathbb{Z}/n. \) The torus \( \mathbb{T}_H \) has equation \( t_2^n = 1 \) in \( (\mathbb{C}^*)^2 \) and \( \Sigma_{G_n*1} = (\mathbb{C}^* \times \{\zeta \mid \zeta \neq 1, \zeta^n = 1\}) \cup \{(1, 1)\}. \)
- Let \( C \) be the curve of equation \( (y^2z - x^3)xz = 0. \) It is not hard to see that \( G_c = \langle a, b \mid [a, b^2] = 1 \rangle, \) \( \mathbb{T}_H = (\mathbb{C}^*)^2 \) and \( \Sigma_{C,1} = (\mathbb{C}^* \times \{-1\}) \cup \{(1, 1)\}. \)
- The mapping \([x : y : z] \mapsto [y^2z : x^3]\) from \( \mathbb{P}^2 \setminus C \) misses only \( \{[1 : 0], [1 : 1]\} \) and the pull-back of \([0 : 1]\) is a double divisor. Then, it defines an orbifold morphism onto \( \mathbb{C}^*_2. \)
- This map pulls back the 1-dimensional component of \( \Sigma_{G_n*,1} \) into \( \Sigma_{C,1}. \)
**Big examples**

**Example**

\[ G^{**n} := \pi_{orb}1(C^{**n}) = F_2 \ast \mathbb{Z}/n \] where \( C^{**n} := C^* \backslash \{1\} \). The torus \( \Sigma G^{**n}, 1 \) has equation \( t^n = 1 \) in \( (C^*)^3, \Sigma G^{**n}, 2 = \left( (C^*)^2 \times \{\zeta | 1 \neq \zeta, \zeta^n = 1\}\right) \cup \{(1, 1)\} \).

We obtain a similar result as for last example taking out another generic fiber.

**Example** Let \( Y \) be an elliptic curve and let \( G^{0, 2} : \pi_{orb}1(Y^{2, 2}) \) with presentation \( \langle a, b, u, v | u^2 = v^2 = [a, b], uv = 1 \rangle \). The torus \( \Sigma H \) has equation \( t^2 = 1 \) in \( (C^*)^3, \Sigma G^{0, 2, 1} = \left( (C^*)^2 \times \{-1\}\right) \cup \{(1, 1)\} \).

We can never have an orbifold map from the complement of a projective curve onto \( C \).
Big examples

Example

- \( G_n^{**} := \pi_1^{\text{orb}}(\mathbb{C}^{**}_n) = F_2 \ast \mathbb{Z}/n \) where \( \mathbb{C}^{**} := \mathbb{C}^* \setminus \{1\} \). The torus \( T_H \) has equation \( t_3^n = 1 \) in \((\mathbb{C}^*)^3\), \( \Sigma_{G_n^{**},1} = T_H \) and \( \Sigma_{G_n^{**},2} = ((\mathbb{C}^*)^2 \times \{\zeta \mid 1 \neq \zeta, \zeta^n = 1\}) \cup \{(1,1)\} \).
Example

- $G_n^{**} := \pi_1^{\text{orb}}(\mathbb{C}^{**}) = \mathbb{F}_2 \ast \mathbb{Z}/n$ where $\mathbb{C}^{**} := \mathbb{C}^* \setminus \{1\}$. The torus $T_H$ has equation $t_3^n = 1$ in $(\mathbb{C}^*)^3$, $\Sigma_{G_n^{**},1} = T_H$ and $\Sigma_{G_n^{**},2} = ((\mathbb{C}^*)^2 \times \{\zeta \mid 1 \neq \zeta, \zeta^n = 1\}) \cup \{(1, 1)\}$.

- We obtain a similar result as for last example taking out another generic fiber.
Big examples

Example

- $G^{**}_n := \pi_1^{\text{orb}}(\mathbb{C}^{**}) = \mathbb{F}_2 \ast \mathbb{Z}/n$ where $\mathbb{C}^{**} := \mathbb{C}^* \setminus \{1\}$. The torus $T_H$ has equation $t^n_3 = 1$ in $(\mathbb{C}^*)^3$, $\Sigma_{G^{**},1} = T_H$ and $\Sigma_{G^{**},2} = ((\mathbb{C}^*)^2 \times \{\zeta \mid 1 \neq \zeta, \zeta^n = 1\}) \cup \{(1,1)\}$.

- We obtain a similar result as for last example taking out another generic fiber.

Example
Big examples

Example

- $G_n^{**} := \pi_1^{\text{orb}}(\mathbb{C}_n^{**}) = F_2 \ast \mathbb{Z}/n$ where $\mathbb{C}_n^{**} := \mathbb{C}^* \setminus \{1\}$. The torus $T_H$ has equation $t_3^n = 1$ in $(\mathbb{C}^*)^3$, $\Sigma_{G_n^{**},1} = T_H$ and
  - $\Sigma_{G_n^{**},2} = ((\mathbb{C}^*)^2 \times \{\zeta \mid 1 \neq \zeta, \zeta^n = 1\}) \cup \{(1,1)\}$.
- We obtain a similar result as for last example taking out another generic fiber.

Example

- Let $Y$ be an elliptic curve and let $G_{2,2}^0 := \pi_1^{\text{orb}}(Y_{2,2})$ with presentation
  $$\langle a, b, u, v \mid u^2 = v^2 = [a, b]uv = 1 \rangle.$$  
  The torus $T_H$ has equation $t_3^2 = 1$ in $(\mathbb{C}^*)^3$,
  - $\Sigma_{G_{2,2}^0,1} = ((\mathbb{C}^*)^2 \times \{-1\}) \cup \{(1,1)\}$. 

Big examples

Example

- $G_{n}^{**} := \pi_{1}^{\text{orb}}(\mathbb{C}^{**}) = \mathbb{F}_2 \ast \mathbb{Z}/n$ where $\mathbb{C}^{**} := \mathbb{C}^* \setminus \{1\}$. The torus $\mathbb{T}_H$ has equation $t_3^n = 1$ in $(\mathbb{C}^*)^3$, $\Sigma_{G_{n}^{**},1} = \mathbb{T}_H$ and $\Sigma_{G_{n}^{**},2} = ((\mathbb{C}^*)^2 \times \{\zeta \mid 1 \neq \zeta, \zeta^n = 1\}) \cup \{(1, 1)\}$.

- We obtain a similar result as for last example taking out another generic fiber.

Example

- Let $Y$ be an elliptic curve and let $G_{2,2}^0 := \pi_{1}^{\text{orb}}(Y_{2,2})$ with presentation

  \[ \langle a, b, u, v \mid u^2 = v^2 = [a, b]uv = 1 \rangle. \]

  The torus $\mathbb{T}_H$ has equation $t_3^2 = 1$ in $(\mathbb{C}^*)^3$, $\Sigma_{G_{2,2}^0,1} = ((\mathbb{C}^*)^2 \times \{-1\}) \cup \{(1, 1)\}$.

- We can never have an orbifold map from the complement of a projective curve onto $\mathbb{C}$. 
Projective examples

Example

$G, p, q, r := \pi_{\text{orb}}(P_1^{p}, q, r)$ with presentation $\langle a, b, c | a^p = b^q = c^r = abc = 1 \rangle$.

Note that $H$ is the kernel of the natural mapping $\mathbb{Z}/p \oplus \mathbb{Z}/q \oplus \mathbb{Z}/r \to \mathbb{Z}/m$, where $m := \text{lcm}(p, q, r)$. For example $H = \mathbb{Z}/6$ for $(2, 3, 6)$, $H = \mathbb{Z}/2 \times \mathbb{Z}/4$ for $(2, 4, 4)$ and $H = \mathbb{Z}/3 \times \mathbb{Z}/3$ for $(3, 3, 3)$.

The torus $T_H$ can be considered in $(\mathbb{C}^\ast)^3$. For $\xi \in T_H$ we define $\ell(\xi)$ the number of non-trivial coordinates. If $\xi \neq 1$ then $\ell(\xi) > 1$.

$\Sigma_{G, p, q, r, 1} = \{ \xi | \ell(\xi) = 3 \}$. These data will be used in the last lecture.
Projective examples

Example

- $G_{p,q,r} := \pi_1^{\text{orb}}(\mathbb{P}^1_{p,q,r})$ with presentation

$$\langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle.$$
Example

- $G_{p,q,r} := \pi_{1}^{\text{orb}}(\mathbb{P}^1_{p,q,r})$ with presentation
  \[
  \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle.
  \]

- Note that $H$ is the kernel of the natural mapping
  \[
  \mathbb{Z}/p \oplus \mathbb{Z}/q \oplus \mathbb{Z}/r \to \mathbb{Z}/m,
  \]
  where $m := \text{lcm}(p, q, r)$. For example:
  - $H = \mathbb{Z}/6$ for $(2, 3, 6)$,
  - $H = \mathbb{Z}/2 \times \mathbb{Z}/4$ for $(2, 4, 4)$ and
  - $H = \mathbb{Z}/3 \times \mathbb{Z}/3$ for $(3, 3, 3)$. 

Projective examples

Example

- \( G_{p, q, r} := \pi_{1}^{\text{orb}}(\mathbb{P}^{1}_{p, q, r}) \) with presentation
  \[ \langle a, b, c \mid a^{p} = b^{q} = c^{r} = abc = 1 \rangle. \]

- Note that \( H \) is the kernel of the natural mapping
  \[ \mathbb{Z}/p \oplus \mathbb{Z}/q \oplus \mathbb{Z}/r \to \mathbb{Z}/m, \]
  where \( m := \text{lcm}(p, q, r) \). For example
  - \( H = \mathbb{Z}/6 \) for \((2, 3, 6)\), \( H = \mathbb{Z}/2 \times \mathbb{Z}/4 \) for \((2, 4, 4)\) and \( H = \mathbb{Z}/3 \times \mathbb{Z}/3 \) for \((3, 3, 3)\).

- The torus \( \mathbb{T}_{H} \) can be considered in \((\mathbb{C}^{*})^{3}\).
Projective examples

Example

- $G_{p, q, r} := \pi_1^{orb}(\mathbb{P}^1_{p, q, r})$ with presentation
  \[
  \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle.
  \]

- Note that $H$ is the kernel of the natural mapping
  $\mathbb{Z}/p \oplus \mathbb{Z}/q \oplus \mathbb{Z}/r \to \mathbb{Z}/m$, where $m := \text{lcm}(p, q, r)$. For example
  $H = \mathbb{Z}/6$ for $(2, 3, 6)$, $H = \mathbb{Z}/2 \times \mathbb{Z}/4$ for $(2, 4, 4)$ and $H = \mathbb{Z}/3 \times \mathbb{Z}/3$ for $(3, 3, 3)$.

- The torus $\mathbb{T}_H$ can be considered in $(\mathbb{C}^*)^3$.

- For $\xi \in \mathbb{T}_H$ we define $\ell(\xi)$ the number of non-trivial coordinates. If $\xi \neq 1$ then $\ell(\xi) > 1$. 
Example

- $G_{p,q,r} := \pi_1^\text{orb}(\mathbb{P}^1_{p,q,r})$ with presentation
  \[ \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle. \]

- Note that $H$ is the kernel of the natural mapping $\mathbb{Z}/p \oplus \mathbb{Z}/q \oplus \mathbb{Z}/r \to \mathbb{Z}/m$, where $m := \text{lcm}(p, q, r)$. For example:
  - $H = \mathbb{Z}/6$ for $(2, 3, 6)$,
  - $H = \mathbb{Z}/2 \times \mathbb{Z}/4$ for $(2, 4, 4)$ and
  - $H = \mathbb{Z}/3 \times \mathbb{Z}/3$ for $(3, 3, 3)$.

- The torus $\mathbb{T}_H$ can be considered in $(\mathbb{C}^*)^3$.

- For $\xi \in \mathbb{T}_H$ we define $\ell(\xi)$ the number of non-trivial coordinates. If $\xi \neq 1$ then $\ell(\xi) > 1$.

- $\Sigma_{G_{p,q,r,1}} = \{ \xi \mid \ell(\xi) = 3 \}$. These data will be used in the last lecture.
Main result

Theorem (Arapura)

Let $\Sigma$ be an irreducible component of $\Sigma_{G,1}$, $G = \pi_1(X)$, $X$ quasi-projective surface. Then,

1. If $\dim \Sigma > 0$ then there exists a primitive surjective morphism $\rho : X \to C$, $C$ algebraic curve, and a torsion element $\sigma$ such that $\Sigma = \sigma \rho^*(H^1(C; \mathbb{C}^*))$.

2. If $\dim \Sigma = 0$ then $\Sigma$ is unitary.

In particular, positive dimensional irreducible components are subtori translated by torsion elements.
Main result

Claim

- $X$ a quasi-projective smooth variety
- $\Sigma := \Sigma_k(X)$ the $k^{th}$ characteristic variety of $X$, $V$ an irreducible component of $\Sigma$.

Then, there exists:

- a primitive surjective orbifold morphism $\rho : X \to C_\varphi$ and
- an irreducible component $V_1$ of $\Sigma_k(\pi_1^{\text{orb}}(C_\varphi))$

such that $V = \rho^*(V_1)$.

In particular, irreducible components are subtori translated by torsion elements.
Main result

Claim

- $X$ a quasi-projective smooth variety
- $\Sigma := \Sigma_k(X)$ the $k^{th}$ characteristic variety of $X$, $V$ an irreducible component of $\Sigma$.

Then, there exists:

- a primitive surjective orbifold morphism $\rho : X \rightarrow C_\varphi$ and
- an irreducible component $V_1$ of $\Sigma_k(\pi_1^{\text{orb}}(C_\varphi))$

such that $V = \rho^*(V_1)$.

In particular, irreducible components are subtori translated by torsion elements.

Remark

The claim is not correct. The Degtyarev curve has as characteristic variety four points of torsion 10 which cannot be obtained as pull-back from an orbifold (–, Cogolludo).
Main result

Theorem (−, Cogolludo, Matei)

Let $X$ be a quasi-projective smooth variety and let $\Sigma$ be the $k^{th}$ characteristic variety of $X$. Let $V$ be an irreducible component of $\Sigma$. Then one of the two following statements holds:

1. There exists a primitive surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component $V_1$ of $\Sigma_k(\pi_1^{\text{orb}}(C_\varphi))$ such that $V = \rho^*(V_1)$.

2. $V$ is an isolated torsion point.
Main result

Theorem (–, Cogolludo, Matei)

Let $X$ be a quasi-projective smooth variety and let $\Sigma$ be the $k^{th}$ characteristic variety of $X$. Let $V$ be an irreducible component of $\Sigma$. Then one of the two following statements holds:

1. There exists a primitive surjective orbifold morphism $\rho : X \to C_{\varphi}$ and an irreducible component $V_1$ of $\Sigma_k(\pi_1^{\text{orb}}(C_{\varphi}))$ such that $V = \rho^*(V_1)$.

2. $V$ is an isolated torsion point.

Remark

The proof uses Deligne-Timmerscheidt theory and follows ideas of Beauville, Arapura and Delzant. One essential ingredient is that for non-unitary characters, some non-trivial elements of the twisted cohomology are represented by twisted logarithmic 1-forms, defining foliations.
Except 1, irreducible components of $\Sigma_k$ are connected components of $T_H$. 

Given a translated subtorus $V$, its shadow $\text{Sh}_V$ is the parallel subtorus passing through 1.

An irreducible component $\Sigma$ of $\Sigma_1$ of dimension $k > 0$ is also an irreducible component of $\Sigma_{k-2}$, if $1 \in \Sigma$, or $\Sigma_k$ if not.

If $\Sigma$ is an irreducible component of $\Sigma_1$ of dimension $k > 2$ then it is also the case for $\text{Sh}_\Sigma$.

For rational orbifolds the same can be assumed for $k \geq 2$.

An irreducible component of dimension 1 never passes through 1.
Properties of characteristic varieties of orbifolds

- Except 1, irreducible components of $\Sigma_k$ are connected components of $T_H$.
- Given a translated subtorus $V$, its shadow $\text{Sh} V$ is the \textit{parallel} subtorus passing through 1.
Properties of characteristic varieties of orbifolds

- Except 1, irreducible components of $\Sigma_k$ are connected components of $T_H$.
- Given a translated subtorus $V$, its shadow $\text{Sh} V$ is the \textit{parallel} subtorus passing through 1.
- An irreducible component $\Sigma$ of $\Sigma_1$ of dimension $k > 0$ is also an irreducible component of $\Sigma_{k-2}$, if $1 \in \Sigma$, or $\Sigma_k$ if not.
Properties of characteristic varieties of orbifolds

- Except 1, irreducible components of $\Sigma_k$ are connected components of $T_H$.
- Given a translated subtorus $V$, its shadow $\text{Sh} V$ is the parallel subtorus passing through 1.
- An irreducible component $\Sigma$ of $\Sigma_1$ of dimension $k > 0$ is also an irreducible component of $\Sigma_{k-2}$, if $1 \in \Sigma$, or $\Sigma_k$ if not.
- If $\Sigma$ is an irreducible component of $\Sigma_1$ of dimension $k > 2$ then it is also the case for $\text{Sh} \Sigma$. 
Properties of characteristic varieties of orbifolds

- Except 1, irreducible components of $\Sigma_k$ are connected components of $T_H$.
- Given a translated subtorus $V$, its shadow $\text{Sh} V$ is the parallel subtorus passing through 1.
- An irreducible component $\Sigma$ of $\Sigma_1$ of dimension $k > 0$ is also an irreducible component of $\Sigma_{k-2}$, if $1 \in \Sigma$, or $\Sigma_k$ if not.
- If $\Sigma$ is an irreducible component of $\Sigma_1$ of dimension $k > 2$ then it is also the case for $\text{Sh} \Sigma$.
- For rational orbifolds the same can be assumed for $k \geq 2$. 
Properties of characteristic varieties of orbifolds

- Except 1, irreducible components of $\Sigma_k$ are connected components of $T_H$.
- Given a translated subtorus $V$, its shadow $Sh V$ is the *parallel* subtorus passing through 1.
- An irreducible component $\Sigma$ of $\Sigma_1$ of dimension $k > 0$ is also an irreducible component of $\Sigma_{k-2}$, if $1 \in \Sigma$, or $\Sigma_k$ if not.
- If $\Sigma$ is an irreducible component of $\Sigma_1$ of dimension $k > 2$ then it is also the case for $Sh \Sigma$.
- For rational orbifolds the same can be assumed for $k \geq 2$.
- An irreducible component of dimension 1 never passes through 1.
Properties of characteristic varieties of quasiprojective groups $G$

- These properties appear in the work of Dimca-Papadima-Suciu and --Cogolludo-Matei.
Properties of characteristic varieties of quasiprojective groups $G$

- These properties appear in the work of Dimca-Papadima-Suciu and --Cogolludo-Matei.
- The properties for orbifolds also hold for quasiprojective groups $G$ (and components of positive dimension).
Properties of characteristic varieties of quasiprojective groups $G$

- These properties appear in the work of Dimca-Papadima-Suciu and Cogolludo-Matei.
- The properties for orbifolds also hold for quasiprojective groups $G$ (and components of positive dimension).
- $\Sigma_1, \Sigma_2$ irreducible components of $\Sigma_{G,1}$, $\dim(\text{Sh} \Sigma_1 \cap \text{Sh} \Sigma_2) > 0 \Rightarrow \Sigma_1 = \Sigma_2$. 
Properties of characteristic varieties of quasiprojective groups $G$

- These properties appear in the work of Dimca-Papadima-Suciu and Cogolludo-Matei.
- The properties for orbifolds also hold for quasiprojective groups $G$ (and components of positive dimension).
- $\Sigma_1, \Sigma_2$ irreducible components of $\Sigma_{G,1}$, $\dim(\text{Sh} \Sigma_1 \cap \text{Sh} \Sigma_2) > 0 \Rightarrow \Sigma_1 = \Sigma_2$.
- If $\Sigma$ is a component of $\Sigma_k$ not in $\Sigma_{k+1}$ and $\xi \in \Sigma$ belongs to $\Sigma_{k+1}$ then $\xi$ is torsion.
Properties of characteristic varieties of quasi-projective groups $G$

- These properties appear in the work of Dimca-Papadima-Suciu and Cogolludo-Matei.
- The properties for orbifolds also hold for quasi-projective groups $G$ (and components of positive dimension).
- $\Sigma_1, \Sigma_2$ irreducible components of $\Sigma_{G,1}$, $\dim(\text{Sh} \Sigma_1 \cap \text{Sh} \Sigma_2) > 0 \Rightarrow \Sigma_1 = \Sigma_2$.
- If $\Sigma$ is a component of $\Sigma_k$ not in $\Sigma_{k+1}$ and $\xi \in \Sigma$ belongs to $\Sigma_{k+1}$ then $\xi$ is torsion.
- Let $\Sigma_1$ be an irreducible component of $\Sigma_k(G)$ and let $\Sigma_2$ be an irreducible component of $\Sigma_\ell(G)$, both of positive dimension. If $\xi \in \Sigma_1 \cap \Sigma_2$ then it is a torsion point and $\xi \in \Sigma_{k+\ell}(G)$.
An Artin group

Example

Let $G := \langle x, y, z | [x, y] = 1, (yz)^2 = (zy)^2, (xz)^3 = (zx)^3 \rangle$; $\Sigma_2(G) = \emptyset$ and $\Sigma_1(G)$ has 5 irreducible components $\Sigma_i$ of dimension 1 such that $\Sigma_i \cap \Sigma_{i+1}$ consists of one point (of torsion type). Then $G$ is not quasiprojective.
Example

Let $G := \langle x, y, z \mid [x, y] = 1, (yz)^2 = (zy)^2, (xz)^3 = (zx)^3 \rangle$; $\Sigma_2(G) = \emptyset$ and $\Sigma_1(G)$ has 5 irreducible components $\Sigma_i$ of dimension 1 such that $\Sigma_i \cap \Sigma_{i+1}$ consists of one point (of torsion type). Then $G$ is not quasiprojective.

Theorem

Let $G_{p,q,r}$ the Artin group associated to a triangle with sides $p, q, r$
An Artin group

Example

Let $G := \langle x, y, z | [x, y] = 1, (yz)^2 = (zy)^2, (xz)^3 = (zx)^3 \rangle$; $\Sigma_2(G) = \emptyset$ and $\Sigma_1(G)$ has 5 irreducible components $\Sigma_i$ of dimension 1 such that $\Sigma_i \cap \Sigma_{i+1}$ consists of one point (of torsion type). Then $G$ is not quasiprojective.

Theorem

Let $G_{p,q,r}$ the Artin group associated to a triangle with sides $p, q, r$

- If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1$ then there exists an affine curve $C_{p,q,r}$ such that $G_{p,q,r} = \pi_1(\mathbb{C}^2 \setminus C_{p,q,r})$
**An Artin group**

**Example**

Let $G := \langle x, y, z | [x, y] = 1, (yz)^2 = (zy)^2, (xz)^3 = (zx)^3 \rangle$; $\Sigma_2(G) = \emptyset$ and $\Sigma_1(G)$ has 5 irreducible components $\Sigma_i$ of dimension 1 such that $\Sigma_i \cap \Sigma_{i+1}$ consists of one point (of torsion type). Then $G$ is not quasiprojective.

**Theorem**

Let $G_{p,q,r}$ the Artin group associated to a triangle with sides $p, q, r$

- **If** $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1$ **then** there exists an affine curve $C_{p,q,r}$ such that $G_{p,q,r} = \pi_1(\mathbb{C}^2 \setminus C_{p,q,r})$

- **If** $p, q, r$ are even, **not all of them equal** and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ **then** the groups $G_{p,q,r}$ are not quasiprojective.