Recent Progress on Topology of Plane Curves: A Quick Trip

Part VI:
The Topology and Geometry of Curves: Torus Type Curves and Quasi-Toric Relations

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Three approaches:

- *Topological*: Braid Monodromy, Fundamental Group, Alexander Polynomial.
- *Geometric*: Morphisms onto curves (De Franchis).
- *Algebraic*: Existence of pencils containing $\mathcal{C}$.

A New Look at a Classical Example

Consider $\mathcal{C} := \{F := h_3^2 + h_2^3 = 0\} \subset \mathbb{P}^2$ a sextic.

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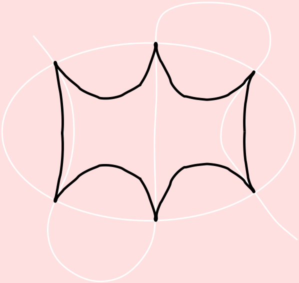

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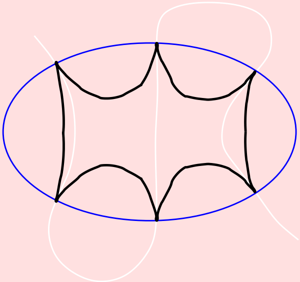

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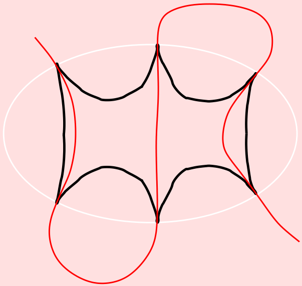

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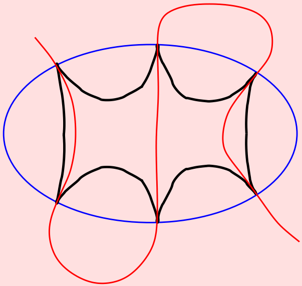

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Orbifolds and Orbifold Fundamental Groups

Definition (Orbifold)

An *orbifold* curve $S_{\bar{m}}$ is a Riemann surface S with a function $\bar{m} : S \to \mathbb{N}$ whose value is 1 outside a finite number of points. A point $P \in S$ for which $\bar{m}(P) > 1$ is called an *orbifold point*.

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Definition (Orbifold Fundamental Group)

For an orbifold $S_{\bar{m}}$, let $P_1, \dots, P_n$ be the orbifold points, $m_j := \bar{m}(P_j) > 1$. Then, the *orbifold fundamental group* of $S_{\bar{m}}$ is

$$\pi_1^{\text{orb}}(S_{\bar{m}}) := \pi_1(S \setminus \{P_1, \dots, P_n\})/\langle \mu_j^{m_j} = 1 \rangle,$$

where μ_j is a meridian of P_j. We will denote $S_{\bar{m}}$ simply by $S_{m_1,\dots,m_n}$.

Orbifold Morphisms

Definition

A dominant algebraic morphism $\varphi : X \to S$ defines an *orbifold morphism* $X \to S_{\bar{m}}$ if for all $P \in S$, the divisor $\varphi^*(P)$ is a $\bar{m}(P)$-multiple.

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Proposition ([1, Proposition 1.5])

Let $\rho : X \to S$ define an orbifold morphism $X \to S_{\bar{m}}$. Then φ induces a morphism $\varphi_* : \pi_1(X) \to \pi_1^{\mathrm{orb}}(S_{\bar{m}})$.
Moreover, if the generic fiber is connected, then φ_* is surjective.

Applications

Example

Consider F equation of $\mathcal{C}_{6,6}$ in Zariski's Example. Since F fits in a functional equation of type

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$$h_3^2 + h_2^3 + F = 0, \tag{1}$$

Then (1) induces a rational map

$$\begin{array}{cccc} \varphi : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \\ & [x : y : z] & \mapsto & [h_2^3 : h_3^2] \end{array}$$

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Then (1) induces a morphism

$$\hat{\varphi} : \quad \widehat{\mathbb{P}}^2 \quad \longrightarrow \quad \mathbb{P}^1$$

such that $\hat{\varphi} = \varphi \circ \varepsilon$.

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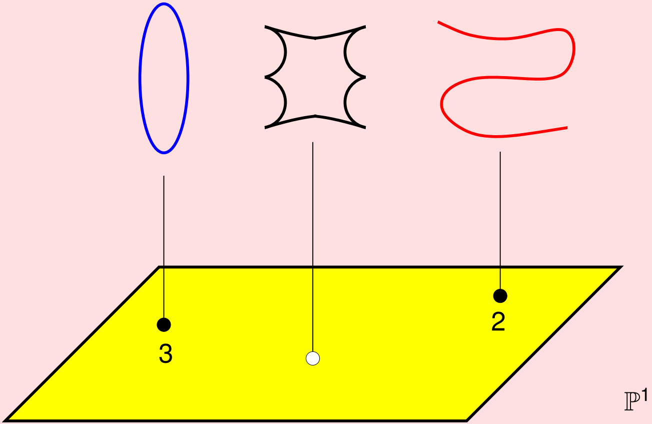
- $\hat{\varphi}|_{\mathbb{P}^2 \setminus \mathcal{C}}$ has two multiple fibers (over $[0:1]$, $[1:0]$).
- $\bar{m}([0:1]) = 2$, $\bar{m}([1:0]) = 3$
- One has an orbifold morphism $\hat{\varphi}_{2,3} : \mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1_{2,3} \setminus \{[1:-1]\}$.
- Since the pencil is primitive, there is an epimorphism

$$\hat{\varphi}_{2,3} : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{2,3} \setminus \{[1:-1]\}) = \mathbb{Z}_2 * \mathbb{Z}_3.$$


3
2
$\mathbb{P}^1$

Applications

Example

In general, suppose F fits in a functional equation of type

$$F_1 h_1^p + F_2 h_2^q + F_3 h_3^r = 0, \tag{2}$$

- Then (2) induces a morphism $\hat{\varphi} : \widehat{\mathbb{P}}^2 \to \mathbb{P}^1$ given by $\varphi([x:y:z]) = [F_1 h_1^p : F_2 h_2^q]$.
- $\hat{\varphi}|_{\mathbb{P}^2 \setminus \mathcal{C}}$ has three multiple fibers (over $[0:1]$, $[1:0]$, and $[1:-1]$).
- $\bar{m}([0:1]) = p$, $\bar{m}([1:0]) = q$, and $\bar{m}([1:-1]) = r$.
- One has an orbifold morphism $\hat{\varphi}_{p,q,r} : \mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1_{p,q,r} \setminus \hat{\varphi}(\{F_1 F_2 F_3 = 0\})$.
- If the pencil is primitive, there is an epimorphism

$$\hat{\varphi}_{p,q,r} : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{p,q,r} \setminus \hat{\varphi}(\{F_1 F_2 F_3 = 0\})) = \frac{\alpha\mathbb{Z}_p * \beta\mathbb{Z}_q}{(\alpha\beta)^r}.$$

Another Application

Corollary

The number of multiple members in a (primitive) pencil of plane curves (with no base components) is at most two.

Functional Relation $F_1 h_1^p + F_2 h_2^q + F_3 h_3^r = 0$

Definition

A curve $\mathcal{C} := \{F = 0\}$ satisfies a *quasi-toric relation* of type (p, q, r) if there exist homogeneous polynomials $h_1, h_2, h_3 \in \mathbb{C}[x, y, z]$ such that

$$F_1 h_1^p + F_2 h_2^q + F_3 h_3^r = 0,$$

where F_1, F_2, F_3 are homogeneous polynomials and $\{F_1 F_2 F_3 = 0\} = \mathcal{C}$.

Main Theorem

Theorem (–, Libgober [3])

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3. *The polynomial F fits in a quasi-toric relation of type $(3,3,3)$ (resp. $(2,4,4)$, $(2,3,6)$).*

Moreover, the set of quasi-toric relations of type $(3,3,3)$ (resp. $(2,4,4)$, $(2,3,6)$) has a group structure, whose rank is twice the multiplicity of ξ as a root of $\Delta_{\mathcal{C},\varepsilon}(t)$.

Examples

Example

Since the 6-cuspidal sextic $\mathcal{C}_{6,6}$ is such that: $\Delta_{\mathcal{C}_{6,6}}(t) = (t^2 - t + 1)$, the decomposition $F = f_3^2 + f_2^3$ the only *primitive* one.

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- Note that there is an *infinite* number of decompositions of F !!!

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- For example, consider

$$\begin{aligned} h_1 &= x^3 + y^3 + z^3 \\ h_2 &= zx - y^2 \\ F &= -(x^6 + 2x^3y^3 + 3x^3z^3 - 3x^2y^2z^2 + 3xzy^4 + z^6 + 2y^3z^3) \end{aligned}$$

one can check that

$$h_1^2 + h_2^3 + F = 0.$$

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- But also, if one considers

$$\begin{aligned}
\tilde{h}_1 &= 25x^3z^3 - 27x^2y^2z^2 + 27xzy^4 - y^6 + 8x^6 + 16x^3y^3 + 16y^3z^3 + 8z^6 \\
\tilde{h}_2 &= -(8x^{12} - y^{12} + 8z^{12} + 297x^2y^8z^2 - 108x^2y^2z^8 - 108x^8y^2z^2 \\
&\quad +621x^4y^4z^4 + 12x^6y^6 + 147x^6z^6 + 12y^6z^6 + 68x^9z^3 - 40y^9x^3 \\
&\quad -40y^9z^3 + 68z^9x^3 + 168x^6y^3z^3 - 216x^5y^5z^2 - 378x^5y^2z^5 \\
&\quad -216x^2y^5z^5 - 480y^6x^3z^3 + 168z^6x^3y^3 + 32x^9y^3 + 32z^9y^3 \\
&\quad -54xzy^10 + 108x^7zy^4 + 216x^4zy^7 + 216xz^4y^7 + 108xz^7y^4) \\
\tilde{h}_3 &= 2(x^3 + y^3 + z^3)
\end{aligned}$$

then

$$\tilde{h}_1^2 + \tilde{h}_2^3 + \tilde{F} h_3^6 = 0.$$

Example

- Consider the tricuspical quartic:
  $\mathcal{C}_{4,3} := \{C_{4,3} = x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x+y+z) = 0\}$,

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- $\{$quasi-toric relations of $F\} = (\mathbb{Z} \oplus \omega_6\mathbb{Z})^2$,
- generated by:

$$\begin{aligned} \sigma_1 &\equiv C_{4,3}L_0^2 = 4C_2^3 + C_3^2 \\ \sigma_2 &\equiv C_{4,3}L_0^2 = 4\tilde{C}_2^3 + \tilde{C}_3^2, \end{aligned}$$

where $C_2 := zx + \omega_3 yz - (1+\omega_3)xy$,
$C_3 := (x^2y - x^2z - y^2x - 3(1+2\omega_3)xyz + y^2z + z^2x - yz^2)$,
$\tilde{C}_2(x,y,z) := C_2(x,z,y)$, and $\tilde{C}_3(x,y,z) := C_3(x,z,y)$.

Examples

Example

$F = (y^3 - z^3)(z^3 - x^3)(x^3 - y^3)$, $\mathcal{C} := \{F = 0\}$, then
$\Delta_{\mathcal{C}}(t) = (t^2 + t + 1)^2 (t - 1)^8$.

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$$x^3(y^3 - z^3) + y^3(z^3 - x^3) + z^3(x^3 - y^3) = 0. \quad (3)$$

However, there should exist another relation independent from (3) of type

$$F_1 \ell_1^3 + F_2 \ell_2^3 + F_3 \ell_3^3 = 0. \quad (4)$$

Examples

...and sure enough, one can check that if:

$$F_i = (y - \omega_3^i z)(z - \omega_3^{i+1} x)(x - \omega_3^{i+2} y), \quad i = 1, 2, 3,$$

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and

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\ell_2 &= (\omega_3 - \omega_3^2)z + (\omega_3 - \omega_3^2)x + (\omega_3^2 - 1)y, \\
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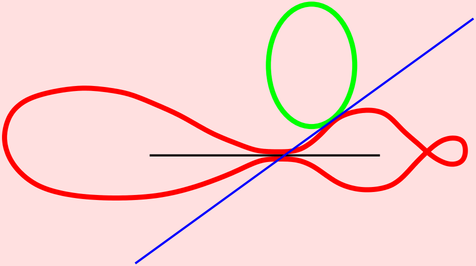
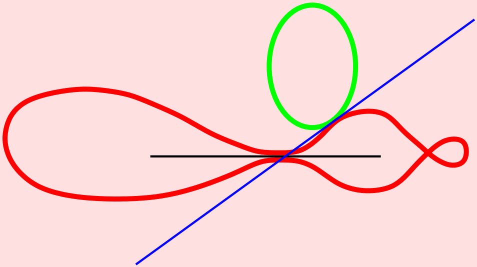
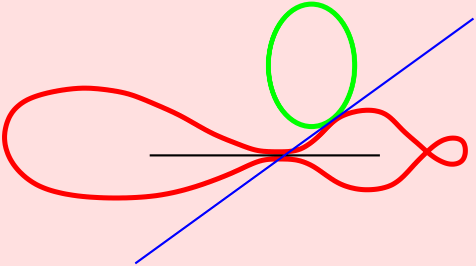


For $\varepsilon = (2, 1)$ one has $\Delta_{\mathcal{C},\varepsilon}(t) = (t - 1)(t^2 + 1)$.



For $\varepsilon = (2, 1)$ one has $\Delta_{\mathcal{C},\varepsilon}(t) = (t-1)(t^2+1)$. There exists an elliptic relation of type $(2, 4, 4)$.

$$\mathcal{C}_2 \ell_1^2 + \mathcal{C}_4 + {\ell_2}^4 = 0$$



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$$\begin{array}{ccc} \Sigma_1(\mathbb{P}^1_{2,4,4}) & \to & \Sigma_1(X) \\ (-1, \sqrt{-1}, \sqrt{-1}) & \mapsto & (-1, \sqrt{-1}) \\ (-1, -\sqrt{-1}, -\sqrt{-1}) & \mapsto & (-1, -\sqrt{-1}) \end{array}$$

The Group Law on Quasi-Toric Relations

Consider the relation

$$F_1 h_1^2 + F_2 h_2^3 + F_3 h_3^6 = 0.$$

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Which is equivalent to

$$\left(\frac{h_1}{h_3^3}\right)^2 + \left(\frac{h_2}{h_6^2}\right)^3 = -F_3.$$

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- Given two points $P_1 = (u_1, v_1)$, $P_2 = (u_2, v_2) \in E_0$, then

$$P_1 + P_2 = \left(\frac{3v_1v_2(u_1v_2 - u_2v_1) + (u_1 - u_2)(u_1u_2 - 3F)}{(v_1 - v_2)^3}, \frac{v_1^2v_2 + v_1v_2^2 + 2u_1u_2 - 2F}{(v_1 - v_2)^2}\right).$$

A Bound on the Degree of the Alexander Polynomial

Theorem (-,Libgober)

The degree of the Alexander polynomial of an irreducible curve $\mathcal{C}$ of degree d, whose singularities are only nodes and cusps satisfies:

$$\deg \Delta_{\mathcal{C}}(t) \leq \frac{10}{3} d - 4.$$

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