On the irreducibility of the moduli spaces of line arrangements.

(j.w. Shaheen Nazir, ArXiv: 1009.0202)

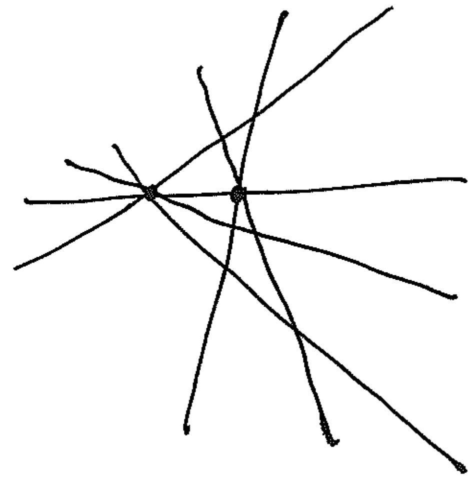
Masahiko Yoshinaga (Kyoto)

2011.3.8 Branched Covering, Degenerations and Related Topics
Tokyo Metropolitan University

Contents :

§1. Intro: $R(\mathcal{I})$, what & why?

Basic Setting:



$$\mathcal{A} = \{H_1, H_2, \cdots, H_n\}$$

$$(H_i \subset \mathbb{P}^2_{\mathbb{C}} \ : \ \text{line on } \mathbb{P}^2_{\mathbb{C}})$$

$$M(\mathcal{A}) = \mathbb{P}^2 \setminus \bigcup_{i=1}^{n} H_i$$

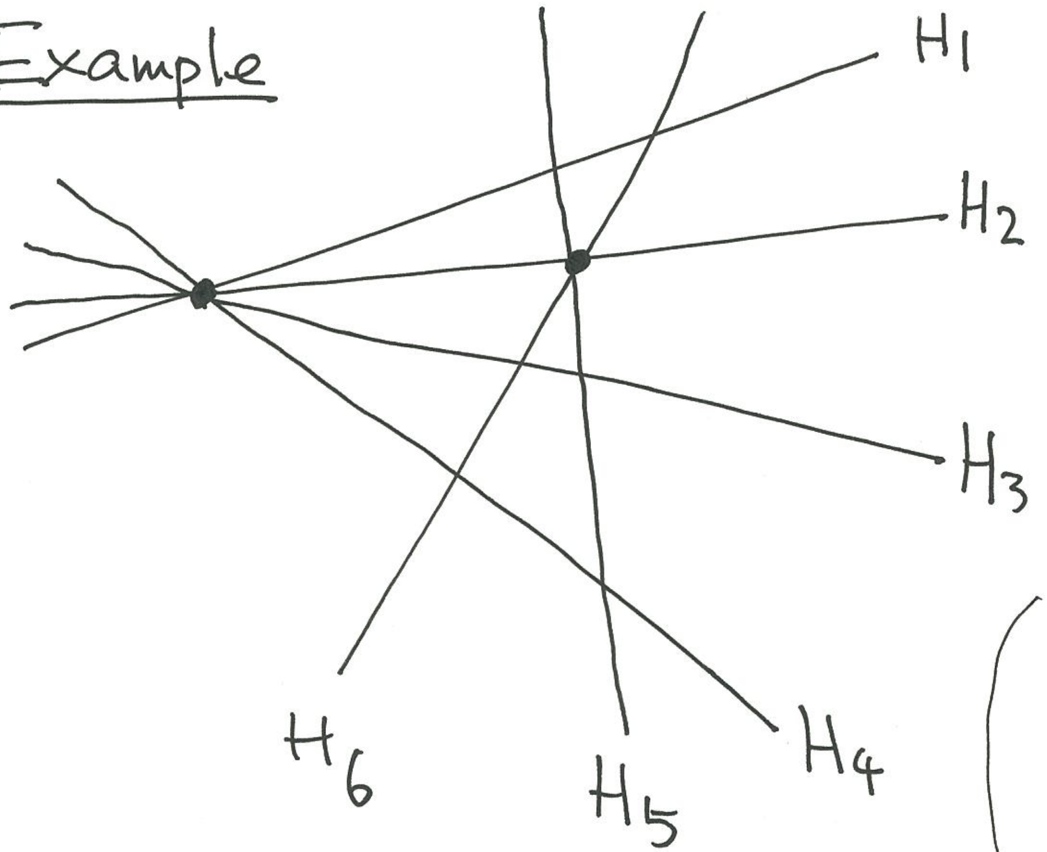
the complement.

Topology of $M(\mathcal{A}) \longleftrightarrow$ How lines are intersecting

Def. (Incidence of $\mathcal{A}$) "the set of intersecting triple"

$$I(\mathcal{A}) := \left\{ \{i, j, k\} \subset \{1, 2, \cdots, n\} \;\middle|\; H_i \cap H_j \cap H_k \neq \phi \right\}$$

Example



$$I(\mathcal{A}) = \{234, 134, 124, 123, 256\}$$

(In this case:

$$M(\mathcal{A}) = \mathbb{P}^2_{\mathbb{C}} \setminus \bigcup_{i=1}^{6} H_i \overset{\text{homeo}}{\cong} \mathbb{C}\setminus\{0,1\} \times \mathbb{C}\setminus\{0,1,2\}$$
)

Notation: $\mathcal{A} = \{H_1, H_2, \cdots, H_n\}$ lines on $\mathbb{P}^2$

$M(\mathcal{A}) := \mathbb{P}^2_{\mathbb{C}} \setminus \bigcup_{i=1}^{n} H_i$

$I(\mathcal{A}) := \{ \{i\,j\,k\} \mid H_i \cap H_j \cap H_k \neq \phi \}$

☆ General Principle ☆

$I(\mathcal{A})$ has (some) topological info's of $M(\mathcal{A})$

<u>General Principle: $I(\mathcal{A})$ has info's of $M(\mathcal{A})$</u>

$I(\mathcal{A})$ determines:

e.g. ① $\underline{H^*(M(\mathcal{A}), \mathbb{Z})}$ (Orlik-Solomon)

② $\underline{H^*(M(\mathcal{A}), \mathcal{L})}$ for certain local system $\mathcal{L}$ (Esnault-Schechtman-Viehweg)

③ $\underline{\pi_1(M(\mathcal{A}))}$ for $\underline{\underline{n \leq 8}}$

(K.-M. Fan, Garber-Teicher-Vishne, etc)

<u>General Principle</u>: $I(\mathcal{A})$ has info's of $M(\mathcal{A})$

$I(\mathcal{A})$ determines : ① $H^*(M(\mathcal{A}), \mathbb{Z})$

e.g. ② $H^*(M(\mathcal{A}), \mathcal{L})$ $\mathcal{L}$: certain loc. sys.

③ $\pi_1(M(\mathcal{A}))$ for $\underline{n \leq 8}$

Counter e.g. ④ (Rybnikov) $n = 13$;

$\exists \mathcal{A}_1, \mathcal{A}_2$ s.t. $I(\mathcal{A}_1) = I(\mathcal{A}_2)$ &

$\pi_1(M(\mathcal{A}_1)) \not\cong \pi_1(M(\mathcal{A}_2))$

Problem : Find other Rybnikov-type pairs.

$I(\mathcal{A})$ determines
$$\begin{cases} ① \; H^*(M(\mathcal{A}), \mathbb{Z}) \\ ② \; H^*(M(\mathcal{A}), \mathcal{L}) \text{ for some } \mathcal{L} \\ ③ \; \pi_1(M(\mathcal{A})) \text{ for } n \leq 8 \end{cases}$$

But ④ $\exists \mathcal{A}_1, \mathcal{A}_2$, $I(\mathcal{A}_1) = I(\mathcal{A}_2)$,
$\pi_1(M(\mathcal{A}_1)) \ncong \pi_1(M(\mathcal{A}_2))$

Lattice-isotopy theorem (Randell)

If the moduli space $R(I)$ is connected (#)

$\Longrightarrow$ ☆ General Principle ☆

The assumption (#) is not true in general.

<u>Today's Focus</u>:

When is the moduli space $R(I)$ connected / disconnected?

<u>Def.</u> (moduli space $R(I)$)

Let $I = I(A) \subset 2^{\{1,2,\cdots,n\}}$: an incidence

$$R(I) := \left\{ (H_1, \cdots, H_n) \in (\mathbb{P}^{2*})^n \;\middle|\; \begin{array}{l} H_i \neq H_j \\ H_i \cap H_j \cap H_k \neq \phi \text{ if } (ijk) \in I \\ H_i \cap H_j \cap H_k = \phi \text{ if } (ijk) \notin I \end{array} \right\}$$

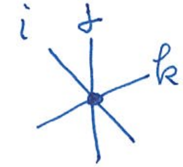
For $I = I(\mathcal{A}) \subset 2^{\{1,2,\dots,n\}}$ an incidence,

$$R(I) := \left\{ (H_1, \dots, H_n) \in (\mathbb{P}^{2*})^n \;\middle|\; \begin{array}{l} H_i \neq H_j \\ H_i \cap H_j \cap H_k \neq \phi \text{ if } (ijk) \in I \\ H_i \cap H_j \cap H_k = \phi \text{ if } (ijk) \notin I \end{array} \right\}$$

<u>Remark</u>

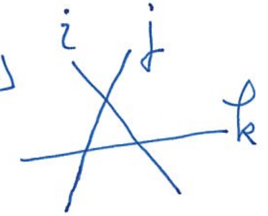
- $PGL_3(\mathbb{C}) \curvearrowright R(I)$ diagonal action.
- $R(I)$ and $R(I) / PGL_3(\mathbb{C})$ are quasi-projective Variety

§2 (dis-) connectivity of $R(I)$



intersecting

$$R(I) = \left\{ (H_1, \ldots, H_n) \in (\mathbb{P}^{2*})^n \;\middle|\; \begin{array}{l} H_i \neq H_j \\ H_i \cap H_j \cap H_k \neq \phi, \text{ if } (ijk) \in I \\ H_i \cap H_j \cap H_k = \phi, \text{ if } (ijk) \notin I \end{array} \right\}$$



NON Intersecting

The simplest examples:

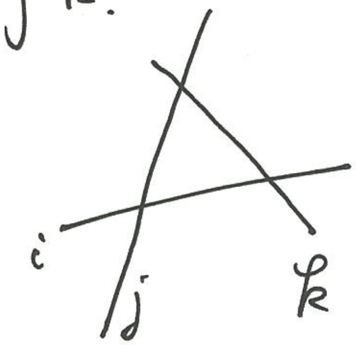
- $\underline{n=1} \Longrightarrow (I = \phi)$, $R(I) = \mathbb{P}^{2*}$
- $\underline{n=2} \Longrightarrow (I = \phi)$, $R(I) = \{(H_1, H_2) \mid H_1 \neq H_2\}$
  $= (\mathbb{P}^{2*})^2 \setminus (\text{diagonal})$

§2 (dis-) connectivity of $R(I)$

Examples of the moduli space

$$R(I) := \left\{ (H_1, H_2, \cdots, H_n) \;\middle|\; \begin{array}{ll} H_i \neq H_j, & \\ H_i \cap H_j \cap H_k \neq \phi & \text{if } (ijk) \in I \\ H_i \cap H_j \cap H_k = \phi & \text{if } (ijk) \notin I \end{array} \right\}$$

<u>Example</u> $I = \phi$, i.e. $H_i \cap H_j \cap H_k = \phi \quad \forall i j k.$



$$R(I) = \left\{ (H_1, \cdots, H_n) \;\middle|\; \begin{array}{l} H_i \neq H_j \\ H_i \cap H_j \cap H_k = \phi \end{array} \right\}$$

$\hookrightarrow (\mathbb{P}^{2*})^n$

Zariski open $\rightsquigarrow R(I)$: irreducible

$\rightsquigarrow$ connected!

<u>Example</u> $n = 8$. $\mathcal{A} = \{H_1, \dots, H_8\}$

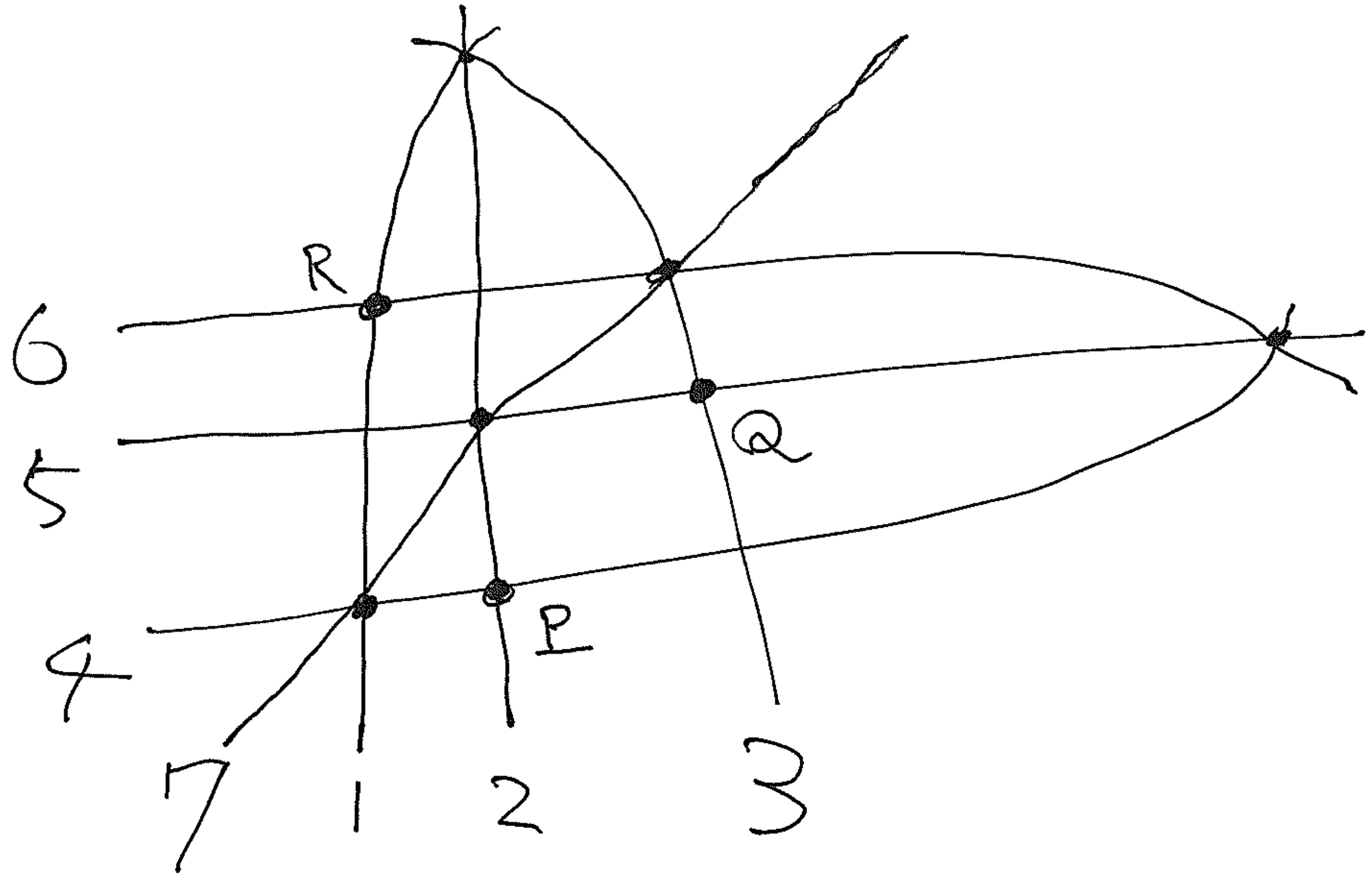
$$I = \{123, 456, 147, 257, 367, 248, 358, 168\}$$

$$R(I)/PGL_3 = ??$$

<u>Example</u> $n=8$, $\mathcal{A}=\{H_1, H_2, \cdots, H_8\}$

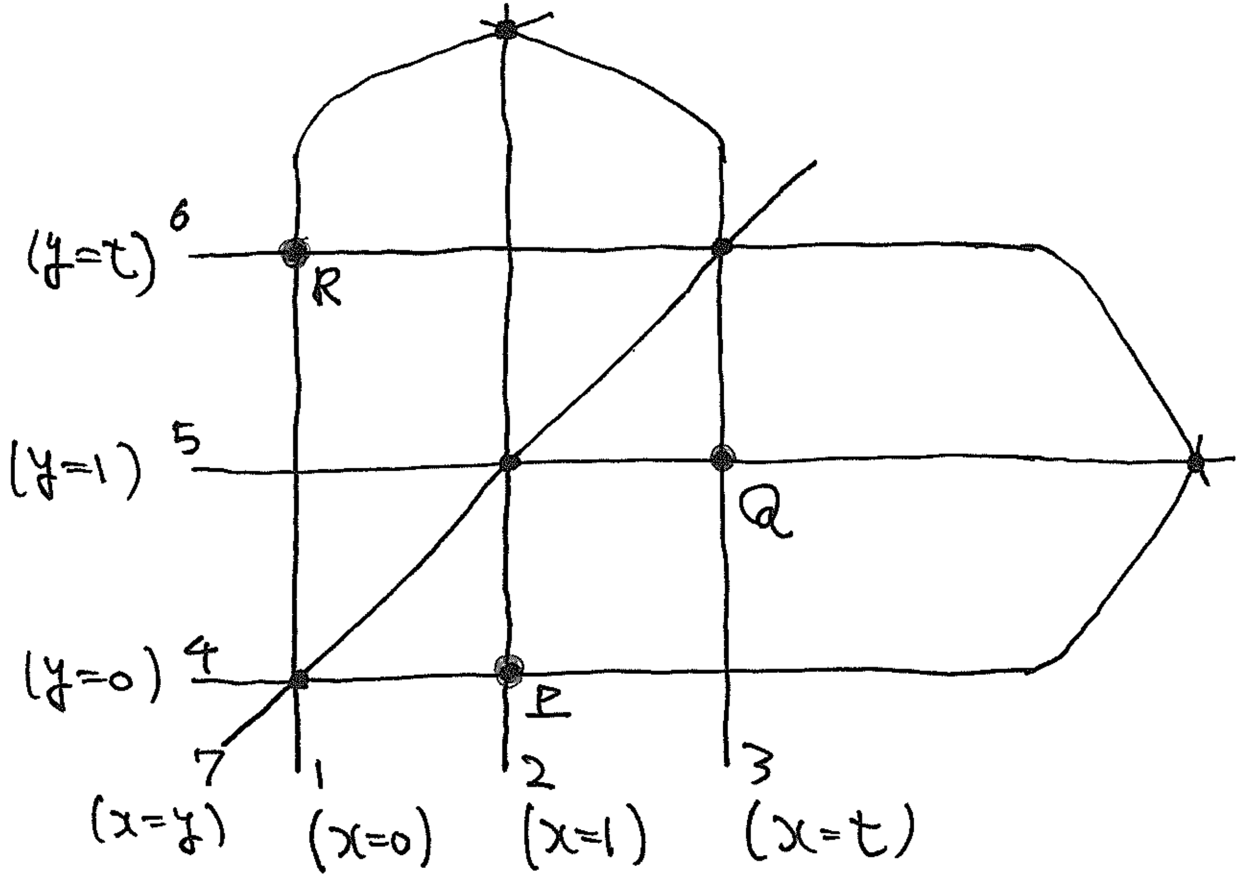
$I=\{\underline{123}, \underline{456}, \underline{147}, 25\underline{7}, \underline{367}, 24\underline{8}, 358, 168\}$

$R(I)/PGL_3(\mathbb{C}) = ??$



<u>Example</u> $n=8$, $\mathcal{A}=\{H_1, H_2, \cdots, H_8\}$

$$I=\{123,\ 456,\ 147,\ 257,\ 367,\ \underset{P}{248},\ \underset{Q}{358},\ \underset{R}{168}\}$$



Points P, Q, R are collinear

$\Updownarrow$

$t^2-t+1=0$, $t=\dfrac{1\pm\sqrt{-3}}{2}$

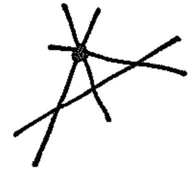
(MacLane arrange-ment)

$$R(I)/PGL_3(\mathbb{C})=\{\mathcal{A}^{+\sqrt{-3}},\ \mathcal{A}^{-\sqrt{-3}}\}\ \text{2-points.}$$

(the simplest example of disconnected $R(I)$)

Def.

$$\mathrm{mult}(\mathcal{A}) := \left\{ p \in \mathbb{P}^2 \;\middle|\; \begin{array}{l} \exists\, i, j, k \text{ distinct} \\ \text{s.t. } p = H_i \cap H_j \cap H_k \end{array} \right\}$$



Theorem 1 (Nazir – Y.)

$\mathcal{A} = \{H_1, H_2, \cdots, H_n\}$, $I := I(\mathcal{A})$, $R(I) \neq \phi$

$\mathcal{A}' = \{H_1, \cdots, H_{n-1}\}$, $I' := I(\mathcal{A}')$

$\mu := |\mathrm{mult}(\mathcal{A}) \cap H_n|$.

If $R(I')$ is irreducible and $\mu \leq 2$

$\Longrightarrow$ $R(I)$ is also irreducible.

<u>Thm. 1</u> $\mathcal{A} = \{H_1, \cdots, H_n\}$, $\mathcal{A}' = \mathcal{A} \setminus \{H_n\}$

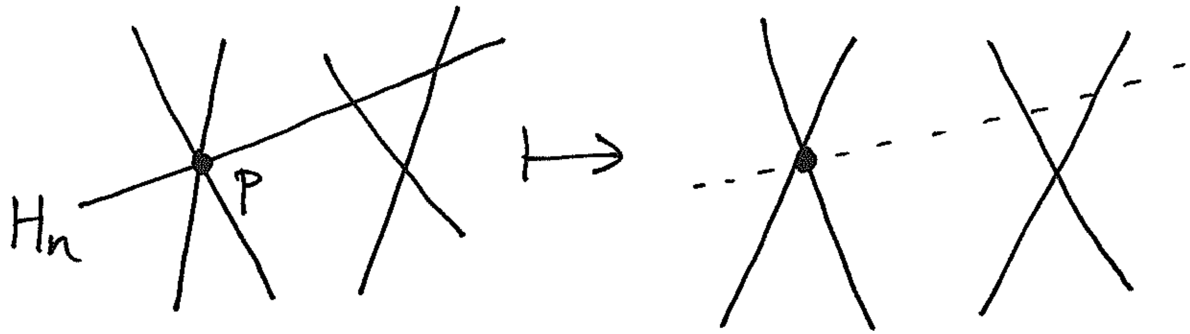
$\mu := |\mathrm{mult}(\mathcal{A}) \cap H_n| \leq 2$ & $R(I')$: irred.

$\Longrightarrow R(I)$: irred.

<u>Sketch of proof for $\mu = 1$</u>

$$\pi : R(I) \longrightarrow R(I')$$

$$R(I) \ni (H_1, \cdots, H_n) \longmapsto (H_1, \cdots, H_{n-1}) \in R(I')$$



$\{P\} = \mathrm{mult}(\mathcal{A}) \cap H_n$

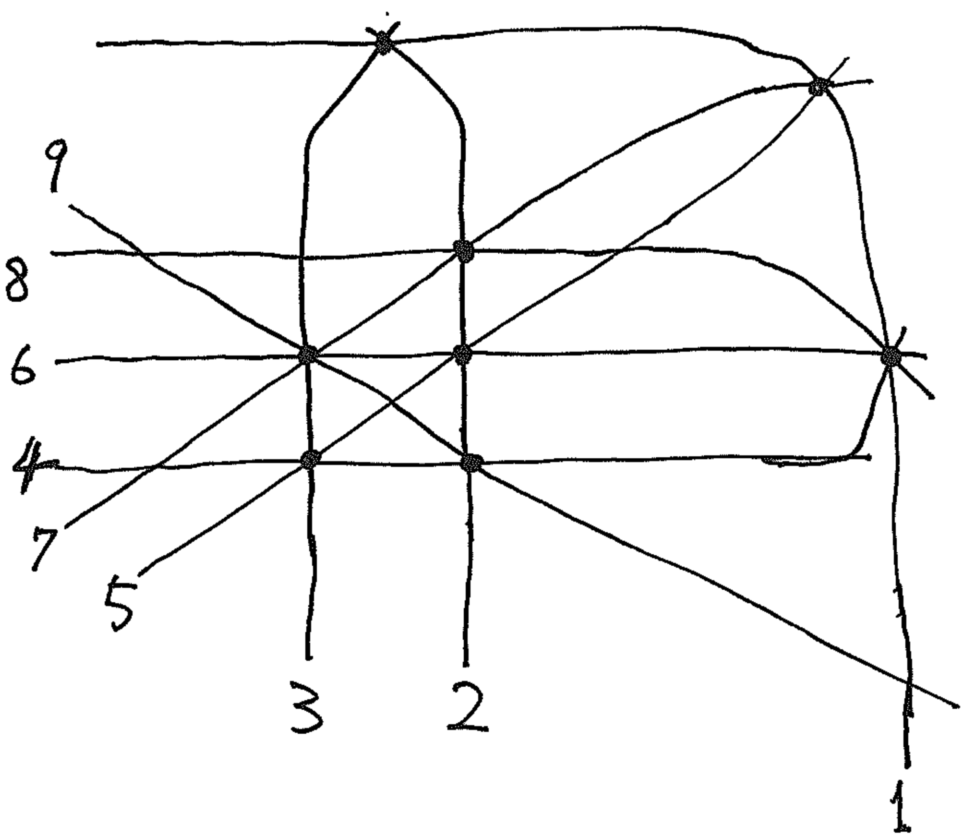
($R(I)$ is a Zariski open of $\mathbb{P}^1$-fibration over $R(I')$)

Thm. 1 $\mathcal{A} = \{H_1, \cdots, H_n\}$, $\mathcal{A}' = \mathcal{A} \setminus \{H_n\}$

$\mu := |\mathrm{mult}(\mathcal{A}) \cap H_n| \leq 2$ & $R(I')$: irred.

$\Longrightarrow$ $R(I)$: irred.

Example

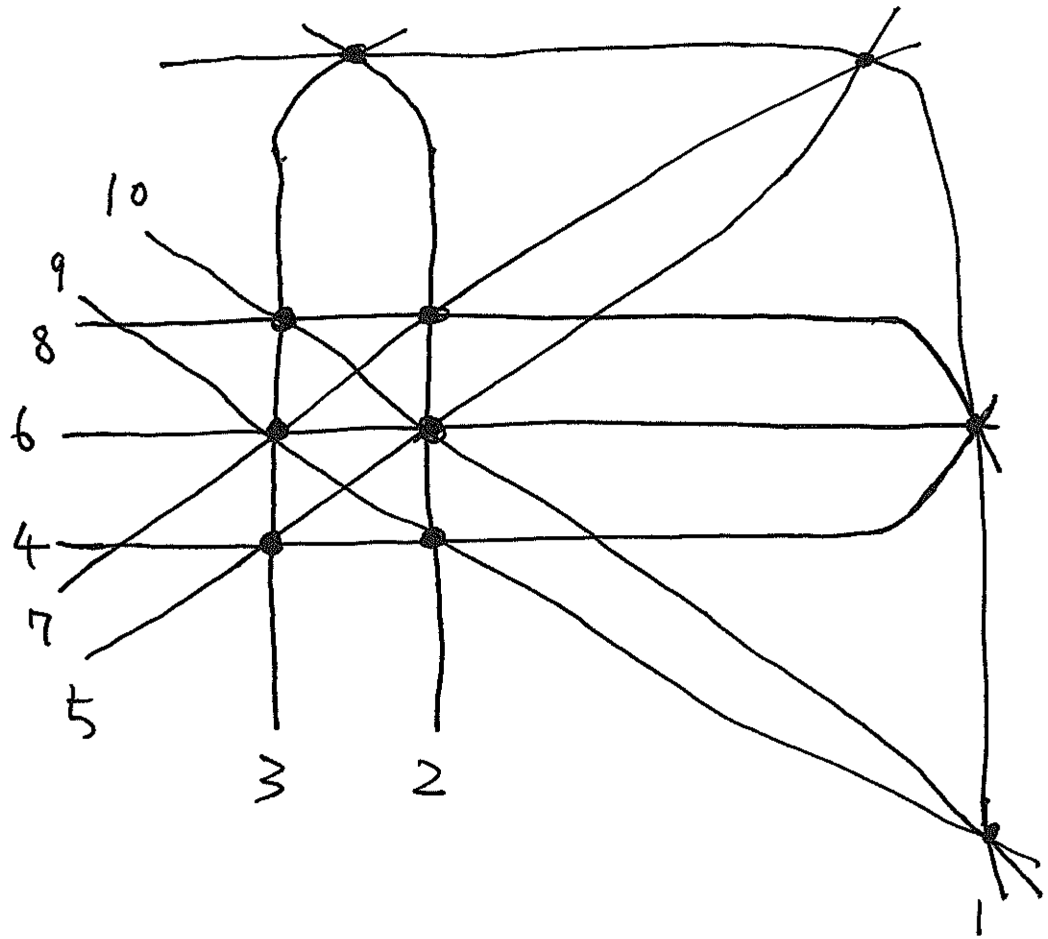


$|\mathrm{mult}(\mathcal{A}) \cap H_9| = 2$

$\Downarrow$ Thm. 1 + inductive argument

$R(I)$: irreducible

<u>Example</u>



$|\mathrm{mult}(\mathcal{A}) \cap H| \geq 3$
for $\forall H \in \mathcal{A}$.

We cannot apply Thm. 1.

But still $R(I)$: irreducible
by the next.

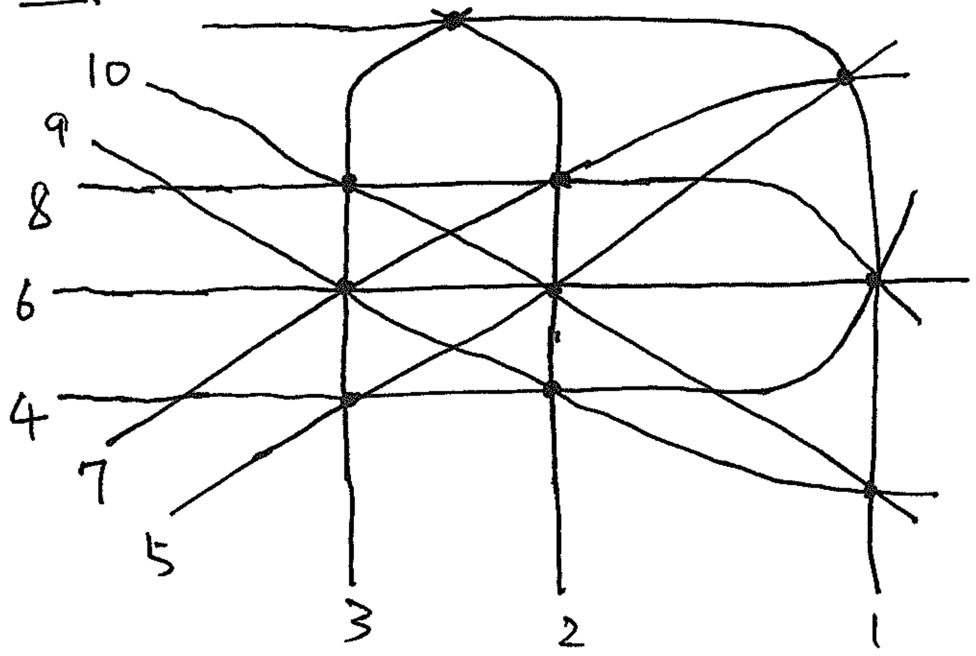
$(\mathrm{mult}(\mathcal{A}) \subset H_1 \cup H_2 \cup H_3)$

Theorem 2 (Nazir–Y.) $\mathcal{A} = \{H_1, \cdots, H_n\}$

$\exists\, i, j, k$ s.t. $\begin{cases} \cdot\ H_i \cap H_j \cap H_k \neq \phi \\ \cdot\ \mathrm{mult}(\mathcal{A}) \subset H_i \cup H_j \cup H_k \end{cases}$

$\Longrightarrow$ $R(I(\mathcal{A}))$ is irreducible

Ex.



$\cdot\ H_1 \cap H_2 \cap H_3 \neq \phi$

$\cdot\ \mathrm{mult}(\mathcal{A}) \subset H_1 \cup H_2 \cup H_3$

$\Big\Downarrow$ Thm. 2

$R(I)$: irreducible

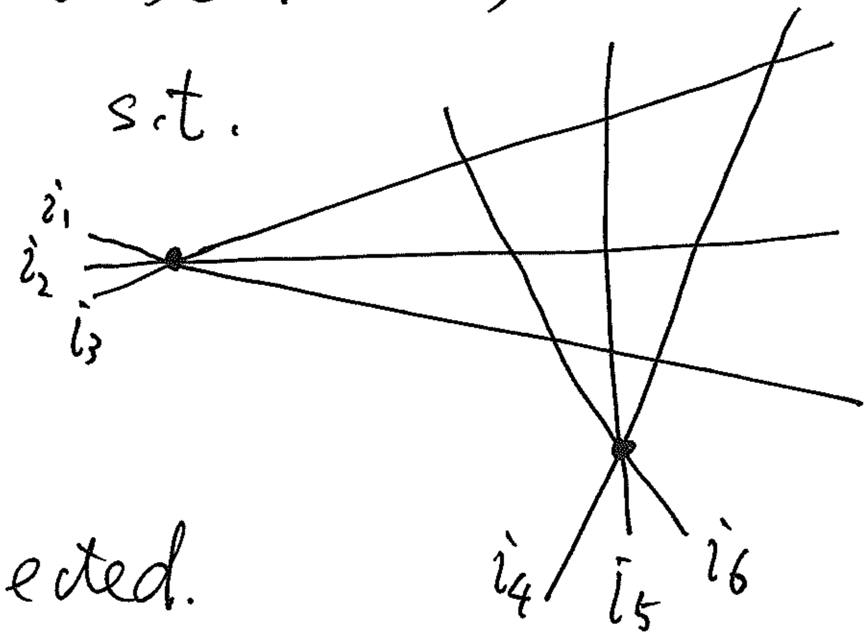
Thm. 2 $\mathcal{A} = \{H_1, \dots, H_n\}$.

$\exists i, j, k$ s.t. $\begin{cases} \cdot\ H_i \cap H_j \cap H_k \neq \phi \\ \cdot\ \mathrm{mult}(\mathcal{A}) \subset H_i \cup H_j \cup H_k \end{cases}$

$\Longrightarrow R(I(\mathcal{A}))$: irreducible

Corollary If $R(I)$ is disconnected,

$\Longrightarrow \exists\, i_1, i_2, \dots, i_6$ s.t.



Cor.

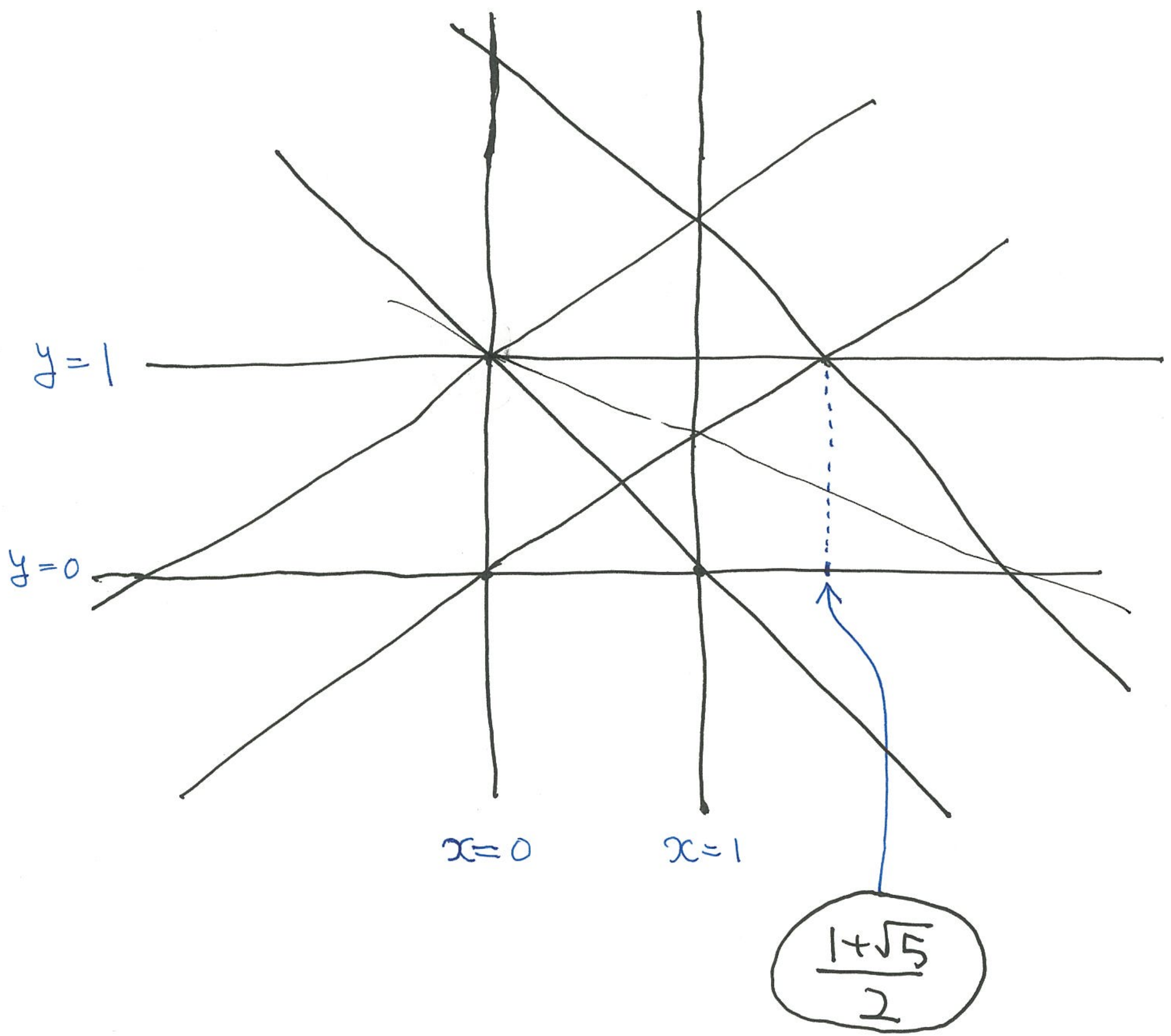
$n \leq 7 \Longrightarrow R(I)$: connected.

§3. Classification up to 9 lines

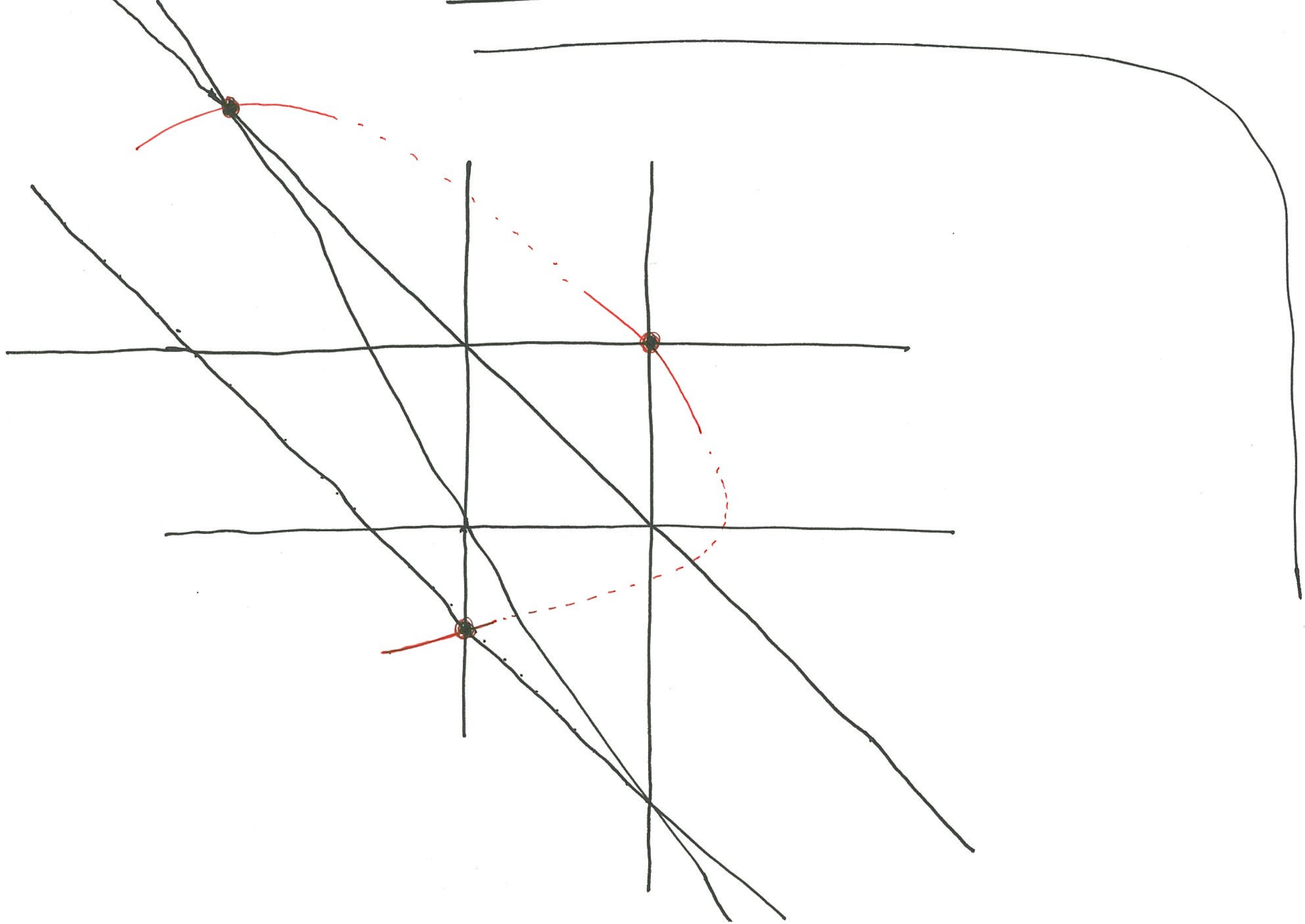
n	disconnected $R(\mathcal{I})$	minimal field of definition
≤ 7	NO	$\mathbb{Q}$
8	$\mathcal{A}^{\pm\sqrt{-3}}$ (MacLane)	$\mathbb{Q}(\sqrt{-3})$
9	$\mathcal{A}^{\pm\sqrt{-3}} \cup \{H_9\}$	$\mathbb{Q}(\sqrt{-3})$
	$\mathcal{A}^{\pm\sqrt{5}}$ (Falk-Sturmfels)	$\mathbb{Q}(\sqrt{5})$
	$\mathcal{A}^{\pm\sqrt{-1}}$	$\mathbb{Q}(\sqrt{-1})$
10	$\mathcal{A}^{\pm\sqrt{2}}$	$\mathbb{Q}(\sqrt{2})$
	and many others	

<u>Cor.</u> For $n \leq 9$, $I(\mathcal{A})$ determines $\pi_1(M(\mathcal{A}))$

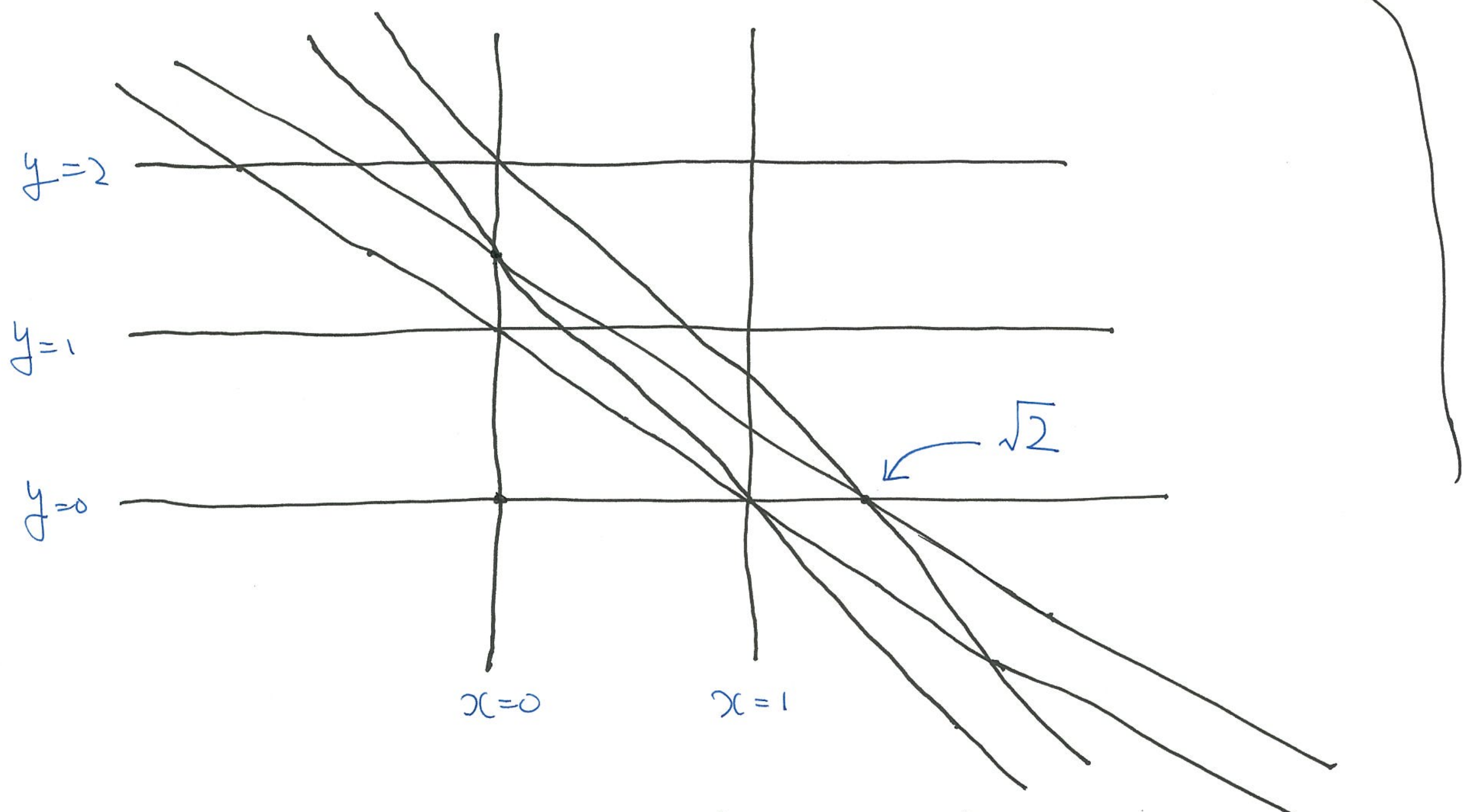
Falk - Sturmfels Arrangement $(n=9)$



$\underline{\mathcal{A}^{\pm\sqrt{-1}}}$ $(n=9)$



$\underline{A^{\pm\sqrt{2}}}$ $(n=10)$



We do not know $\pi_1(M(A^{+\sqrt{2}})) \cong \pi_1(M(A^{-\sqrt{2}}))$ or $\ncong$