Conjugation-free geometric presentation of fundamental groups of arrangements and its applications

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Based on joint works with Meital Eliyahu and Mina Teicher and with Michael Friedman

Importance

• Distinguishing between connected components of the moduli space of surfaces.

• Computation the fundamental group of complements of hypersurfaces in $\mathbb{CP}^N$.

• New examples of Zariski pairs.

• Exploring new examples for finite non-abelian fundamental groups.

• Computing the fundamental group of the Galois cover of a surface.

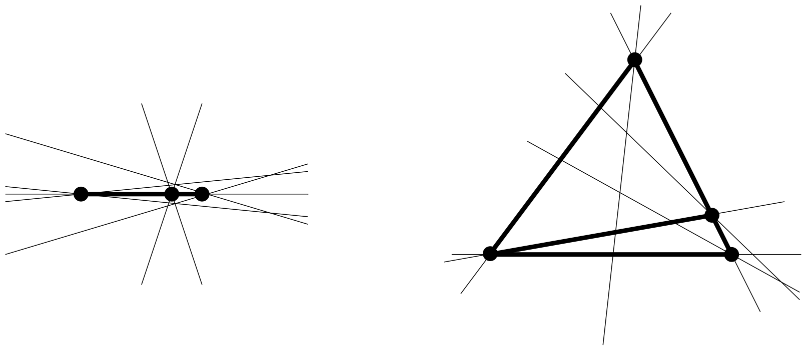
Graph of multiple points

Line arrangement in $\mathbb{CP}^2$: An algebraic curve in $\mathbb{CP}^2$ which is a union of projective lines. An arrangement is called *real* if its defining equations can be written with real coefficients.

$G(\mathcal{L})$:

Vertices: Multiple points

Edges: Segments on lines with more than two multiple points.



(a) (b)

Fan (1997): Let $\mathcal{L}$ be an arrangement of n lines and $S = \{a_1, \dots, a_p\}$ be the set of multiple points of $\mathcal{L}$ (multiplicity ≥ 3). Suppose that $\beta(\mathcal{L}) = 0$ (i.e. the graph $G(\mathcal{L})$ has no cycles). Then:

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^r \oplus \mathbb{F}_{m(a_1)-1} \oplus \cdots \oplus \mathbb{F}_{m(a_p)-1}$$

where $r = n + p - 1 - m(a_1) - \cdots - m(a_p)$.

G-Teicher (2000): Part of this result by braid monodromy techniques.

Eliyahu-Liberman-Schaps-Teicher (2009): If the fundamental group is a sum of free groups, then $G(\mathcal{L})$ has no cycles.

A generic presentation of the fundamental group

(Arvola, Randell, Cohen-Suciu, ...)

Let $\mathcal{L}$ be an arrangement of n lines.

Then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ is generated by $x_1, \ldots, x_n$ - the natural topological generators.

The relations: for each intersection point of multiplicity k:

$$x_{i_k}^{s_k} x_{i_{k-1}}^{s_{k-1}} \cdots x_{i_1}^{s_1} = x_{i_{k-1}}^{s_{k-1}} \cdots x_{i_1}^{s_1} x_{i_k}^{s_k} = \cdots = x_{i_1}^{s_1} x_{i_k}^{s_k} \cdots x_{i_2}^{s_2}$$

where $a^b := b^{-1}ab$ and s_i are words in $\langle x_1, \ldots, x_n \rangle$ $(1 \leq i \leq k)$.

Conjugation-free geometric presentation of fundamental group

A *conjugation-free geometric presentation* of a fundamental group is a presentation with the natural topological generators $x_1, \ldots, x_n$ and the cyclic relations:

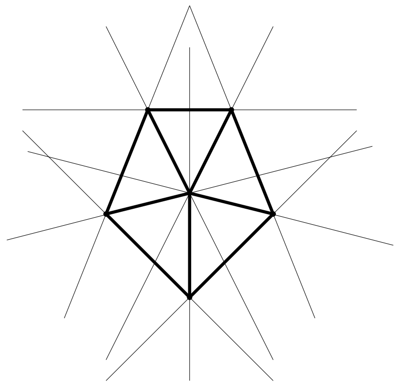
$$x_{i_k} x_{i_{k-1}} \cdots x_{i_1} = x_{i_{k-1}} \cdots x_{i_1} x_{i_k} = \cdots = x_{i_1} x_{i_k} \cdots x_{i_2}$$

with no conjugations on the generators.

Main importance: For this family the lattice determines the fundamental group. Moreover, one can read this presentation directly from the arrangement.

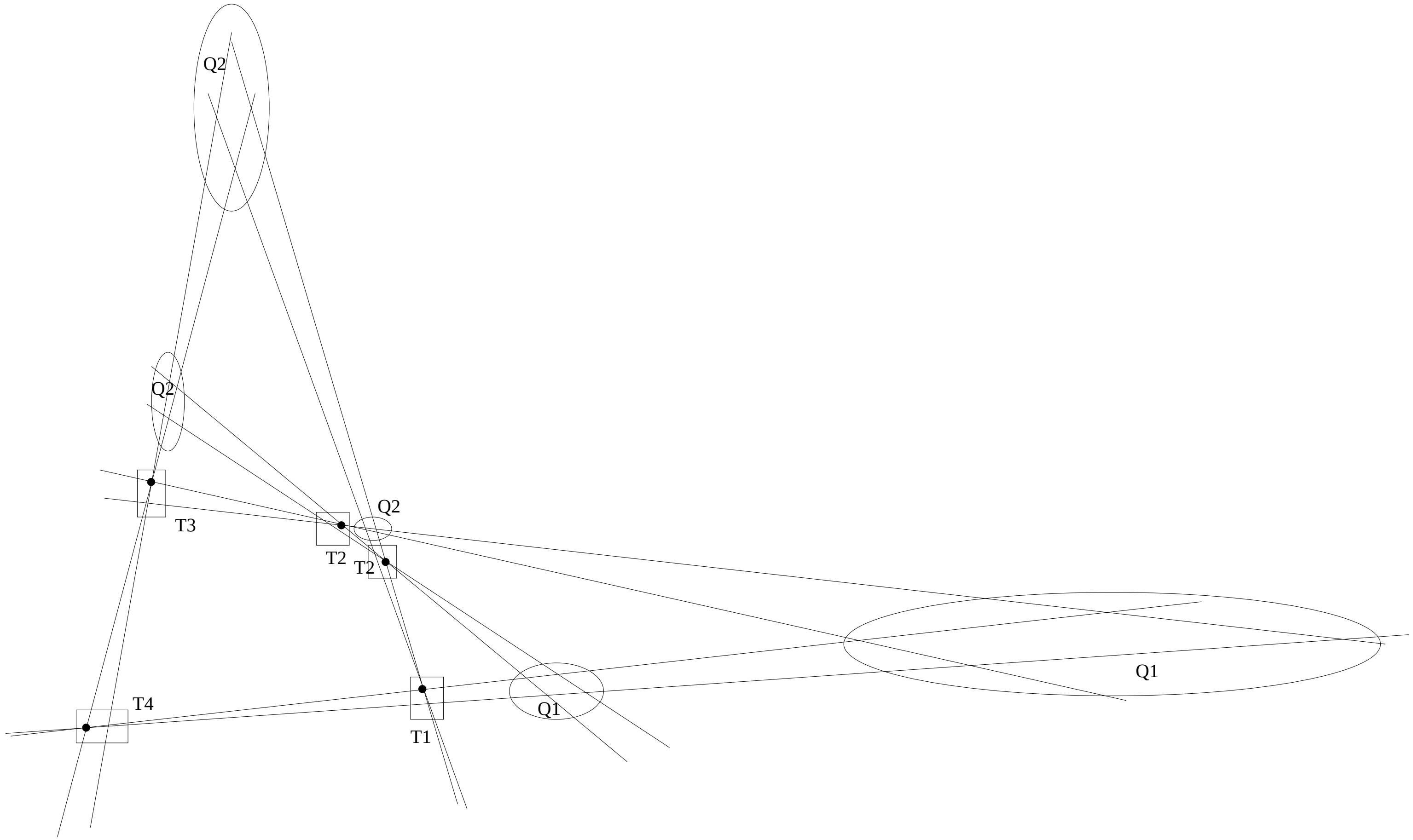
Eliyahu-G-Teicher (2008): if $G(\mathcal{L})$ is a disjoint union of cycles, then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ has a conjugation-free geometric presentation.

Family A_n:



Computationally proved: A_5, A_6 have a conjugation-free geometric presentation. A_3 (Ceva) and A_7 have no conjugation-free geometric presentation (computational proof by Testisom package).

Direct proof for an arrangement whose graph is a cycle



Proof of an arrangement whose graph is a **union** of cycles

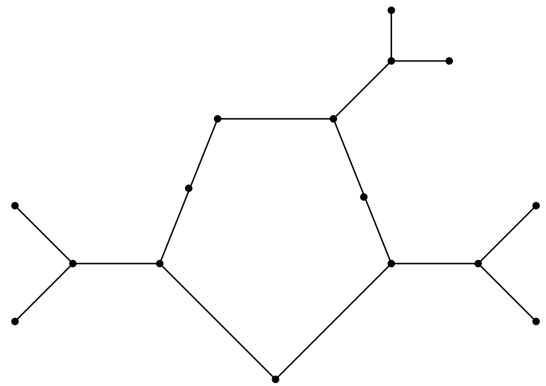
We use:

Oka-Sakamoto (1978): Let C_1 and C_2 be algebraic plane curves in $\mathbb{C}^2$. Assume that the intersection $C_1 \cap C_2$ consists of distinct $d_1 \cdot d_2$ points, where d_i $(i \in \{1,2\})$ are the respective degrees of C_1 and C_2. Then:

$$\pi_1(\mathbb{C}^2 - (C_1 \cup C_2)) \cong \pi_1(\mathbb{C}^2 - C_1) \oplus \pi_1(\mathbb{C}^2 - C_2)$$

Expansion of the conjugation-free geometric presentation's family

A **cycle-tree** graph is a graph which consists of a cycle, where each vertex of the cycle can be a root of a tree.



Eliyahu-G-Teicher (2010): if $G(\mathcal{L})$ is a disjoint union of cycle-tree graphs, then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ has a conjugation-free geometric presentation.

Key lemma: Adding a line which passes through only one multiple point preserves the conjugation-free property.

Applications of Key Lemma

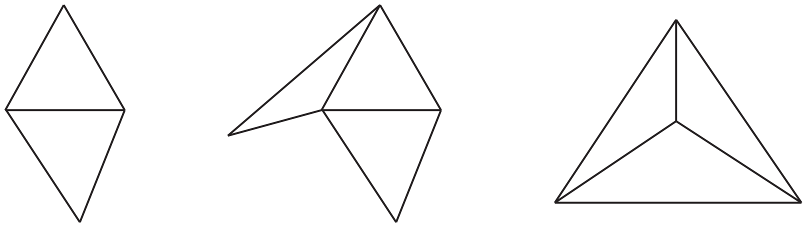
Eliyahu-Friedman-G-Teicher (2012)

Main theorem of **Eliyahu-G-Teicher** (if G is a cycle, then we have conjugation-free) can be reproved in an inductive way:

(1) We first draw only the lines that are the edges of the cycle $G(\mathcal{L})$. Since we have only nodes, the fundamental group is abelian and conjugation-free.

(2) Add inductively the lines which pass through the multiple points.

Definition: Let G be a planar graph. G is called a **CF-graph** if:
(1) $\beta(G(\mathcal{L})) \leq 1$, or
(2) Let $\{v_i\}_{i=1}^m \subseteq V(G)$ satisfying deg$(v_i) \leq 2$. For each v_i, let V_i be a subgraph of G composed of v_i and the edge(s) exiting from it. Let $X = X(G) \doteq \bigcup_{i_1}^m V_i$. Then G is CF graph if $G - X$ is.



(a) (b) (c)

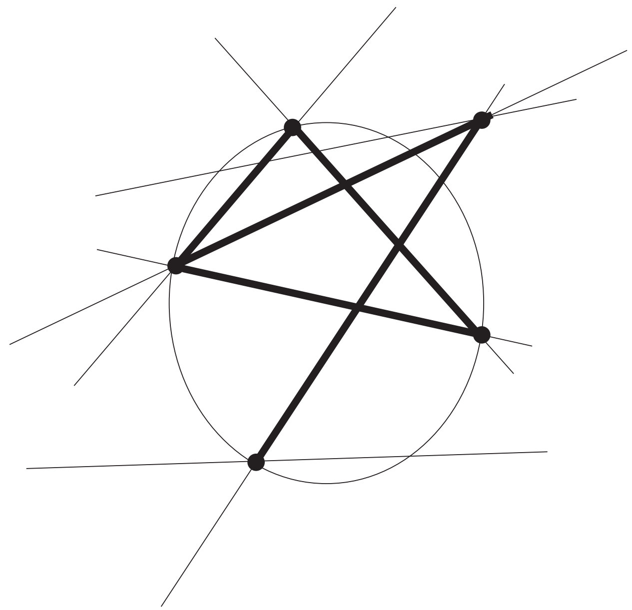
Theorem: Let $\mathcal{L}$ be a line arrangement. If $G(\mathcal{L})$ is CF-graph, then the fundamental group of $\mathcal{L}$ is conjugation-free.

Remark: There exist arrangements whose fundamental group is conjugation-free whose graph is not a CF graph, e.g. A_5.

Real conic-line arrangements

Definition: A *real conic-line (CL) arrangement* $\mathcal{A}$ is a collection of conics and lines in $\mathbb{C}^2$, where all the conics and the lines are defined over $\mathbb{R}$ and every singular point of the arrangement is in $\mathbb{R}^2$. In addition, for every conic C, $C \cap \mathbb{R}^2$ is not an empty set, neither a point nor a (double) line.

The graph for a real CL arrangement: the vertices of the graph will be the multiple points (with multiplicity larger than 2), and the edges will be the segments **on the lines** connecting these points.



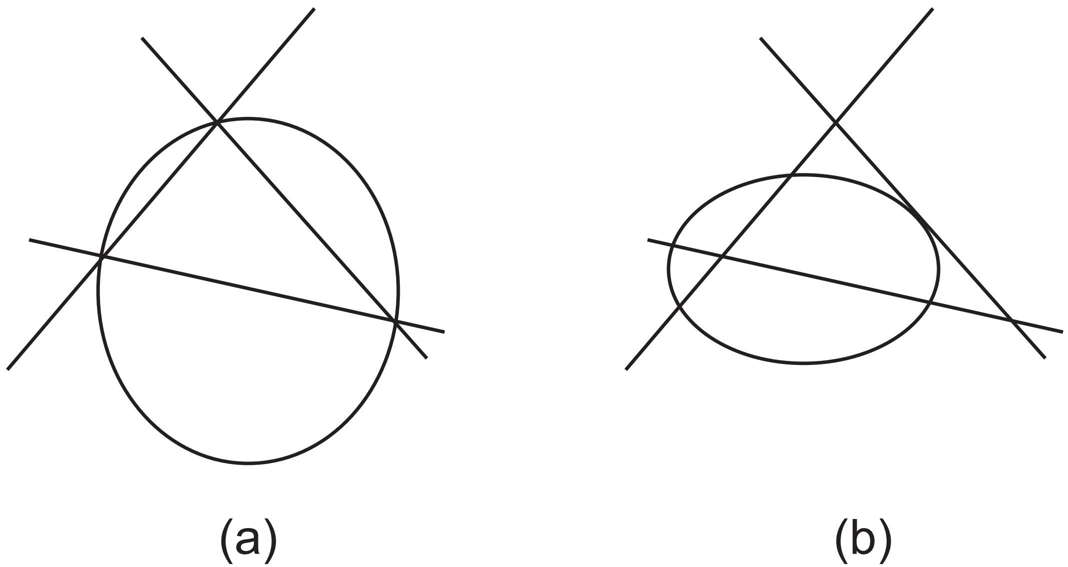
Real conic-line arrangements whose graphs are cycle-free

Theorem (G-Friedman, 2012): Let $\mathcal{A}$ be a real CL arrangement with one conic and k lines, and $S = \{a_1, \dots, a_p, b_1, \dots, b_q\}$ be the set of all multiple points of $\mathcal{A}$, where the conic is passing through the intersection points $a_1, \dots, a_p$. Suppose that the graph $G(\mathcal{A})$ has no cycles. Then:

$$\pi_1(\mathbb{CP}^2 - \mathcal{A}) \simeq \mathbb{Z}^r \oplus \bigoplus_{i=1}^{p} \mathbb{F}_{m(a_i)-2} \oplus \bigoplus_{i=1}^{q} \mathbb{F}_{m(b_i)-1},$$

where $m(a_i)$ is the multiplicity of the intersection point a_i and $r = k + 2p + q - \sum_{i=1}^{p} m(a_i) - \sum_{i=1}^{q} m(b_i)$.

Remark: For line arrangements - the inverse direction is correct (a direct sum implies no cycles), but for CL arrangements it is not true anymore, since there exist CL arrangements with cycles and tangent points whose fundamental groups are still abelian.



Extension of the notion of conjugation-free geometric presentation to CL arrangements

Definition: Let G be a fundamental group of the affine or projective complements of some CL arrangement with k lines and n conics. We say that G has *a conjugation-free geometric presentation* if G has the following presentation:

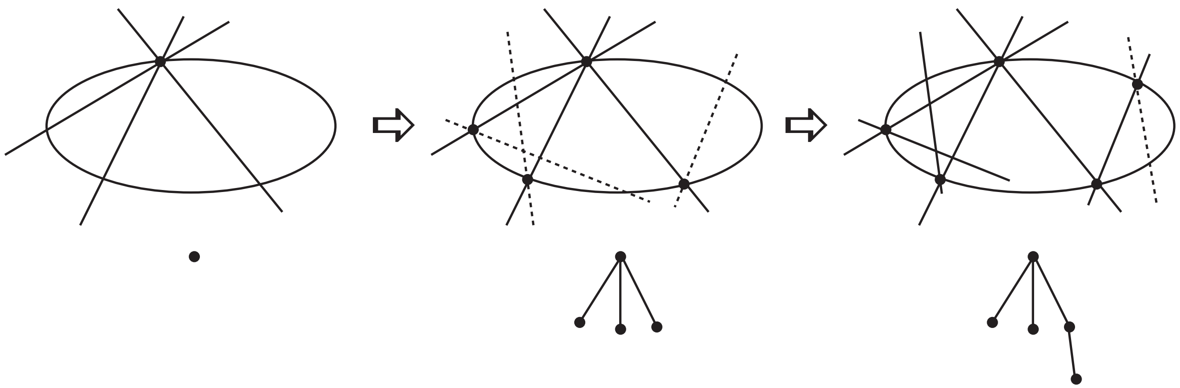
- The generators $\{x_1, \ldots, x_m\}$ are the meridians of lines and conics at some far side of the arrangement, where $m = k + 2n$ in the affine case and $m = k + 2n - 1$ in the projective case.
- The relations are of the following types:

  $x_{i_t} x_{i_{t-1}} \cdots x_{i_1} = x_{i_{t-1}} \cdots x_{i_1} x_{i_t} = \cdots = x_{i_1} x_{i_t} \cdots x_{i_2}$ or $x_{i_1} = x_{i_2}$.
- In the projective case, we have an extra relation that a specific multiplication of all the generators is equal to the identity element.

Idea of the proof of the theorem:

Key lemma: Let $\mathcal{A}$ be a real CL arrangement with one conic such that $\pi_1(\mathbb{CP}^2 - \mathcal{A})$ is conjugation-free. Let L be a line that passes through a single multiple point of $\mathcal{A}$. Then $\pi_1(\mathbb{CP}^2 - (\mathcal{A} \cup L))$ is conjugation-free.

Proposition: Let $\mathcal{A}$ be a real CL arrangement with one conic and k lines. Suppose that $\beta(\mathcal{A}) = 0$. Then $\pi_1(\mathbb{CP}^2 - \mathcal{A})$ is conjugation-free.



(A) (B) (C)

Proposition: Let $y_1, \ldots, y_k$ be the geometric generators associated to the k lines and let x be the generator of the conic. Then for each $1 \leq i \leq k$,

$$[x, y_i] \doteq x y_i x^{-1} y_i^{-1} = e.$$

From the proposition, we get:

$$\pi_1(\mathbb{CP}^2 - \mathcal{A}) \simeq \langle x \rangle \oplus \pi_1(\mathbb{CP}^2 - (\mathcal{A} - C)),$$

which implies the theorem, by Fan's result.

Corollary: Let $\mathcal{A}$ be a real CL arrangement with one conic and k lines, with only nodes and triple points as singularities. If $\beta(\mathcal{A}) = 0$ and all the triple points are on the conic, then $\pi_1(\mathbb{CP}^2 - \mathcal{A})$ and $\pi_1(\mathbb{C}^2 - \mathcal{A})$ are abelian.

Real conic-line arrangements whose graphs is a cycle

First case: The conic does not pass through all the points of the cycle.

Theorem (Friedman-G, 2012): Let $\mathcal{A}$ be a real CL arrangement with one conic such that $\beta(G) = 1$, the conic C does not pass through all the points of the graph G and $G(\mathcal{A} - C)$ is a tree. Then:

$$\pi_1(\mathbb{CP}^2 - \mathcal{A}) \simeq \mathbb{Z} \oplus \pi_1(\mathbb{CP}^2 - (\mathcal{A} - C)),$$

and the structure of $\pi_1(\mathbb{CP}^2 - (\mathcal{A} - C))$ is known by Fan's theorem.

Idea of proof:

1. By inductive construction and the key lemma, we show that the presentation of the fundamental group is conjugation-free.

2. By similar arguments to previous results, we show that the generator of the conic commutes with the generators of the other lines.

3. Fan's result yields the structure of the group.

Second case: The conic passes through all the points of the cycle.

The real CL arrangement $\mathcal{D}_n$: Take a polygon with n edges and pass an ellipse through its n vertices. We then extend the edges to be infinite straight lines. Thus: $G(\mathcal{D}_n)$ is a cycle of length n. Let $G_n \doteq \pi_1(\mathbb{C}^2 - \mathcal{D}_n)$.

Proposition:
(1) G_3 is abelian and is not conjugation-free.
(2) $G_4 \simeq \mathbb{Z}^3 \oplus \mathbb{F}_2$ and is not conjugation-free.
(3) G_6 is not isomorphic to a direct sum of free groups and an abelian group.

The generalization:

Theorem (Friedman-G, 2012): For odd n, G_n is abelian and is not conjugation-free.

Idea of proof: One can show that the relation induced from one of branch points becomes a commutative relation, which implies the needed commutation between all the generators.

Thank you!