A combinatorial approach to broken Lefschetz fibrations via mapping class groups

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$\diamondsuit$ Main Results (Roughly)

1. Classification of genus-1 SBLF.
2. $^{\exists}$sections of SBLF w/ properties that sections of LF would not have.

◇ The Plan of Talk

§.1. Introduction

§.2. Main Results

§.3. A Combinatorial Approach

§.4. Outline of Proofs

* We will work in **smooth category**

* We will always assume that **manifolds are oriented**.

§.1. Introduction

◇ Background

Donaldson '99, Gompf '04

symplectic 4-manifold $\Longleftrightarrow$ Lefschetz pencil $\xrightarrow{\text{blow up}}$ Lefschetz fibration

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near-symplectic 4-manifold $\Longleftrightarrow$ **BLP** with $^{\exists}$**condition** $\xrightarrow{\text{blow up}}$ **BLF** with $^{\exists}$**condition**

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symplectic 4-manifold	$\Longleftrightarrow$	Lefschetz pencil	$\xrightarrow{\text{blow up}}$	Lefschetz fibration

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near-symplectic 4-manifold	$\Longleftrightarrow$	**BLP** with $^{\exists}$**condition**	$\xrightarrow{\text{blow up}}$	**BLF** with $^{\exists}$**condition**

$^{\exists}$**BLF without a condition. Indeed,**

> **Theorem 1.1 (Williams '10 e.t.c.)**
> $^{\forall}$closed ori. conn. 4-mfd admits a (**simplified**) **BLF**

Definition 1.2

$f : M^4 \to B^2$ is called a **broken Lefschetz fibration** (**BLF**) if it satisfies the following conditions:

(1) $f^{-1}(\partial B) = \partial M$,

(2) f has at most the following types of singularities:

L) $(z_1, z_2) \mapsto \xi = z_1 z_2$ (**Lefschetz singularity**),

I) $(t, x, y, z) \mapsto (s, w) = (t, x^2 + y^2 - z^2)$ (**indefinite fold**),

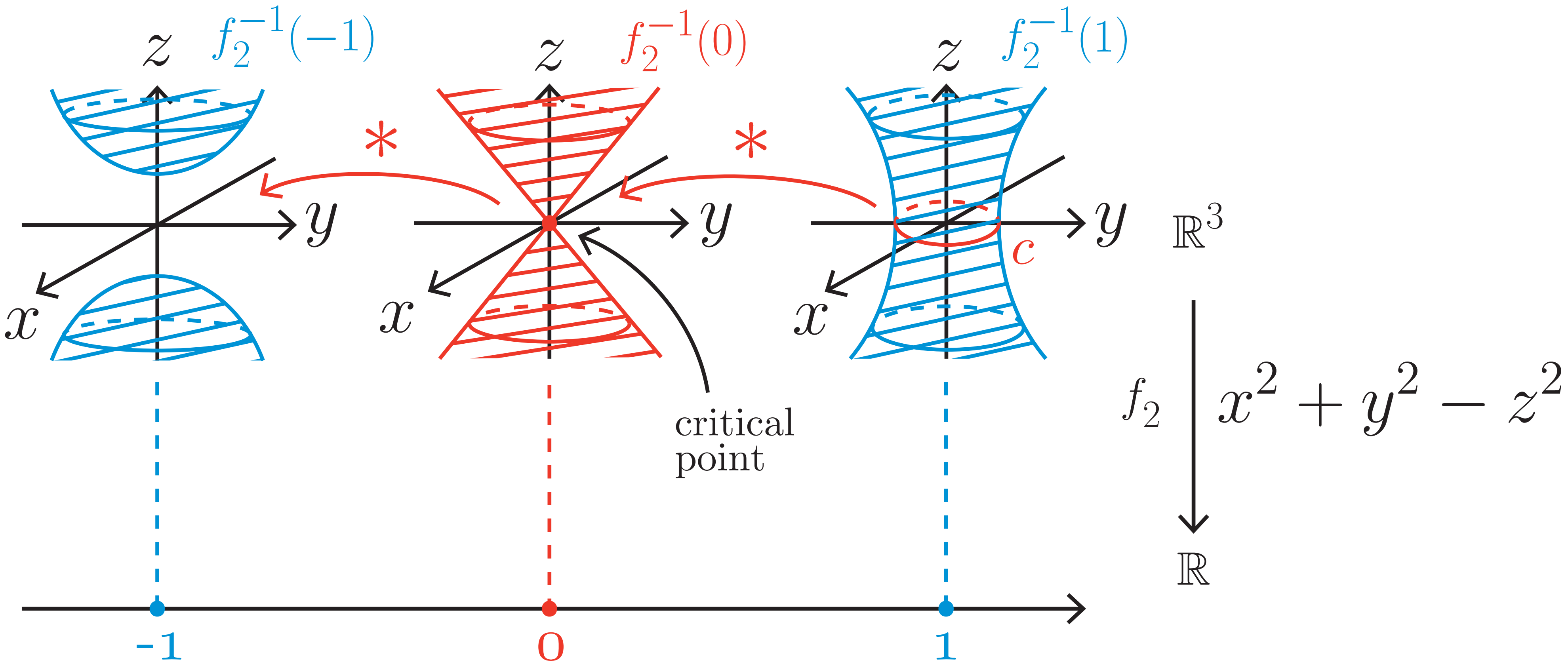
(3) $f|_{\mathcal{C}_f}$: injective and $f|_{Z_f}$: immersion, where

Z_f :the set of indefinite folds of f,

$\mathcal{C}_f$:the set of Lefschetz singularities of f.

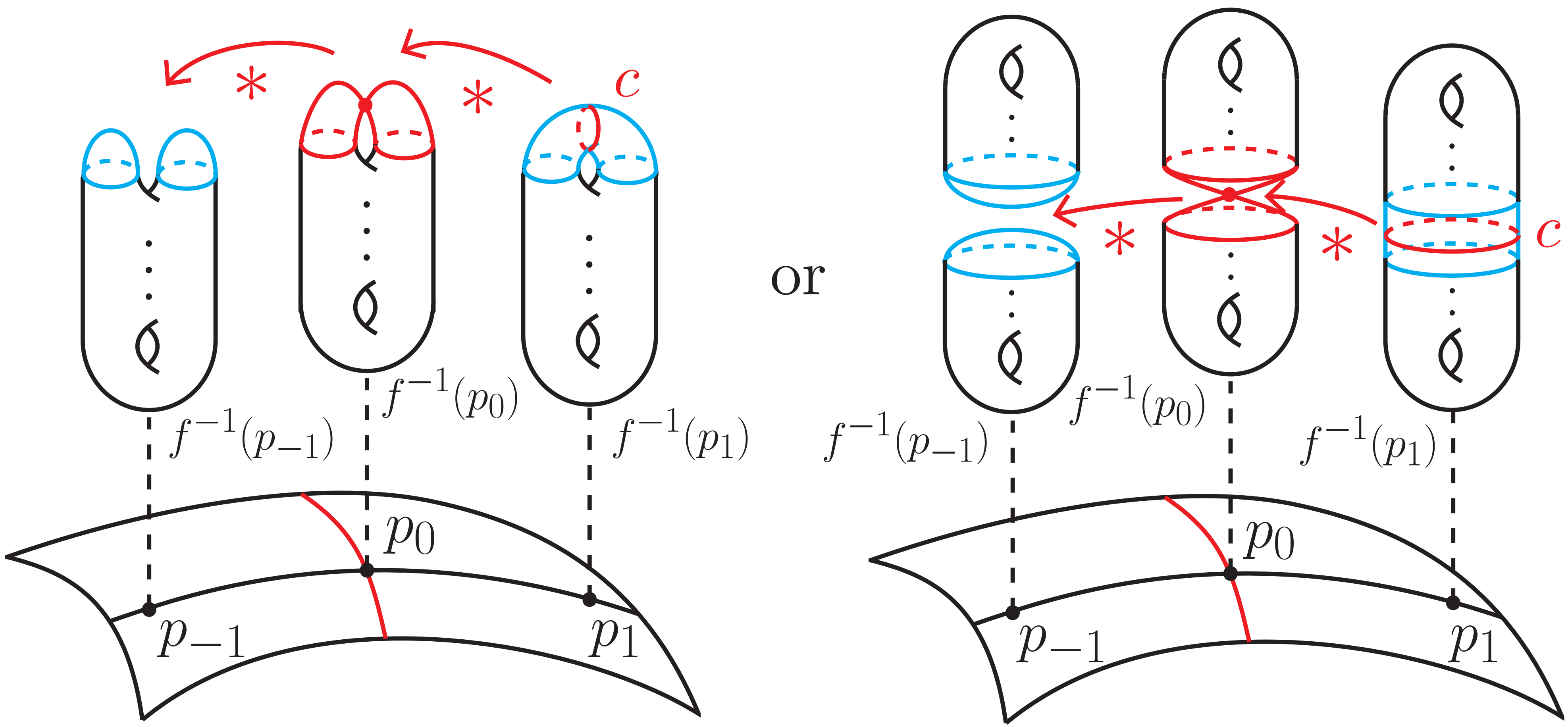
∗A BLF f is called a **Lefschetz fibration** (**LF**) if $Z_f = \emptyset$.

◇ Fibers around indefinite folds (2nd component)



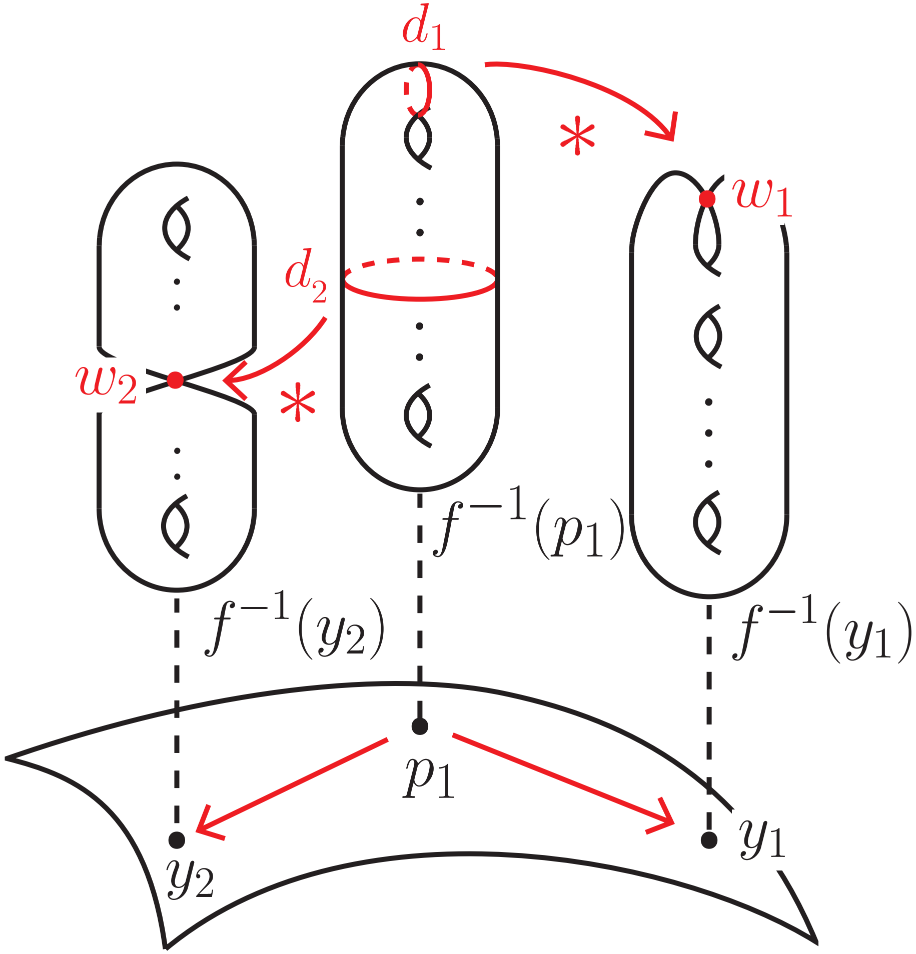
∗ the curve c collapses to the point and then vanishes.

◇ Fibers around indefinite folds (global model)



$*$ a curve c is called a **vanishing cycle** of the indefinite folds.

◇ Fibers around a Lefschetz singularity

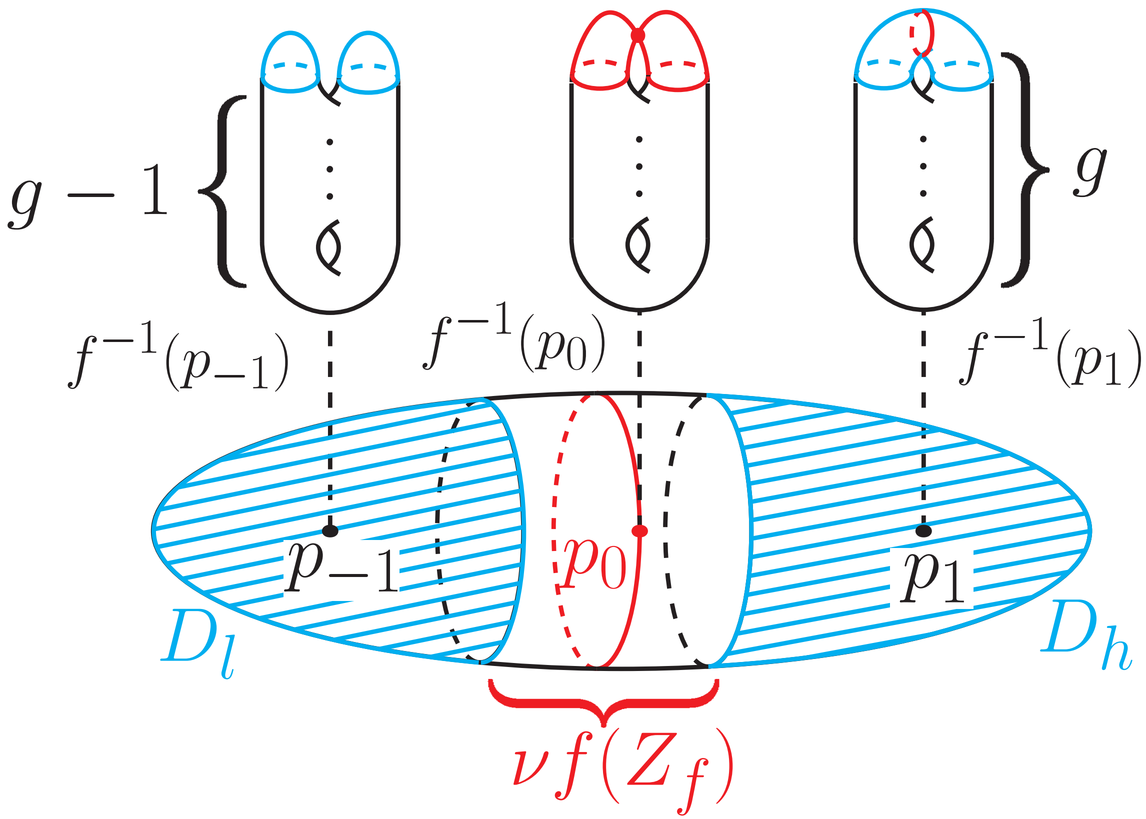


$*$ a curve $\boldsymbol{d_i} \subset \boldsymbol{f^{-1}(p_1)}$ is called a **vanishing cycle** of Lefschetz singularity $\boldsymbol{w_i}$.

$*$ an isotopy class of a vanishing cycle depends on choice of a path on the base space.

$f : M \to S^2$:BLF. Assume

- Z_f :conn., $\neq \emptyset$,
- fibers of f :conn., then,
- $f|_{Z_f}$: injective,



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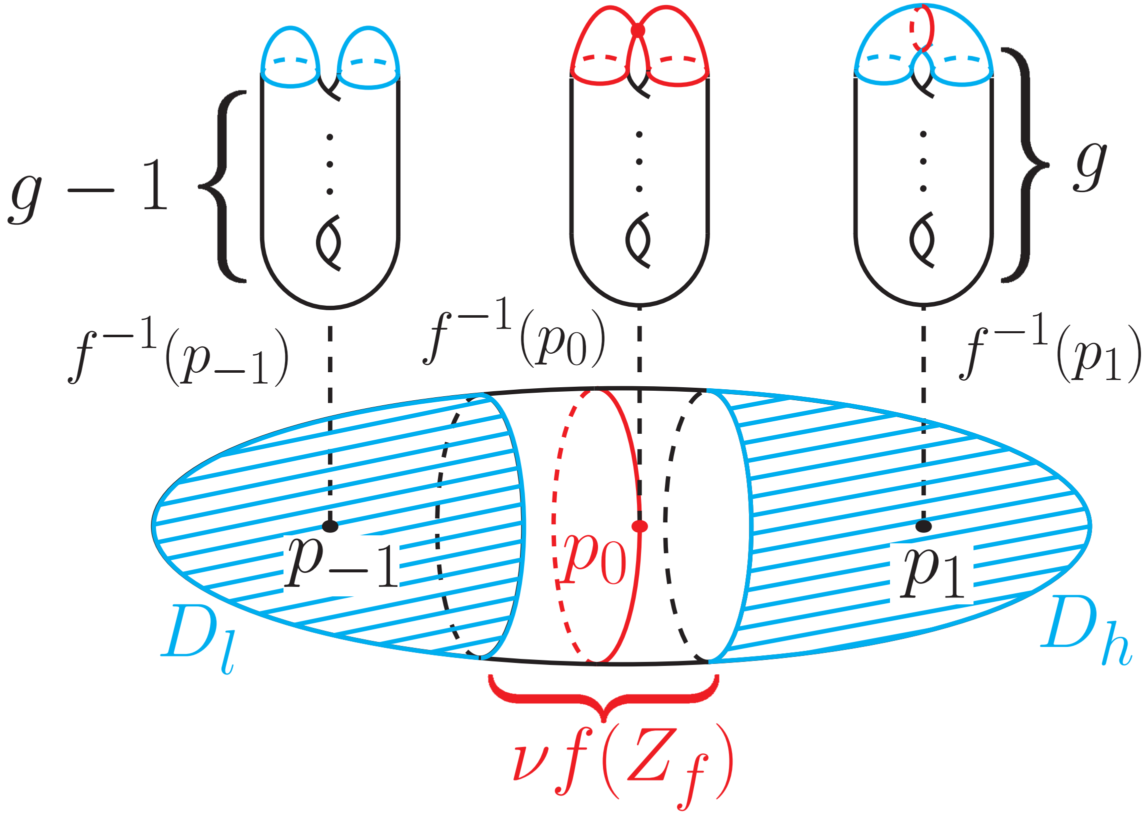

$f^{-1}(D_h)$ (resp. $f^{-1}(D_l)$) :the **higher** (resp. **lower**) **side** of f

Definition 1.3 (Baykur '09)

$f : M \to S^2$:BLF is said to be **simplified** (**SBLF**) if

(1) Z_f is conn., fiber of f is conn. and $f|_{Z_f}$: injective,

(2) if $Z_f \neq \emptyset$, $\mathcal{C}_f \subset f^{-1}(D_h)$.

The genus g of a regular fiber (in $f^{-1}(D_h)$ if $Z_f \neq \emptyset$) is called the **genus** of f.

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$*$ $f^{-1}(D_l)$ is the trivial Σ_{g-1}-bundle if $Z_f \neq \emptyset$.

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* $f^{-1}(D_l)$ is the trivial Σ_{g-1}-bundle if $Z_f \neq \emptyset$.

* $^\forall$closed ori. conn. 4-mfd. admits an SBLF (Williams '10).

§.2. Main Results

Definition 2.1 $f : M \to S^2$: genus-g SBLF.

1. f is **relatively minimal**
   $\overset{\text{def}}{\Longleftrightarrow}$ no fiber contains (-1)-spheres.
2. f is **trivial**
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* [non-relatively minimal SBLF] $\xrightarrow[\text{the } (-1)\text{-spheres}]{\text{blow-down at}}$ [relatively minimal SBLF]

* $f : M \to S^2$: relatively minimal genus-g SBLF w/ $Z_f \neq \emptyset$
  $\longrightarrow$ $^{\exists} f^{(m)} : M \# m\overline{\mathbb{CP}^2} \to S^2$: **rel. min.** genus-$g$ SBLF
  (Auroux-Donaldson-Katzarkov '05).

$\diamondsuit$ Classification of genus-1 SBLF

We tried to generalize the following result.

Theorem 2.2 (Kas '77, Moishezon '77)

$f : M \to S^2$: genus-1 LF w/ $\begin{cases} \text{relatively minimal} \\ \sharp \mathcal{C}_f = r > 0 \end{cases}$

$\Longrightarrow r \equiv 0 \pmod{12}$ and $M \cong E\left(\frac{r}{12}\right)$.

$*$ $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$: the elliptic surface, and

$$E(n) = \underbrace{E(1) \#_f \cdots \#_f E(1)}_{n} \to \mathbb{CP}^1 \text{ (fiber sum).}$$

Theorem 2.3 (H. '11)

The following 4-mfds admit a relatively minimal genus-1 SBLF w/ $Z_f \neq \emptyset$, $\sharp \mathcal{C}_f = r$:

- $\# k\mathbb{CP}^2 \#(r-k)\overline{\mathbb{CP}^2}$ $(0 \leq k \leq r-1)$,
- $\# \frac{r}{2} S^2 \times S^2$, for even r,
- $L \# r\overline{\mathbb{CP}^2}$ $(L = L_n$ or L'_n: defined by Pao '77).

* $S^1 \times S^3 \# S \# r\overline{\mathbb{CP}^2}$ (S: S^2-bundle over S^2) also admits a genus-1 SBLF w/ the above properties (ADK '05).

Theorem 2.4 (H.)

Let $f : M \to S^2$ be a genus-1 SBLF w/ $Z_f \neq \emptyset$, $\sharp\mathcal{C}_f = r$.

1. If $r \leq 5$, M is diffeo. to one of the following 4-mfds:
   - $\#k\mathbb{CP}^2\#(r-k)\overline{\mathbb{CP}^2}$ $(0 \leq k \leq r-1)$,
   - $\#\frac{r}{2}S^2 \times S^2$, for even r,
   - $L\#r\overline{\mathbb{CP}^2}$ $(L = L_n$ or L'_n: defined by Pao '77),
   - $S^1 \times S^3\#S\#r\overline{\mathbb{CP}^2}$ (S:S^2-bundle over S^2).
2. If M is spin and $r \neq 0$, then r is even and

$$M \cong \#\frac{r}{2}S^2 \times S^2.$$

◇ Remarks on Theorem 2.3 and Theorem 2.4

1. Baykur-Kamada also studied genus-1 SBLFs. They proved a part of 1. of Theorem 2.4 (the case $r = 0$) and,

Theorem 2.5 (Baykur-Kamada)

$f : M \to S^2$: rel. min. genus-1 SBLF w/ $Z_f, \mathcal{C}_f \neq \emptyset$.

$\pi_1(M) = 1$

$\Longrightarrow$ $^\exists m \gg 0$ s.t. $f^{(m)} : M \# m\overline{\mathbb{CP}^2} \to S^2$ is "equiv." to

$\tilde{f} : \# k\mathbb{CP}^2 \# l\overline{\mathbb{CP}^2} \to S^2$: *canonical* genus-1 SBLF.

for some $k, l \geq 0$.

In particular, $M \# m\overline{\mathbb{CP}^2} \cong \# k\mathbb{CP}^2 \# l\overline{\mathbb{CP}^2}$.

2. $\#k\overline{\mathbb{CP}^2}$ $(k \geq 1)$ cannot admit any genus-1 SBLFs (H. '11), but admits a genus-2 SBLF (Sato-H.).

◇ Sections of SBLFs

Definition 2.6

$f : M \to S^2$: genus-g SBLF

$\sigma : S^2 \to M$ is a **section** of $f \overset{\text{def}}{\Longleftrightarrow} f \circ \sigma = \mathrm{id}_{S^2}$.

$*$ Sections of LFs were studied well. For example,

Theorem 2.7 (Smith '01)
${}^\forall f : M \to S^2$: rel. min. non-triv. genus-$g \geq 2$ LF,
f has only finitely many homotopy classes of sections,
that is, there exist only finitely many elements of

$$[S^2, M] = \{h : S^2 \to M : C^\infty\text{-map}\}/(\text{homotopy})$$

represented by a section of f.

Theorem 2.8 (Smith '01, Stipsicz '01)
${}^\forall f : M \to S^2$: rel. min. non-triv. genus-$g \geq 2$ LF,
${}^\forall \sigma : S^2 \to M$: section of f,
$[\sigma(S^2)]^2 < 0$.

$*$ These results **CANNOT** be generalized to SBLFs.

Theorem 2.9 (H.) $^\forall \boldsymbol{g} \geq \mathbf{2}$,

$^\exists \boldsymbol{f} : \boldsymbol{M} \to \boldsymbol{S^2}$: SBLF w/ $\begin{cases} \text{genus-}\boldsymbol{g}, \\ \text{non-trivial}, \\ \text{relatively minimal}, \end{cases}$ s.t.

there exist infinitely many homotopy classes of sections.

Theorem 2.10 (H.) $^\forall \boldsymbol{g} \geq \mathbf{2}$, $^\forall \boldsymbol{n} \geq \mathbf{0}$,

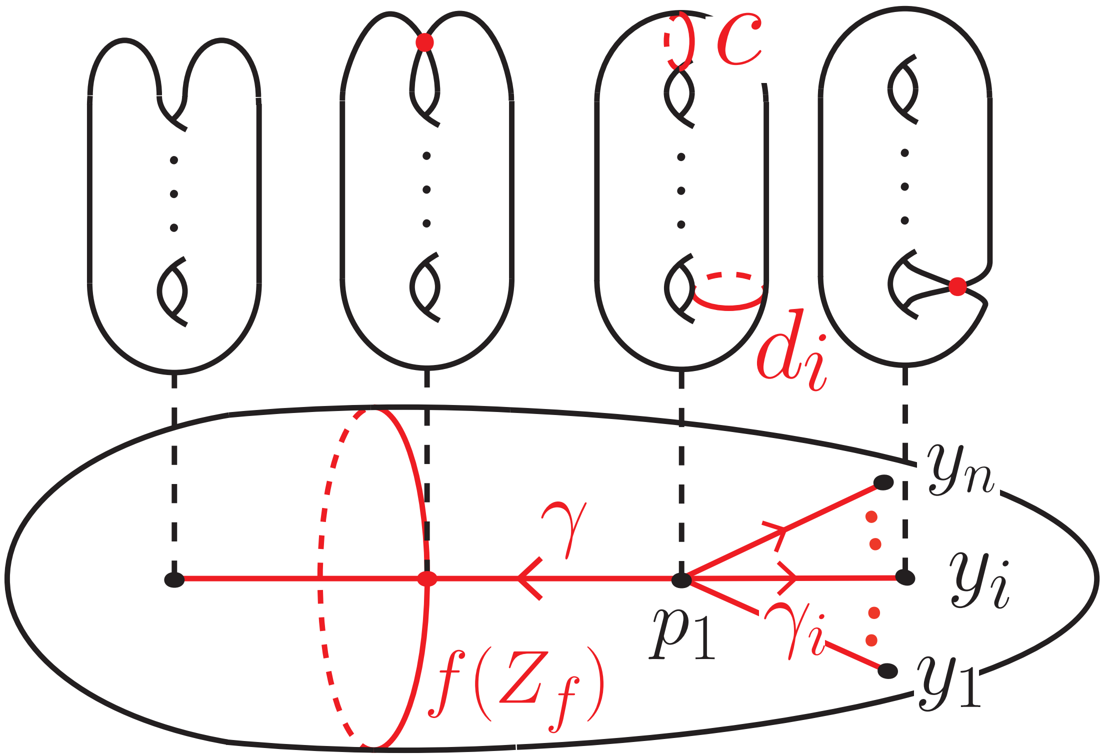
$^\exists \boldsymbol{f} : \boldsymbol{M} \to \boldsymbol{S^2}$: SBLF w/ $\begin{cases} \text{genus-}\boldsymbol{g}, \\ \text{non-trivial}, \\ \text{relatively minimal}, \end{cases}$

$^\exists \boldsymbol{\sigma} : \boldsymbol{S^2} \to \boldsymbol{M}$: section of $\boldsymbol{f}$ s.t.

$$[\boldsymbol{\sigma}(\boldsymbol{S^2})]^{\mathbf{2}} = \boldsymbol{n}.$$

§.3. A Combinatorial Approach

$f : M \to S^2$: genus-g SBLF, $Z_f \neq \emptyset$, $f(\mathcal{C}_f) = \{y_1, \ldots, y_n\}$



$*$We call $(c, d_1, \ldots, d_n)$ a **Hurwitz cycle system** of f.

Theorem 3.1 (ADK '05, Baykur '09)

1. $f : M \to S^2$: genus-g SBLF, $Z_f \neq \emptyset$

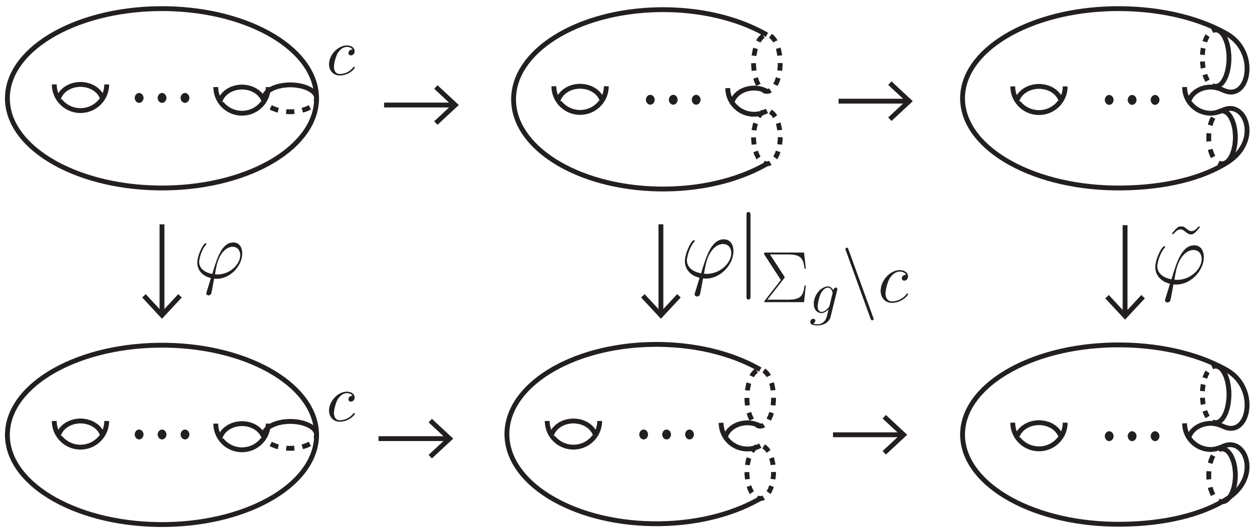
   $(c, d_1, \ldots, d_n)$:Hurwitz cycle system of f

   then $t_{d_1} \cdot \cdots \cdot t_{d_n} \in \operatorname{Ker} \Phi_c$ ($\subset \mathcal{M}_g(c)$)

2.

$*\boldsymbol{\mathcal{M}_g(c)} = \{[\varphi] \in \boldsymbol{\mathcal{M}_g} \mid \boldsymbol{\varphi(c) = c}\}$. $\boldsymbol{\mathcal{M}_g := \pi_0(\mathrm{Diff}^+ \Sigma_g)}$.

$*\ \boldsymbol{\Phi_c : \mathcal{M}_g(c) \to \mathcal{M}_{g-1}}$ is defined as follows:



$[\varphi] \in \mathcal{M}_g(c)$ $\qquad$ $\Phi_c([\varphi]) = [\tilde{\varphi}]$

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   then $t_{d_1} \cdot \cdots \cdot t_{d_n} \in \mathrm{Ker}\,\Phi_c$ $(\subset \mathcal{M}_g(c))$

2. Conversely,

   $d_1, \ldots, d_n \subset \Sigma_g$:s.c.c.

   $c \subset \Sigma_g$:non-separating s.c.c.

   s.t. $t_{d_1} \cdot \cdots \cdot t_{d_n} \in \mathrm{Ker}\,\Phi_c$

   $\Longrightarrow {}^{\exists}f : M \to S^2$: genus-$g$ SBLF w/ $Z_f \neq \emptyset$, Hurwitz cycle system $(c, d_1, \ldots, d_n)$.

Theorem 3.1 (ADK '05, Baykur '09)

1. $f: M \to S^2$: genus-g SBLF, $Z_f \neq \emptyset$

   $(c, d_1, \ldots, d_n)$:Hurwitz cycle system of f

   then $t_{d_1} \cdot \cdots \cdot t_{d_n} \in \mathrm{Ker}\,\Phi_c \; (\subset \mathcal{M}_g(c))$

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   s.t. $t_{d_1} \cdot \cdots \cdot t_{d_n} \in \mathrm{Ker}\,\Phi_c$

   $\Longrightarrow {}^{\exists} f: M \to S^2$: genus-$g$ SBLF w/ $Z_f \neq \emptyset$, Hurwitz cycle system $(c, d_1, \ldots, d_n)$.

* f is uniquely determined by $(c, d_1, \ldots, d_n)$ **when $g \geq 3$**.

$\diamondsuit$ The case $Z_f = \emptyset$ (i.e. Lefschetz fibrations)

Theorem 3.2

1. $f : M \to S^2$: rel. min. genus-g LF, $\mathcal{C}_f \neq \emptyset$

   $d_1, \ldots, d_n \subset \Sigma_g$: vanishing cycles of f.

   Then, $t_{d_1} \cdot \cdots \cdot t_{d_n} = 1$.

2. (Kas '80, Matsumoto '96) Conversely,

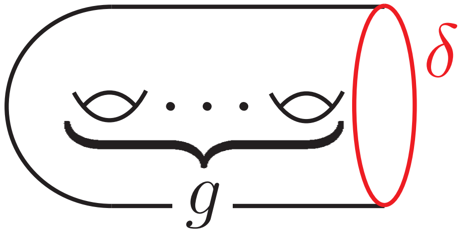
   $d_1, \ldots, d_n \subset \Sigma_g$: essential s.c.c. s.t. $t_{d_1} \cdot \cdots \cdot t_{d_n} = 1$

   $\Longrightarrow {}^{\exists} f : M \to S^2$: rel. min. genus-$g$ LF

   w/ vanishing cycles $d_1, \ldots, d_n$

   Moreover, such an f is unique up to "equivalence".

$*$ We can extend Theorem 3.1 to SBLFs **with sections**.

$\boldsymbol{\Sigma_{g,1}} =$ 

$\subset \boldsymbol{\Sigma_g}$, $\quad \boldsymbol{i : \Sigma_{g,1} \hookrightarrow \Sigma_g}$: inclusion

$\boldsymbol{\overline{\Phi}_c : \mathcal{M}_{g,1}(c) \to \mathcal{M}_{g-1,1}}$: defined as we define $\boldsymbol{\Phi_c}$.

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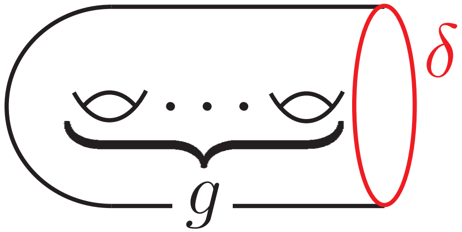
$\Sigma_{g,1} =$ 

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$\overline{\Phi}_c : \mathcal{M}_{g,1}(c) \to \mathcal{M}_{g-1,1}$: defined as we define Φ_c.

Lemma 3.3 (H.)

$d_1, \ldots, d_n \subset \Sigma_{g,1}$:s.c.c.

$c \subset \Sigma_{g,1}$:non-separating s.c.c.

s.t. $\overline{\Phi}_c(t_{d_1} \cdot \cdots \cdot t_{d_n}) = t_\delta^{-m}$

$\Longrightarrow {}^\exists f : M \to S^2$: genus-$g$ SBLF w/ Hurwitz cycle system $(i(c), i(d_1), \ldots, i(d_n))$

and

${}^\exists \sigma : S^2 \to M$: section of f w/ $[\sigma(S^2)] = m$.

§.4. Outline of Proofs

◇ Classification of genus-1 SBLFs

∗ We recall the main results

Theorem 2.3 (H. '11)

The following 4-mfds admit a relatively minimal genus-1 SBLF w/ $Z_f \neq \emptyset$, $\sharp\mathcal{C}_f = r$:

- $\#k\mathbb{CP}^2\#(r-k)\overline{\mathbb{CP}^2}$ $(0 \leq k \leq r-1)$,
- $\#\frac{r}{2}S^2 \times S^2$, for even r,
- $L\#r\overline{\mathbb{CP}^2}$ ($L = L_n$ or L'_n: defined by Pao '77).

Theorem 2.4 (H.)

Let $f : M \to S^2$ be a genus-1 SBLF w/ $Z_f \neq \emptyset$, $\sharp\mathcal{C}_f = r$.

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   - $\#k\mathbb{CP}^2\#(r-k)\overline{\mathbb{CP}^2}$ $(0 \leq k \leq r-1)$,
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   - $S^1 \times S^3\#S\#r\overline{\mathbb{CP}^2}$ (S:S^2-bundle over S^2).
2. If M is spin and $r \neq 0$, then r is even and

$$M \cong \#\frac{r}{2}S^2 \times S^2.$$

(Outline of Proof of Theorem 2.3 and 2.4)

$f : M \to S^2$: relatively minimal genus-1 SBLF, $Z_f \neq \emptyset$

$(c, d_1, \ldots, d_n)$: Hurwitz cycle system of f.

$W_f := (t_{d_1}, \ldots, t_{d_n}) \in \mathcal{M}_1{}^n$, $b(W_f) := t_{d_1} \cdot \cdots \cdot t_{d_n} \in \mathrm{Ker}\,\Phi_c$.

$\mu_1, \mu_2 \subset T^2$: generator of $\pi_1(T^2)$. Assume $c = \mu_1$.

(Outline of Proof of Theorem 2.3 and 2.4)

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$\mu_1, \mu_2 \subset T^2$: generator of $\pi_1(T^2)$. Assume $c = \mu_1$.

Step.1: Prove that W_f can be changed into:

$$"S_r T(m_1, \ldots, m_s) \; = (t_{\mu_1}, \ldots, t_{\mu_1}, t_{t_{\mu_1}^{m_1}(\mu_2)}, \ldots, t_{t_{\mu_1}^{m_s}(\mu_2)}) \in \mathcal{M}_1{}^{r+s}" \; (r \geq 0, m_1, \ldots m_s \in \mathbb{Z})$$

by "Hurwitz action" (using **charts** introduced by Kamada).

Step.2: Construct examples of $S_r T(m_1, \dots, m_s)$ w/

$$b(S_r T(m_1, \dots, m_s)) \in \mathrm{Ker}\, \Phi_{\mu_1}.$$

- $b(S_r) = {t_{\mu_1}}^r \in \mathrm{Ker}\, \Phi_{\mu_1}$,
- $T_s^k := T(\dots, \tilde{m}_i + k, \dots)$ (for some $\tilde{m}_i \in \mathbb{Z}$), $b(T_s^k) \in \mathrm{Ker}\, \Phi_{\mu_1}$

<u>Step.2</u>: Construct examples of $S_rT(m_1,\ldots,m_s)$ w/

$$b(S_rT(m_1,\ldots,m_s)) \in \mathrm{Ker}\,\Phi_{\mu_1}.$$

- $b(S_r) = {t_{\mu_1}}^r \in \mathrm{Ker}\,\Phi_{\mu_1}$,
- $T_s^k := T(\ldots,\tilde{m}_i + k,\ldots)$ (for some $\tilde{m}_i \in \mathbb{Z}$), $b(T_s^k) \in \mathrm{Ker}\,\Phi_{\mu_1}$

We then prove (by Kirby calculus) that:

- $W_f \sim S_r \Longrightarrow M = \#r\overline{\mathbb{CP}^2}, L\#r\overline{\mathbb{CP}^2}$ or $S^1 \times S^3\#S\#r\overline{\mathbb{CP}^2}$,
- $W_f \sim T_s^k \Longrightarrow M = S\#(s-2)\mathbb{CP}^2$,
- $W_f \sim S_rT_{s_1}^{k_1}\cdots T_{s_l}^{k_l} \Longrightarrow M = \#r\overline{\mathbb{CP}^2} \mathop{\#}_{i=1}^{l} (S\#(s_i-2)\mathbb{CP}^2)$.

In particular, we can prove Theorem 2.3.

Step.3: Try to classify standard forms $S_rT(m_1, \ldots, m_s)$ up to "Hurwitz equivalence" by using:

$$\pi : SL(2, \mathbb{Z}) \xrightarrow{\text{quotient}} PSL(2, \mathbb{Z}).$$

$PSL(2, \mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$ (free product)

$\Longrightarrow$ It becomes easy to solve Hurwitz eq. problem. Indeed,

- $r + s \leq 5$, $S_rT(n_1, \ldots, n_s) \sim {}^{\exists}W = S_tT_{s_1}^{k_1} \cdots T_{s_l}^{k_l}$,
- M: spin, $\mathcal{C}_f \neq \emptyset \Longrightarrow W_f \sim T_2^{k_1} \cdots T_2^{k_l}$.

  (by using a generalization of Stipsicz's result on spin str. on LF)

This completes the proof of Theorem 2.4.

◇ Sections of SBLFs

Theorem 2.9 (H.) ${}^{\forall}g \geq 2$,

${}^{\exists}f : M \to S^2$: SBLF w/ genus-$g$, non-triv. rel. min. s.t.

there exist infinitely many elements of

$$[S^2, M] = \{h : S^2 \to M : C^\infty\text{-map}\}/(\text{homotopy})$$

represented by a section of f.

Theorem 2.10 (H.) ${}^{\forall}g \geq 2$, ${}^{\forall}n \geq 0$,

${}^{\exists}f : M \to S^2$: SBLF w/ genus-$g$, non-triv. rel. min.

${}^{\exists}\sigma : S^2 \to M$: section of f s.t.

$$[\sigma(S^2)]^2 = n.$$

(Outline of Proof of Theorem 2.9)

Construct SBLF f_g w/ sections σ_n ($n \in \mathbb{Z}$) by Lemma 3.3

$$f_g : M_g := S^2 \times \Sigma_{g-1} \# S^1 \times S^3 \# 2\overline{\mathbb{CP}^2} \to S^2.$$

By computing $\pi_2(M_g)$ (and $\curvearrowleft \pi_1(M_g)$), we prove

$$\sigma_n \not\simeq \sigma_m \text{ if } n \neq m.$$

(Outline of Proof of Theorem 2.9)

Construct SBLF f_g w/ sections σ_n ($n \in \mathbb{Z}$) by Lemma 3.3

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By computing $\pi_2(M_g)$ (and $\curvearrowleft \pi_1(M_g)$), we prove

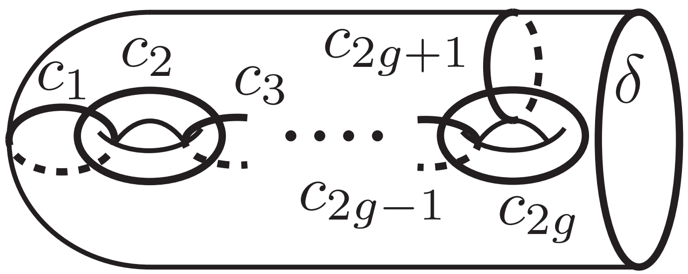
$$\sigma_n \not\simeq \sigma_m \text{ if } n \neq m.$$

(Outline of Proof of Theorem 2.10)

The following equalities imply Theorem 2.10:

$$\Phi_{c_{2g+1}}((t_{c_{2g}} \cdot \cdots \cdot t_{c_2} \cdot t_{c_1}^2 \cdot t_{c_2} \cdot \cdots \cdot t_{c_{2g}})^2 \cdot (t_{c_1} \cdot \cdots \cdot t_{c_{2g-2}})^{2(2g-1)}) = 1,$$

$$\Phi_{c_{2g+1}}((t_{c_{2g}} \cdot \cdots \cdot t_{c_2} \cdot t_{c_1}^2 \cdot t_{c_2} \cdot \cdots \cdot t_{c_{2g}})^{2n}) = t_\delta^{-n}.$$



Thank you for your attention !