Fibered Faces and Dynamics of Mapping Classes II

Branched Coverings, Degenerations, and Related Topics 2012

Hiroshima University

Eriko Hironaka
Florida State University/Tokyo Institute of Technology

March 5-7, 2012

I. Pseudo-Anosov mapping classes

1. Introduction
2. Visualizing pseudo-Anosov mapping classes
3. Train tracks
4. Minimum dilatation problem

II. Fibered faces and applications

1. Introduction
2. Fibered face theory
3. Alexander and Teichmüller polynomials
4. First application: Orientable mapping classes

III. Families of mapping classes with small dilatations

1. Introduction
2. Deformations of mapping classes on fibered faces
3. Penner sequences and applications
4. Quasiperiodic mapping classes

Introduction

Setup:

- S compact oriented surface, g genus, n number of boundary components, $\chi(S) = 2 - 2g - n < 0$.
- $\mathrm{Mod}(S)$ mapping class group, $\mathrm{Mod}(S) = \mathrm{Homeo}^+ / \sim$
- $\mathcal{P}_S$ pseudo-Anosov elements of $\mathrm{Mod}(S)$

Goal in this lecture:

Set $\mathcal{P} = \bigcup_S \mathcal{P}_S$

- Describe an embedding $\mathcal{P} \hookrightarrow \bigcup F_\alpha$, where F_α are some convex polyhedra in Eucldiean space (fibered faces), and the image of $\mathcal{P}$ are the union of rational points in the interiors of F_α.
- Describe invariants of $\mathcal{P}$ like homological and geomeric dilatations from this point of view.

Horizontal and vertical theory of mapping classes

Think of

$\mathcal{P}_S \leftrightsquigarrow$ *horizontal theory of pA maps*

(Mapping class groups, Teichmüller space, moduli space)

$F_\alpha \leftrightsquigarrow$ *vertical theory of pA maps*

(3-manifold geometry and topology)

This point of view stems from work of *W. Thurston, D. Fried, C. McMullen*

Vertical Theory for $\mathcal{P}$

Mapping Tori

The *mapping torus* M_ϕ of a mapping class (S, ϕ) is the 3-manifold:

$$M_\phi = S \times [0,1] / \sim$$

where $(x, 1) \sim (\phi(x), 0)$.

- The homeomorphism type of M_ϕ is determined by the isotopy class of ϕ.
- M_ϕ is hyperbolic $\Leftrightarrow$ ϕ is pseudo-Anosov.

Let $\Phi(M) = \{(S, \phi) \mid M_\phi = M\}$, the *monodromies* of M.

Vertical partition 1:

$\{\Phi(M)\}$ defines partitions of

$$\mathrm{Mod} = \bigcup_S \mathrm{Mod}(S) \qquad \text{and} \qquad \mathcal{P} = \bigcup_S \mathcal{P}_S.$$

Add some structure on $\Phi(M)$.

Thurston norm

Fix a 3-manifold M.

Given a connected subsurface $\Sigma \subset M$, define
$\chi_-(\Sigma) = \max\{0, -\chi(\Sigma)\}$.

For $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k$, Σ_i connected, define
$\chi_-(\Sigma) = \sum_{i=1}^{k} \chi_-(\Sigma_i)$.

For $\alpha \in H^1(M; \mathbb{Z})$,
$||\alpha|| = \min\{\chi_-(\Sigma) \mid [\Sigma] \in H_2(M; \mathbb{Z}) \text{ is dual to } \alpha\}$.

$|| \; ||$ extends to the *Thurston (semi-) norm* on $H^1(M; \mathbb{R})$.

If M is hyperbolic, the Thurston norm extends to a norm on
$H^1(M; \mathbb{R})$. (Assume hereafter that M is hyperbolic.)

Fibered faces

The *Thurston norm ball*

$$\{\alpha \in H^1(M;\mathbb{R}) \ : \ ||\alpha|| \leq 1\}$$

is a convex polyhedron with integer vertices.
For each top dimensional face F, let C_F be the positive cone over F.
Let $H^1(M;\mathbb{Z})^{\text{prim}}$ be the set of *primitive* elements
$\Leftrightarrow$ has a simply connected dual surface.
Then either:

- $\Phi(M) \cap C_F = H^1(M;\mathbb{Z})^{\text{prim}} \cap C_F$; or
- $\Phi(M) \cap C_F = \emptyset$.

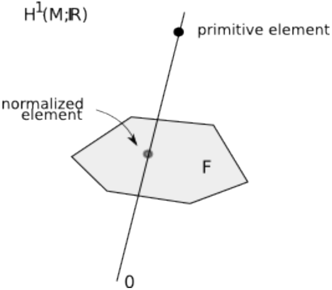
In the former case we say F is a *fibered face* of M.

Vertical partition 2

The sets

$$\Phi(M, F) = \Phi(M) \cap C_F \subset \Phi(M)$$

define a subpartition of $\mathcal{P}$.



- Identify each $\Phi(M, F)$ with rational points on F.
- Identify *closure* $\overline{\mathcal{P}}$ with the disjoint union of closures of fibered faces. These are homeomorphic to closed disks of dimension $d = \dim H^1(M; \mathbb{R})$ -1.

Invariants of mapping classes

We are interested in the following invariants of $(S, \phi) \in \mathcal{P}$:

- Dilatation $\lambda(\phi)$
- *Homological dilatation*

$$\lambda_{\text{hom}}(\phi) = \text{Spec.Rad.}(\phi_* : H_1(S;\mathbb{R}) \to H_1(S;\mathbb{R})).$$

- *Normalized dilatation* of (S, ϕ)

$$L(S, \phi) = \lambda(\phi)^{|\chi(S)|}.$$

(Useful for studying families of mapping classes with asymptotically small dilatations.)

Invariants of mapping classes

We are interested in the following invariants of $(S, \phi) \in \mathcal{P}$:

- Dilatation $\lambda(\phi)$
- *Homological dilatation*

$$\lambda_{\text{hom}}(\phi) = \text{Spec.Rad.}(\phi_* : H_1(S;\mathbb{R}) \to H_1(S;\mathbb{R})).$$

- *Normalized dilatation* of (S, ϕ)

$$L(S,\phi) = \lambda(\phi)^{|\chi(S)|}.$$

(Useful for studying families of mapping classes with asymptotically small dilatations.)

Invariants of mapping classes

We are interested in the following invariants of $(S, \phi) \in \mathcal{P}$:

- Dilatation $\lambda(\phi)$
- *Homological dilatation*

$$\lambda_{\mathrm{hom}}(\phi) = \mathrm{Spec.Rad.}(\phi_* : H_1(S; \mathbb{R}) \to H_1(S; \mathbb{R})).$$

- *Normalized dilatation* of (S, ϕ)

$$L(S, \phi) = \lambda(\phi)^{|\chi(S)|}.$$

(Useful for studying families of mapping classes with asymptotically small dilatations.)

Invariants of mapping classes

We are interested in the following invariants of $(S, \phi) \in \mathcal{P}$:

- Dilatation $\lambda(\phi)$
- *Homological dilatation*

$$\lambda_{\text{hom}}(\phi) = \text{Spec.Rad.}(\phi_* : H_1(S; \mathbb{R}) \to H_1(S; \mathbb{R})).$$

- *Normalized dilatation* of (S, ϕ)

$$L(S, \phi) = \lambda(\phi)^{|\chi(S)|}.$$

(Useful for studying families of mapping classes with asymptotically small dilatations.)

Specializing Laurent polynomials

Let $G = \mathbb{Z}^d$, and let

$$f = \sum_{g \in G} a_g g \in \mathbb{Z}[G]$$

be a Laurent polynomial. Here we take $a_g \in \mathbb{Z}$ to be nonzero for only a finite number of g.

Let $\psi : G \to \mathbb{Z}$. The *specialization* of f at ψ is:

$$f^\psi = \sum_{g \in G} a_g t^{\psi(g)}.$$

Given a single variable polynomial $f(t) \in \mathbb{R}[t]$, the *house* is:

$$|f| = \max\{|\mu| \;:\; f(\mu) = 0\}.$$

Specializing Laurent polynomials

Let $G = \mathbb{Z}^d$, and let

$$f = \sum_{g \in G} a_g g \in \mathbb{Z}[G]$$

be a Laurent polynomial. Here we take $a_g \in \mathbb{Z}$ to be nonzero for only a finite number of g.

Let $\psi : G \to \mathbb{Z}$. The *specialization* of f at ψ is:

$$f^\psi = \sum_{g \in G} a_g t^{\psi(g)}.$$

Given a single variable polynomial $f(t) \in \mathbb{R}[t]$, the *house* is:

$$\overline{|f|} = \max\{|\mu| \ : \ f(\mu) = 0\}.$$

Specializing Laurent polynomials

Let $G = \mathbb{Z}^d$, and let

$$f = \sum_{g \in G} a_g g \in \mathbb{Z}[G]$$

be a Laurent polynomial. Here we take $a_g \in \mathbb{Z}$ to be nonzero for only a finite number of g.

Let $\psi : G \to \mathbb{Z}$. The *specialization* of f at ψ is:

$$f^\psi = \sum_{g \in G} a_g t^{\psi(g)}.$$

Given a single variable polynomial $f(t) \in \mathbb{R}[t]$, the *house* is:

$$|f| = \max\{|\mu| \ : \ f(\mu) = 0\}.$$

Alexander and Teichmüller polynomials

$G = H_1(M_\phi; \mathbb{Z})/\text{Torsion}$ $(\simeq \mathbb{Z}^d)$
$\Delta \in \mathbb{Z}[G]$ the *Alexander polynomial* of M_ϕ. (E.g., use Fox calculus.)
For $(S, \phi_a) \in \Phi(M_\phi, F_\phi)$, let $\psi_a : G \to \mathbb{Z}$ be the map associated to the corresponding fibration $M_{\phi_a} \to S^1$. Then we have:

$$\lambda_{\text{hom}}(\phi_a) = |\Delta^{\psi_a}|.$$

Analogously,...
(C. McMullen '00) Given a pseudo-Anosov mapping class (S, ϕ) and associated fibered face $(M, F) = (M_\phi, F_\phi)$, there is a *Teichmüller polynomial* $\Theta \in \mathbb{Z}[G]$ such that for all $(S_\alpha, \phi_\alpha) \in \Phi(M, F)$

$$\lambda(\phi_a) = |\Theta^{\psi_a}|.$$

Alexander and Teichmüller polynomials

$G = H_1(M_\phi; \mathbb{Z})/\text{Torsion}$ ($\simeq \mathbb{Z}^d$)
$\Delta \in \mathbb{Z}[G]$ the *Alexander polynomial* of M_ϕ. (E.g., use Fox calculus.)
For $(S, \phi_a) \in \Phi(M_\phi, F_\phi)$, let $\psi_a : G \to \mathbb{Z}$ be the map associated to the corresponding fibration $M_{\phi_a} \to S^1$. Then we have:

$$\lambda_{\text{hom}}(\phi_a) = |\Delta^{\psi_a}|.$$

Analogously,...
(C. McMullen '00) Given a pseudo-Anosov mapping class (S, ϕ) and associated fibered face $(M, F) = (M_\phi, F_\phi)$, there is a *Teichmüller polynomial* $\Theta \in \mathbb{Z}[G]$ such that for all $(S_\alpha, \phi_\alpha) \in \Phi(M, F)$

$$\lambda(\phi_a) = |\Theta^{\psi_a}|.$$

Alexander and Teichmüller polynomials

Consequences:

- On connected components of $\text{int}(\overline{\mathcal{P}})$, it is possible to find defining equations for homological and geometric dilatations so that the coefficient strings of defining polynomials for homological and geometric dilatations are the same.
- The algebraic integers that realize the homological and geometric dilatations belong to particular kinds of algebraic families.
- This property can help to identify when two mapping classes lie on the same fibered face.

Alexander and Teichmüller polynomials

Consequences:

- On connected components of $\text{int}(\overline{\mathcal{P}})$, it is possible to find defining equations for homological and geometric dilatations so that the coefficient strings of defining polynomials for homological and geometric dilatations are the same.

- The algebraic integers that realize the homological and geometric dilatations belong to particular kinds of algebraic families.

- This property can help to identify when two mapping classes lie on the same fibered face.

Alexander and Teichmüller polynomials

Consequences:

- On connected components of $\mathrm{int}(\overline{\mathcal{P}})$, it is possible to find defining equations for homological and geometric dilatations so that the coefficient strings of defining polynomials for homological and geometric dilatations are the same.

- The algebraic integers that realize the homological and geometric dilatations belong to particular kinds of algebraic families.

- This property can help to identify when two mapping classes lie on the same fibered face.

Normalized dilatation and fibered faces

(D. Fried '82, C. McMullen '00) The *normalized dilatation*

$$\begin{array}{rcl} L : \mathcal{P} & \to & \mathbb{R} \\ (S, \phi) & \mapsto & \lambda(\phi)^{|\chi(S)|} \end{array}$$

extends to a continuous convex function on $\mathrm{int}(\overline{\mathcal{P}})$ that goes to infinity toward the boundary of $\overline{\mathcal{P}}$.

Effect of puncturing at singularities:

For $(S, \phi) \in \mathcal{P}$, let $S^0 = S \setminus \mathrm{Sing}(\phi)$ and $\phi^0 = \phi|_{S^0}$.

Equivalence relation on $\mathcal{P}$:
Write $(S_1, \phi_1) \sim (S_2, \phi_2)$ if $(S_1^0, \phi_1^0) = (S_2^0, \phi_2^0)$.

Lemma
If $(S_1, \phi_1) \sim (S_2, \phi_2)$, then $\lambda(\phi_1) = \lambda(\phi_2)$.

Remark: One is tempted to mod $\mathcal{P}$ out by this equivalence relations. One problem: the normalized dilatation $L(S, \phi)$ does not behave well with respect to this equivalence relation. (Subject of further study.)

Effect of puncturing at singularities:

For $(S, \phi) \in \mathcal{P}$, let $S^0 = S \setminus \mathrm{Sing}(\phi)$ and $\phi^0 = \phi|_{S^0}$.

Equivalence relation on $\mathcal{P}$:
Write $(S_1, \phi_1) \sim (S_2, \phi_2)$ if $(S_1^0, \phi_1^0) = (S_2^0, \phi_2^0)$.

Lemma
If $(S_1, \phi_1) \sim (S_2, \phi_2)$, then $\lambda(\phi_1) = \lambda(\phi_2)$.

Remark: One is tempted to mod $\mathcal{P}$ out by this equivalence relations. One problem: the normalized dilatation $L(S, \phi)$ does not behave well with respect to this equivalence relation. (Subject of further study.)

Effect of puncturing at singularities:

For $(S, \phi) \in \mathcal{P}$, let $S^0 = S \setminus \mathrm{Sing}(\phi)$ and $\phi^0 = \phi|_{S^0}$.

Equivalence relation on $\mathcal{P}$:
Write $(S_1, \phi_1) \sim (S_2, \phi_2)$ if $(S_1^0, \phi_1^0) = (S_2^0, \phi_2^0)$.

Lemma
If $(S_1, \phi_1) \sim (S_2, \phi_2)$, then $\lambda(\phi_1) = \lambda(\phi_2)$.

Remark: One is tempted to mod $\mathcal{P}$ out by this equivalence relations. One problem: the normalized dilatation $L(S, \phi)$ does not behave well with respect to this equivalence relation. (Subject of further study.)

Effect of puncturing at singularities:

For $(S, \phi) \in \mathcal{P}$, let $S^0 = S \setminus \mathrm{Sing}(\phi)$ and $\phi^0 = \phi|_{S^0}$.

Equivalence relation on $\mathcal{P}$:
Write $(S_1, \phi_1) \sim (S_2, \phi_2)$ if $(S_1^0, \phi_1^0) = (S_2^0, \phi_2^0)$.

Lemma
If $(S_1, \phi_1) \sim (S_2, \phi_2)$, then $\lambda(\phi_1) = \lambda(\phi_2)$.

Remark: One is tempted to mod $\mathcal{P}$ out by this equivalence relations. One problem: the normalized dilatation $L(S, \phi)$ does not behave well with respect to this equivalence relation. (Subject of further study.)

Effect of puncturing at singularities:

For $(S, \phi) \in \mathcal{P}$, let $S^0 = S \setminus \text{Sing}(\phi)$ and $\phi^0 = \phi|_{S^0}$.

Equivalence relation on $\mathcal{P}$:
Write $(S_1, \phi_1) \sim (S_2, \phi_2)$ if $(S_1^0, \phi_1^0) = (S_2^0, \phi_2^0)$.

Lemma
If $(S_1, \phi_1) \sim (S_2, \phi_2)$, then $\lambda(\phi_1) = \lambda(\phi_2)$.

Remark: One is tempted to mod $\mathcal{P}$ out by this equivalence relations. One problem: the normalized dilatation $L(S, \phi)$ does not behave well with respect to this equivalence relation. (Subject of further study.)

Universal Finiteness theorem

(Farb-Leininger-Margalit '09, Agol '10) For any $L > 1$, there is a finite collection (M_i, F_i), $i = 1, \ldots, k$ so that

$$L(S, \phi) \leq L \quad \Rightarrow \quad (S^0, \phi^0) \in \Phi(M_i, F_i) \quad \text{for some } i.$$

(Penner) $\Rightarrow$ The minimum dilatation mapping classes on genus g surfaces have mapping tori coming from a finite collection of 3-manifolds.

Universal Finiteness theorem

(Farb-Leininger-Margalit '09, Agol '10) For any $L > 1$, there is a finite collection (M_i, F_i), $i = 1, \dots, k$ so that

$$L(S, \phi) \leq L \quad \Rightarrow \quad (S^0, \phi^0) \in \Phi(M_i, F_i) \quad \text{for some } i.$$

(Penner) $\Rightarrow$ The minimum dilatation mapping classes on genus g surfaces have mapping tori coming from a finite collection of 3-manifolds.

Universal Finiteness theorem

(Farb-Leininger-Margalit '09, Agol '10) For any $L > 1$, there is a finite collection (M_i, F_i), $i = 1, \ldots, k$ so that

$$L(S, \phi) \leq L \quad \Rightarrow \quad (S^0, \phi^0) \in \Phi(M_i, F_i) \quad \text{for some } i.$$

(Penner) $\Rightarrow$ The minimum dilatation mapping classes on genus g surfaces have mapping tori coming from a finite collection of 3-manifolds.

Some immediate consequences and questions:

(Brinkman, Penner) As $g \to \infty$,

$$\delta(S_g) = \min\{\lambda(\phi) \; ; \; \phi \in \mathcal{P}_S\} \to 1$$

(In fact, $\log(\delta(S_g)) \asymp \frac{1}{g}$.)

UFT $\Rightarrow$ the minimum value ℓ for L is greater than one.

Question 1: Is the minimum ℓ of L attained by some (S, ϕ)?

(H_) $\limsup \ell(S_g) \leq (\frac{3+\sqrt{5}}{2})^2 = L(S_0\phi_0)$, where (S_0, ϕ_0) is the simplest pseudo-Anosov braid. (see also Kin-Takasawa, Aaber-Dunfield)

Golden Mean Conjecture: $\lim_{g\to\infty} \ell(S_g) = L(S_0, \phi_0)$.

Question 2 (McMullen): Are the local minima of L in $\overline{\mathcal{P}}$ attained at rational points (i.e., points in $\mathcal{P}$)?

If Question 2 is true, then UFT would imply Question 1.

Some immediate consequences and questions:

(Brinkman, Penner) As $g \to \infty$,

$$\delta(S_g) = \min\{\lambda(\phi)\ ;\ \phi \in \mathcal{P}_S\} \to 1$$

(In fact, $\log(\delta(S_g)) \asymp \frac{1}{g}$.)

UFT $\Rightarrow$ the minimum value ℓ for L is greater than one.

Question 1: Is the minimum ℓ of L attained by some (S, ϕ)?

(H_) $\limsup \ell(S_g) \leq (\frac{3+\sqrt{5}}{2})^2 = L(S_0\phi_0)$, where (S_0, ϕ_0) is the simplest pseudo-Anosov braid. (see also Kin-Takasawa, Aaber-Dunfield)

Golden Mean Conjecture: $\lim_{g\to\infty} \ell(S_g) = L(S_0, \phi_0)$.

Question 2 (McMullen): Are the local minima of L in $\overline{\mathcal{P}}$ attained at rational points (i.e., points in $\mathcal{P}$)?

If Question 2 is true, then UFT would imply Question 1.

Some immediate consequences and questions:

(Brinkman, Penner) As $g \to \infty$,

$$\delta(S_g) = \min\{\lambda(\phi) \; ; \; \phi \in \mathcal{P}_S\} \to 1$$

(In fact, $\log(\delta(S_g)) \asymp \frac{1}{g}$.)

UFT $\Rightarrow$ the minimum value ℓ for L is greater than one.

Question 1: Is the minimum ℓ of L attained by some (S, ϕ)?

(H_) $\limsup \ell(S_g) \leq (\frac{3+\sqrt{5}}{2})^2 = L(S_0\phi_0)$, where (S_0, ϕ_0) is the simplest pseudo-Anosov braid. (see also Kin-Takasawa, Aaber-Dunfield)

Golden Mean Conjecture: $\lim_{g\to\infty} \ell(S_g) = L(S_0, \phi_0)$.

Question 2 (McMullen): Are the local minima of L in $\overline{\mathcal{P}}$ attained at rational points (i.e., points in $\mathcal{P}$)?

If Question 2 is true, then UFT would imply Question 1.

Some immediate consequences and questions:

(Brinkman, Penner) As $g \to \infty$,

$$\delta(S_g) = \min\{\lambda(\phi)\ ;\ \phi \in \mathcal{P}_S\} \to 1$$

(In fact, $\log(\delta(S_g)) \asymp \frac{1}{g}$.)

UFT $\Rightarrow$ the minimum value ℓ for L is greater than one.

Question 1: Is the minimum ℓ of L attained by some (S, ϕ)?

(H_-) $\limsup \ell(S_g) \leq (\frac{3+\sqrt{5}}{2})^2 = L(S_0\phi_0)$, where (S_0, ϕ_0) is the simplest pseudo-Anosov braid. (see also Kin-Takasawa, Aaber-Dunfield)

Golden Mean Conjecture: $\lim_{g\to\infty} \ell(S_g) = L(S_0, \phi_0)$.

Question 2 (McMullen): Are the local minima of L in $\overline{\mathcal{P}}$ attained at rational points (i.e., points in $\mathcal{P}$)?

If Question 2 is true, then UFT would imply Question 1.

Some immediate consequences and questions:

(Brinkman, Penner) As $g \to \infty$,

$$\delta(S_g) = \min\{\lambda(\phi)\ ;\ \phi \in \mathcal{P}_S\} \to 1$$

(In fact, $\log(\delta(S_g)) \asymp \frac{1}{g}$.)

UFT $\Rightarrow$ the minimum value ℓ for L is greater than one.

Question 1: Is the minimum ℓ of L attained by some (S, ϕ)?

(H_) $\limsup \ell(S_g) \leq (\frac{3+\sqrt{5}}{2})^2 = L(S_0\phi_0)$, where (S_0, ϕ_0) is the simplest pseudo-Anosov braid. (see also Kin-Takasawa, Aaber-Dunfield)

Golden Mean Conjecture: $\lim_{g\to\infty} \ell(S_g) = L(S_0, \phi_0)$.

Question 2 (McMullen): Are the local minima of L in $\overline{\mathcal{P}}$ attained at rational points (i.e., points in $\mathcal{P}$)?

If Question 2 is true, then UFT would imply Question 1.

Some immediate consequences and questions:

(Brinkman, Penner) As $g \to \infty$,

$$\delta(S_g) = \min\{\lambda(\phi)\ ;\ \phi \in \mathcal{P}_S\} \to 1$$

(In fact, $\log(\delta(S_g)) \asymp \frac{1}{g}$.)

UFT $\Rightarrow$ the minimum value ℓ for L is greater than one.

Question 1: Is the minimum ℓ of L attained by some (S, ϕ)?

(H_) $\limsup \ell(S_g) \leq (\frac{3+\sqrt{5}}{2})^2 = L(S_0\phi_0)$, where (S_0, ϕ_0) is the simplest pseudo-Anosov braid. (see also Kin-Takasawa, Aaber-Dunfield)

Golden Mean Conjecture: $\lim_{g\to\infty} \ell(S_g) = L(S_0, \phi_0)$.

Question 2 (McMullen): Are the local minima of L in $\overline{\mathcal{P}}$ attained at rational points (i.e., points in $\mathcal{P}$)?

If Question 2 is true, then UFT would imply Question 1.

Families with asymptotically small dilatations

A family $\mathcal{F} \subset \mathcal{P}$ is said to have *asymptotically small dilatation elements* if $\mathcal{F}$ contains a subfamily $\mathcal{F}_1 = \{(S, \phi)\}$ where

- $\chi(S)$ is unbounded, and
- normalized dilatation $L(S, \phi) = \lambda(\phi)^{|\chi(S)|}$ is bounded. i.e.,

$$\log(\lambda(\phi)) \asymp \frac{1}{|\chi(S)|}.$$

Problem: Which natural subsets of $\mathcal{P}_S$ have asymptotically small dilatation elements?

Examples:

- Torelli subgroups $\cap\ \mathcal{P}$? No (Farb-Leininger-Margalit '08)
- Hyperelliptic elements of $\mathcal{P}$? Orientable elements of $\mathcal{P}$? Yes (H-Kin '06)

Families with asymptotically small dilatations

A family $\mathcal{F} \subset \mathcal{P}$ is said to have *asymptotically small dilatation elements* if $\mathcal{F}$ contains a subfamily $\mathcal{F}_1 = \{(S, \phi)\}$ where

- $\chi(S)$ is unbounded, and
- normalized dilatation $L(S, \phi) = \lambda(\phi)^{|\chi(S)|}$ is bounded. i.e.,

$$\log(\lambda(\phi)) \asymp \frac{1}{|\chi(S)|}.$$

Problem: Which natural subsets of $\mathcal{P}_S$ have asymptotically small dilatation elements?

Examples:

- Torelli subgroups $\cap\, \mathcal{P}$? No (Farb-Leininger-Margalit '08)
- Hyperelliptic elements of $\mathcal{P}$? Orientable elements of $\mathcal{P}$? Yes (H-Kin '06)

Families with asymptotically small dilatations

A family $\mathcal{F} \subset \mathcal{P}$ is said to have *asymptotically small dilatation elements* if $\mathcal{F}$ contains a subfamily $\mathcal{F}_1 = \{(S, \phi)\}$ where

- $\chi(S)$ is unbounded, and
- normalized dilatation $L(S, \phi) = \lambda(\phi)^{|\chi(S)|}$ is bounded.
  i.e.,
  $$\log(\lambda(\phi)) \asymp \frac{1}{|\chi(S)|}.$$

Problem: Which natural subsets of $\mathcal{P}_S$ have asymptotically small dilatation elements?
Examples:

- Torelli subgroups $\cap\, \mathcal{P}$? No (Farb-Leininger-Margalit '08)
- Hyperelliptic elements of $\mathcal{P}$? Orientable elements of $\mathcal{P}$? Yes (H-Kin '06)

I. Pseudo-Anosov mapping classes

1. Introduction
2. Visualizing pseudo-Anosov mapping classes
3. Train tracks
4. Minimum dilatation problem

II. Fibered faces and applications

1. Introduction
2. Fibered face theory
3. Alexander and Teichmüller polynomials
4. First application: Orientable mapping classes

III. Families of mapping classes with small dilatations

1. Introduction
2. Deformations of mapping classes on fibered faces
3. Penner sequences and applications
4. Quasiperiodic mapping classes

Orientable examples

$(S, \phi) \in \mathcal{P}$ is an *orientable mapping class* if the stable foliation $\mathcal{F}^s$ is orientable

(Rykken) For any $(S, \phi) \in \mathcal{P}$

$$\lambda_{hom}(\phi) \leq \lambda(\phi),$$

with equality if and only if ϕ is orientable.

Let $\delta^+(S_g)$ be the smallest dilatation amongst orientable elements of $\mathcal{P}_{S_g}$.

Orientable examples

$(S, \phi) \in \mathcal{P}$ is an *orientable mapping class* if the stable foliation $\mathcal{F}^s$ is orientable

(Rykken) For any $(S, \phi) \in \mathcal{P}$

$$\lambda_{hom}(\phi) \leq \lambda(\phi),$$

with equality if and only if ϕ is orientable.

Let $\delta^+(S_g)$ be the smallest dilatation amongst orientable elements of $\mathcal{P}_{S_g}$.

Orientable examples

$(S, \phi) \in \mathcal{P}$ is an *orientable mapping class* if the stable foliation $\mathcal{F}^s$ is orientable

(Rykken) For any $(S, \phi) \in \mathcal{P}$

$$\lambda_{hom}(\phi) \leq \lambda(\phi),$$

with equality if and only if ϕ is orientable.

Let $\delta^+(S_g)$ be the smallest dilatation amongst orientable elements of $\mathcal{P}_{S_g}$.

LT polynomials

LT-polynomials:

$$LT_{a,n}(x) = x^{2n} - x^{n+a} - x^n - x^{n-a} + 1.$$

Let $\lambda_{a,n} = |LT_{n,a}|$ be the house of $LT_{n,a}$.

(Lanneau-Thiffeault '09) For $g = 2, 3, 4, 6, 8$,

$$\lambda_{1,g} \leq \delta^+(S_g).$$

with equality for $g = 2, 3, 4$.

LT-Question: Is it true that for all even g,

$$\delta^+(S_g) = \lambda_{1,g}?$$

LT polynomials

LT-polynomials:

$$LT_{a,n}(x) = x^{2n} - x^{n+a} - x^n - x^{n-a} + 1.$$

Let $\lambda_{a,n} = |LT_{n,a}|$ be the house of $LT_{n,a}$.

(Lanneau-Thiffeault '09) For $g = 2, 3, 4, 6, 8$,

$$\lambda_{1,g} \leq \delta^+(S_g).$$

with equality for $g = 2, 3, 4$.

LT-Question: Is it true that for all even g,

$$\delta^+(S_g) = \lambda_{1,g}?$$

LT polynomials

LT-polynomials:

$$LT_{a,n}(x) = x^{2n} - x^{n+a} - x^n - x^{n-a} + 1.$$

Let $\lambda_{a,n} = |LT_{n,a}|$ be the house of $LT_{n,a}$.

(Lanneau-Thiffeault '09) For $g = 2, 3, 4, 6, 8$,

$$\lambda_{1,g} \leq \delta^+(S_g).$$

with equality for $g = 2, 3, 4$.

LT-Question: Is it true that for all even g,

$$\delta^+(S_g) = \lambda_{1,g}?$$

Other known minima for $\delta^+(S_g)$

Using Lanneau and Thiffeault's lower bounds (for $g \leq 8$) and examples:

- (Lanneau-Thiffeault) $\delta^+(S_5) = \lambda_{1,6}$ (= Lehmer's number $\approx$ 1.17628).
- (H '10) $\delta^+(S_8) = \lambda_{1,8}$
- (Aaber-Dunfield '10, Kin-Takasawa '11) $\delta^+(S_7) = \lambda_{2,9}$.

Other known minima for $\delta^+(S_g)$

Using Lanneau and Thiffeault's lower bounds (for $g \leq 8$) and examples:

- (Lanneau-Thiffeault) $\delta^+(S_5) = \lambda_{1,6}$ (= Lehmer's number ≈ 1.17628).
- (H '10) $\delta^+(S_8) = \lambda_{1,8}$
- (Aaber-Dunfield '10, Kin-Takasawa '11) $\delta^+(S_7) = \lambda_{2,9}$.

Other known minima for $\delta^+(S_g)$

Using Lanneau and Thiffeault's lower bounds (for $g \leq 8$) and examples:

- (Lanneau-Thiffeault) $\delta^+(S_5) = \lambda_{1,6}$ (= Lehmer's number ≈ 1.17628).
- (H '10) $\delta^+(S_8) = \lambda_{1,8}$
- (Aaber-Dunfield '10, Kin-Takasawa '11) $\delta^+(S_7) = \lambda_{2,9}$.

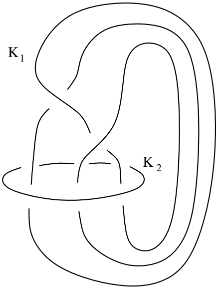
Other known minima for $\delta^+(S_g)$

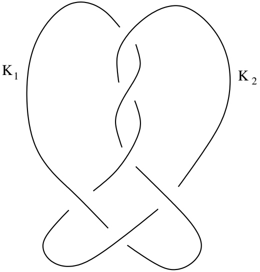
Using Lanneau and Thiffeault's lower bounds (for $g \leq 8$) and examples:

- (Lanneau-Thiffeault) $\delta^+(S_5) = \lambda_{1,6}$ (= Lehmer's number ≈ 1.17628).
- (H '10) $\delta^+(S_8) = \lambda_{1,8}$
- (Aaber-Dunfield '10, Kin-Takasawa '11) $\delta^+(S_7) = \lambda_{2,9}$.

Simplest pseudo-Anosov braid revisited

Let (S, ϕ) be the simplest pseudo-Anosov braid, let (M, F) be its mapping torus and fibered face, so that $(S, \phi) \in \Phi(M, F) \subset F$.

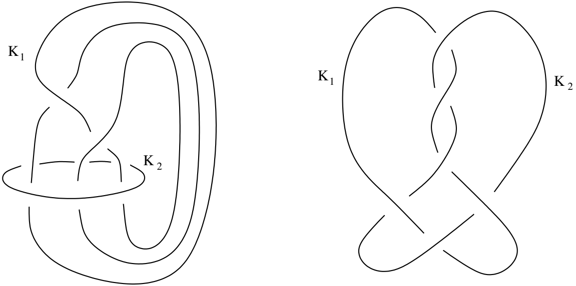




The meridians μ_1, μ_2 of K_1 and K_2 determine coordinates (t, u) for $H^1(M; \mathbb{Z})$.

Simplest pseudo-Anosov braid revisited

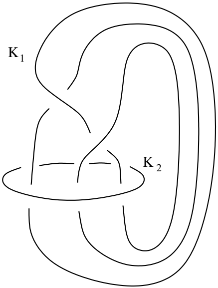
Let (S, ϕ) be the simplest pseudo-Anosov braid, let (M, F) be its mapping torus and fibered face, so that $(S, \phi) \in \Phi(M, F) \subset F$.

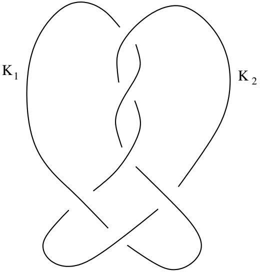


The meridians μ_1, μ_2 of K_1 and K_2 determine coordinates (t, u) for $H^1(M; \mathbb{Z})$.

Simplest pseudo-Anosov braid revisited

Let (S, ϕ) be the simplest pseudo-Anosov braid, let (M, F) be its mapping torus and fibered face, so that $(S, \phi) \in \Phi(M, F) \subset F$.





The meridians μ_1, μ_2 of K_1 and K_2 determine coordinates (t, u) for $H^1(M; \mathbb{Z})$.

Polynomial invariants

The Alexander polynomial and Teichmüller polynomials are given in terms of these coordinates by:

$$\Delta(t,u) = u^2 - u(1 - t - t^{-1}) + 1$$

and

$$\Theta(t,u) = u^2 - u(1 + t + t^{-1}) + 1.$$

and their specializations

$$\Delta^{(a,b)}(x) = \Delta(x^a, x^b) \qquad \Theta^{(a,b)}(x) = \Theta(x^a, x^b) = LT_{a,b}(x).$$

It follows that $\phi_{a,b}$ is orientable for a odd and b even.

Consequences

Evidence for LT-Question:

Theorem (H_-)

For g even and $6 \nmid g$, there is a sequence of orientable mapping classes ϕ_g defined on a closed genus g surfaces so that with $\lambda(\phi_g) = \lambda_{1,g}$.

Evidence for Golden Mean Conjecture:

Theorem (H_-)

There is an infinite sequence of mapping classes (S_g, ϕ_g) where S_g is a closed genus g surface, such that

$$\lim_{g\to\infty} L(S, \phi_g) = \lim_{g\to\infty} \lambda(\phi_g)^{2g} = \left(\frac{3+\sqrt{5}}{2}\right)^2$$

Consequences

Evidence for LT-Question:

Theorem (H_)

For g even and $6 \not| g$, *there is a sequence of orientable mapping classes* ϕ_g *defined on a closed genus g surfaces so that with* $\lambda(\phi_g) = \lambda_{1,g}$.

Evidence for Golden Mean Conjecture:

Theorem (H_)

There is an infinite sequence of mapping classes (S_g, ϕ_g) *where* S_g *is a closed genus g surface, such that*

$$\lim_{g\to\infty} L(S, \phi_g) = \lim_{g\to\infty} \lambda(\phi_g)^{2g} = \left(\frac{3+\sqrt{5}}{2}\right)^2$$

Fibered face for simplest pseudo-Anosov braid