Non-left-orderability of double branched coverings via "coarse" presentation

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[(reference)

Non-left-orderable double branched coverings, arXiv : 1106.1499]

§ I. Introduction

G : group

$<_G$: Total ordering of G

- $<_G$ is a **left-ordering**

$\iff$ $a <_G b \Rightarrow ca <_G cb$ $\forall a, b, c \in G$

- G is **left-orderable (LO)**

$\iff$ G has at least one left-ordering

example.

- $(\mathbb{Z}, <)$: natural ordering of $\mathbb{Z}$ is a left-ordering.

- Σ : compact, orientable surface with **non-empty boundaries**

$MCG(\Sigma, \mathrm{rel}\ \partial\Sigma)$: Mapping Class group of Σ (fixing $\partial\Sigma$)

$MCG(\Sigma, \mathrm{rel}\ \partial\Sigma)$ is LO.

Motivating Problem

M : compact orientable 3-manifold

When $\pi_1(M)$ is LO ?

[Background]

$$G \text{ is LO} \iff G \hookrightarrow \mathrm{Homeo}^+(\mathbb{R})$$

- When $\widetilde{M}$ "compress" to $\mathbb{R}$?

$$\begin{tikzcd} \pi_1(M) \arrow[r, bend left] \arrow[dr, dashed, bend right, "?"'] & \widetilde{M} \arrow[d, dashed, "\text{“compress”}"] \\ & \mathbb{R} \end{tikzcd}$$

(faithful, orientation preserving)

Theorem (Boyer-Rolfsen-Wiest '05)

M is not $\mathbb{Q}$HS $\Rightarrow$ $\pi_1(M)$ is LO

So, the problem is :

Motivating Problem'

M : $\mathbb{Q}$HS

When $\pi_1(M)$ is LO ?

L-space conjecture

$\pi_1(M)$ is **NOT LO** $\Longleftrightarrow$ M is an L-space

(hat version)

$\widehat{HF}(M)$: Heegaard Floer homology

For $\mathbb{Q}HS$, rank $\widehat{HF}(M) \geqq |H_1(M;\mathbb{Z})|$

M is an <u>L-space</u>

$\overset{\text{def}}{\Longleftrightarrow}$ rank $\widehat{HF}(M) = |H_1(M;\mathbb{Z})|$

"Heegaard-Floer theoretical generalization of Lens spaces"

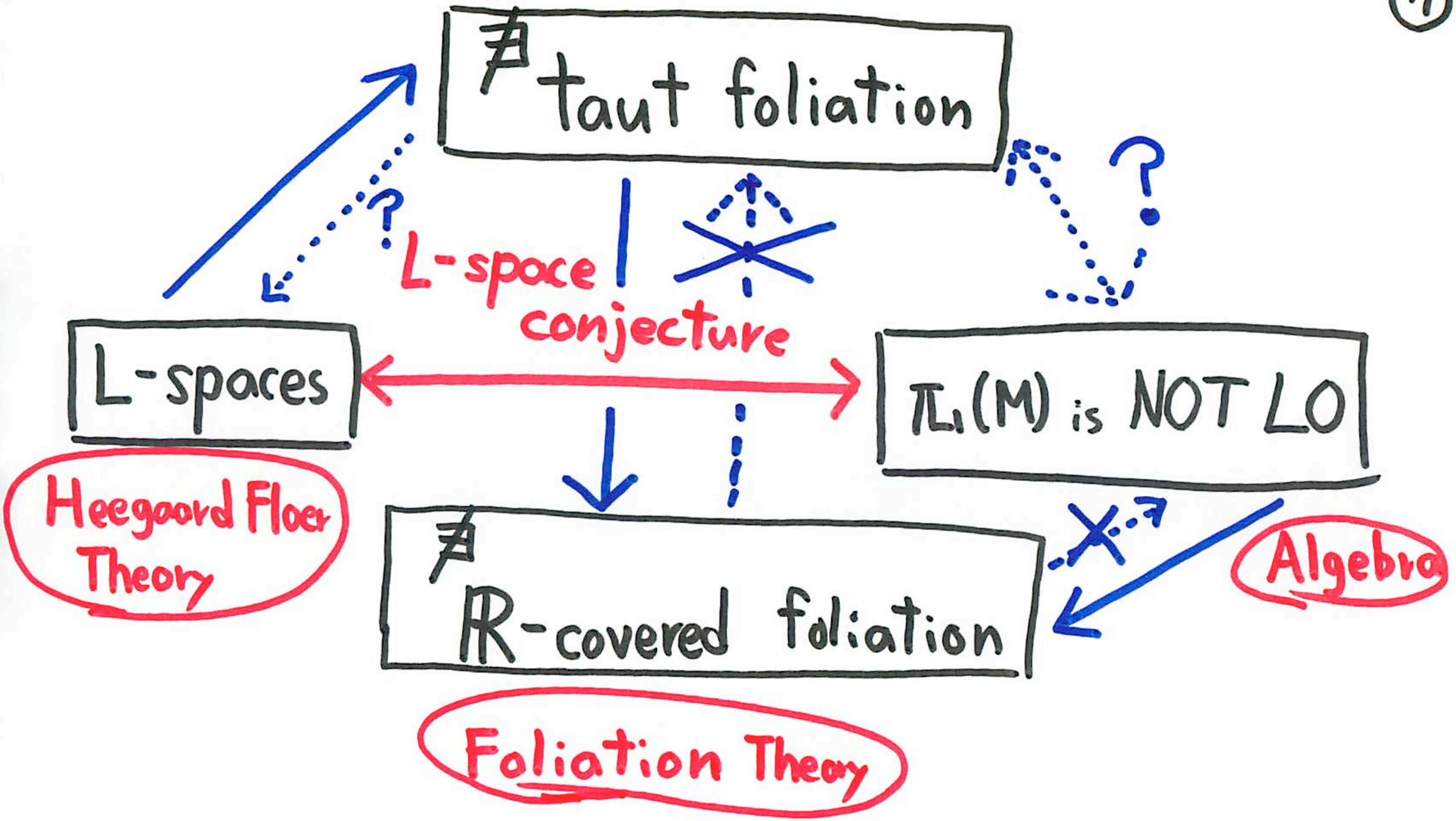
- L-space conjecture
  was confirmed for many cases.

※ Seifert fibered 3-manifolds

※ 3-manifolds admitting
  Sol geometry

※ Graph manifolds which are $\mathbb{Z}HS$

※ Many known L-spaces



$\nexists$ taut foliation
?
?
L-space conjecture
L-spaces
$\pi_1(M)$ is NOT LO
Heegaard Floer Theory
$\nexists$ $\mathbb{R}$-covered foliation
Algebra
Foliation Theory

Today's talk

$L \subset S^3$: (oriented) link

$M = \Sigma_2(L)$: double branched covering

A method to show $\pi_1(M)$ is NOT LO via "coarse" presentation

presentation $\dashrightarrow$ "coarse" presentation

$x = z$ (equality) $\dashrightarrow$ $x \leq z$ (inequality)

Why $\Sigma_2(L)$?

- $\{$L-spaces$\} \supseteq \{$branched double covering of **quasi-alternating Links**$\}$

  (Heegaard-Floer theoretical generalization of alternating links)

- Montesinos trick

  $\rightsquigarrow$

  $\Sigma_2(L)$ is related to Dehn surgery.

∘ How to prove G is not LO via usual presentation

※ Assume $^{\exists}$ left-ordering $<_G$ of G

$$\xrightarrow{\text{presentatin, left-invariace}} \text{Contradiction}$$

Ex. $G = \langle x, \cdots \mid x^n = 1, \cdots \rangle$

If $1 <_G x$, then

$$1 <_G x <_G x^2 <_G \cdots <_G x^n = 1 \Rightarrow x = 1$$

If $1 >_G x$, then

$$1 >_G x >_G x^2 >_G \cdots >_G x^n = 1 \Rightarrow x = 1$$

$\boxed{\text{Contradiction}}$

Problem to use presentation

- ad hoc (not general).
- presentation might be complex.

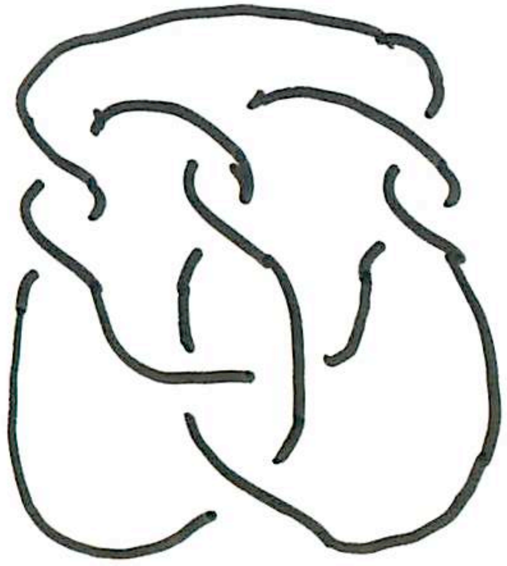
↳ it is not easy to apply NON-LO proof for "similar" presentation

§ II Coarse presentation of $\pi_1(\Sigma_2(L))$ ⑫

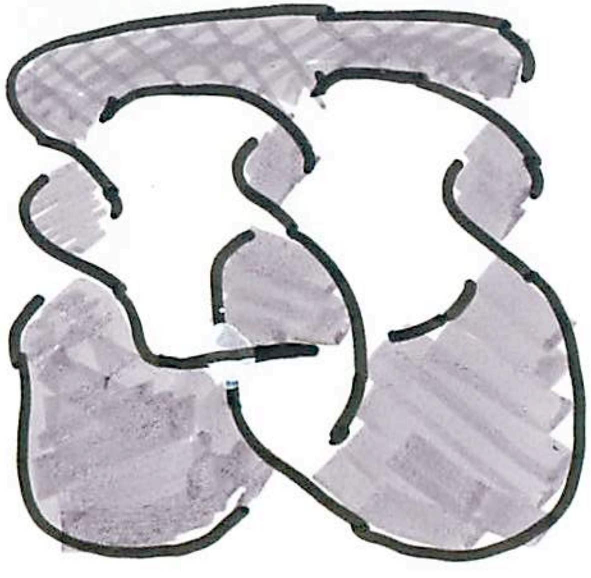
※ Presentation of $\pi_1(\Sigma_2(L))$ [Brunner]

D : link diagram of L

$S(D)$: Checker board surface of D



D



$S(D)$

$S'(D) = [\text{Discs}] \cup [\text{Twisted bands}]$

n twists

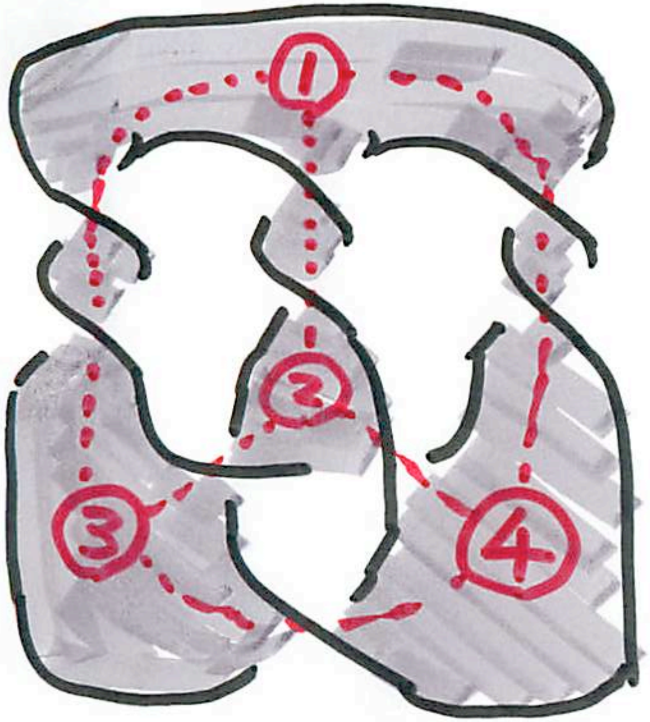
n twists

Disc $\longrightarrow$ vertex

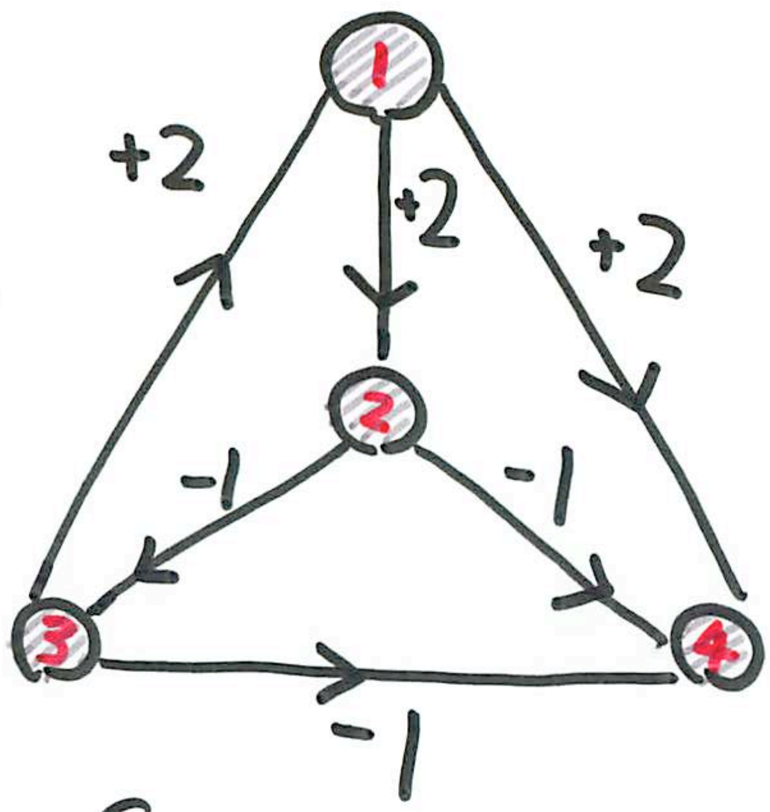
band $\longrightarrow$ labelled edge ⇢ planer graph G

"decomposition Graph"

ex.)



$\longrightarrow$



- orient edges of G.
- region $\overset{\text{def}}{=}$ component of $\mathbb{R}^2 - G$.

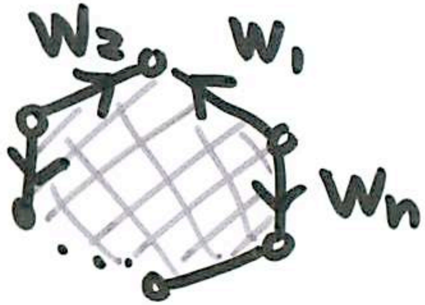
Brunner's Presentation of $\pi_1(\Sigma_2(L))$ ⑮

- **Generator**
  - $\{W_i\}$: edge of G
  - $\{R_i\}$: regions
- **relation**

(i) $R_0 = 1$ R_0 : unbounded region

(ii) [Global cycle relation]

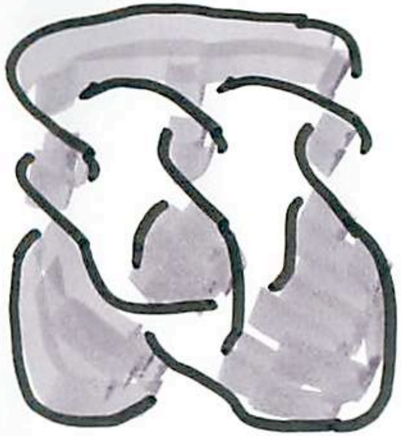
$W_n^{\pm 1} \cdots W_1^{\pm 1} = 1$ if

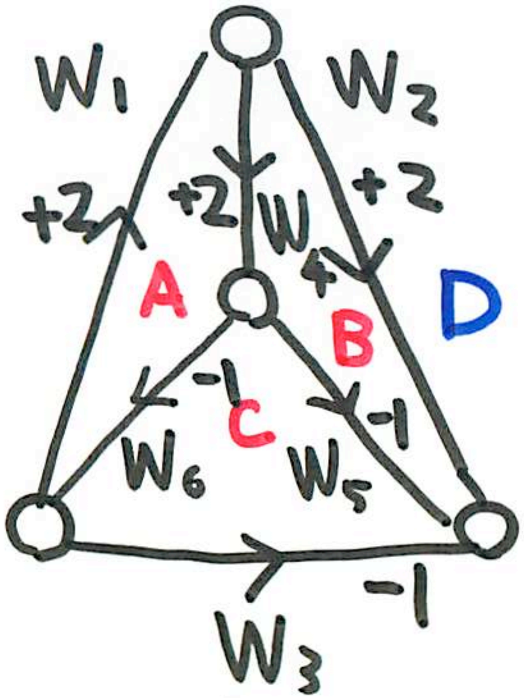


(iii) [local edge relation]

$W = (R_\ell^{-1} R_r)^a$ if

[example:]





[Generator]

$W_1, \cdots, W_6, A, B, C, (D=1)$

[relation]

$W_1 = A^2$

$W_2 = B^2$

$W_3 = C$

$W_4 = (B^{-1}A)^2$

$W_5 = (B^{-1}C)^{-1}$

$W_6 = (C^{-1}A)^{-1}$

$W_6 W_4 W_1 = 1$

$W_4^{-1} W_5^{-1} W_2 = 1$

$W_6^{-1} W_3^{-1} W_5 = 1$

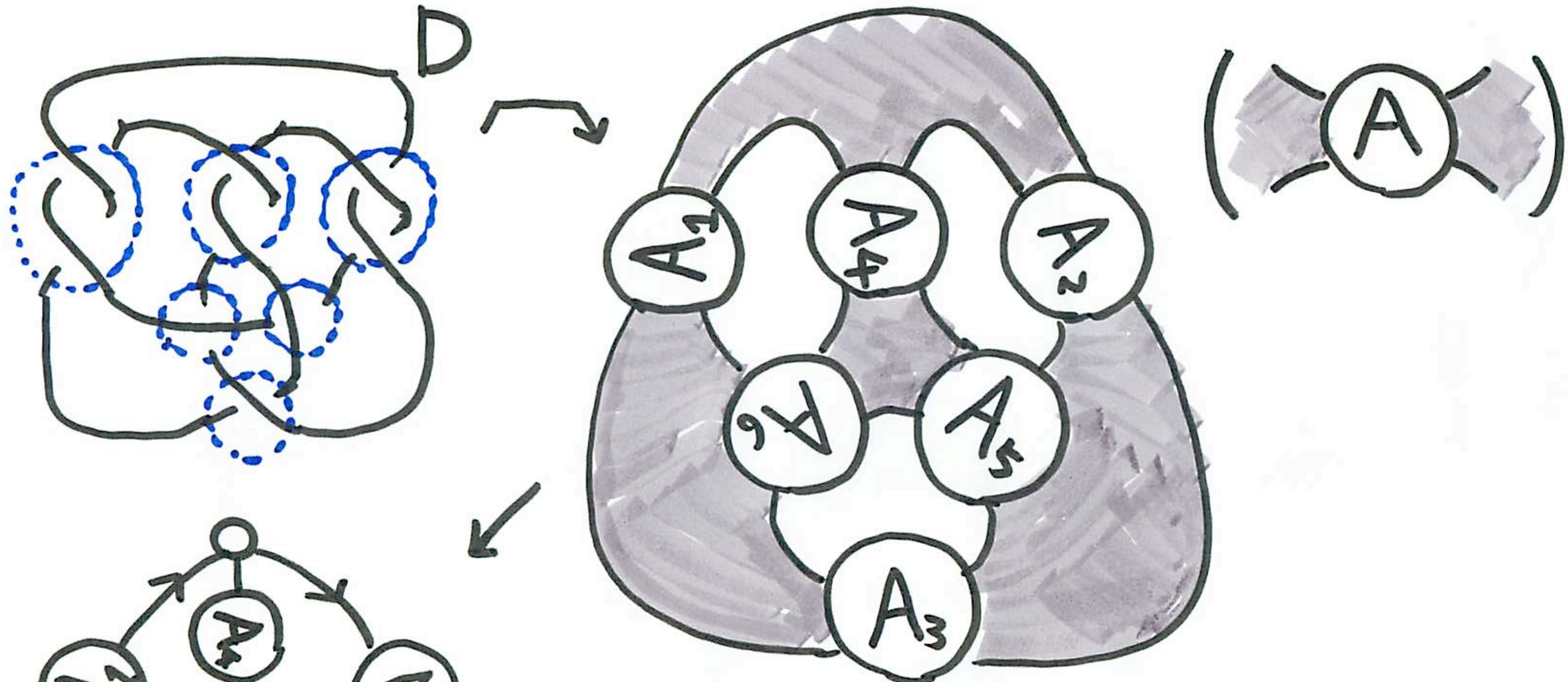
- Brunner's presentation has simple relations → suitable for algebraic/combinatorial arguments
- "similar" diagrams yield "similar" presentation

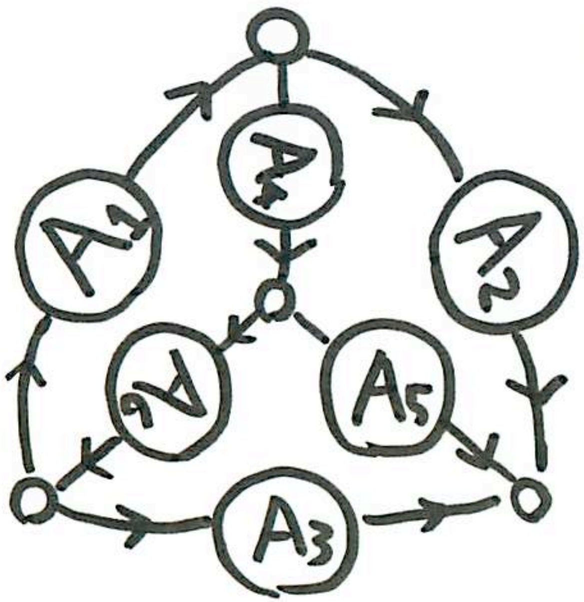
Problem

complex diagram gives complex Brunner's presentation!

↳ extract "essential information" needed to prove NON-LO.

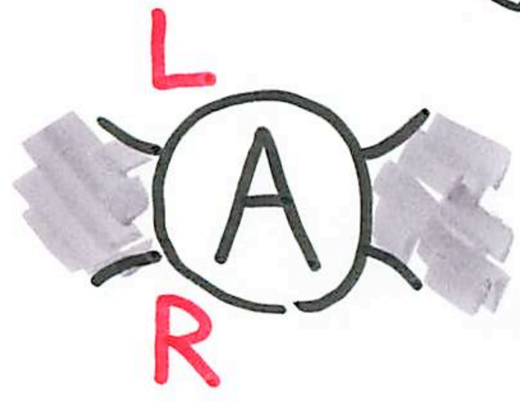
- Decompose link diagram as Tangles / strands





"coarse decomposition Graph"

Key observation.



A : algebraic tangle

$\exists\, W_A \in \pi_1(\Sigma_2(L))$, $\frac{q}{p}, \frac{s}{r} \in \mathbb{Q}$

s.t. for all left-ordering $<_G$ of $\pi_1(\Sigma_2(L))$

$$(*)\begin{cases} \bullet\ W_A^{q} \leq_G (L^{-1}R)^{p},\ (L^{-1}R)^{r} \leq_G W_A^{s} & (\text{if } 1 \leq_G W_A) \\ W_A^{s} \leq_G (L^{-1}R)^{r},\ (L^{-1}R)^{p} \leq W_A^{q} & (\text{if } 1 \geq W_A) \end{cases}$$

- $(L^{-1}R)\, W_A = W_A\, (L^{-1}R)$

Universal inequality

We express (*) by $W_A \in [[\frac{q}{p}, \frac{s}{r}]]_{L^{-1}R}$

(*) ... "universal" inequality
which holds for all left-orderings.

- rationals $\frac{q}{p}, \frac{s}{r}$ are computed (estimated) for an algebraic tangle A

ex.) $A = Q(\frac{q}{p}) \Rightarrow W_A \in [[\frac{q}{p}, \frac{q}{p}]]_{L^{-1}R}$

Coarse Brunner's Presentation

Generator.

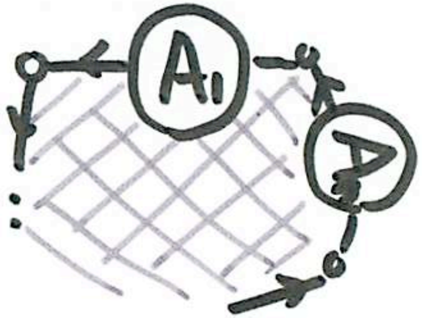
$\{W_A\}$, R (regions of coarse decomposition graph)

Relation

(i) $R_0 = 1$ $\quad$ R_0 : unbounded region

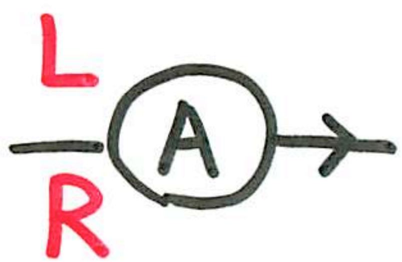
(ii) [Global cycle relation]

$$W_{A_1}^{\pm 1} \cdots W_{A_n}^{\pm 1} = 1 \quad \text{if}$$



(iii) [Local coarse "relation"]

$$W_A \in \left[\left[\frac{q}{p}, \frac{s}{r}\right]\right]_{L^{-1}R}$$



※ Remark

- "Coarse" presentation does not determine a group.

↳ Different link diagram might have the same coarse presentation.

(Different (algebraic) tangles  ,  might give the same local coarse relation

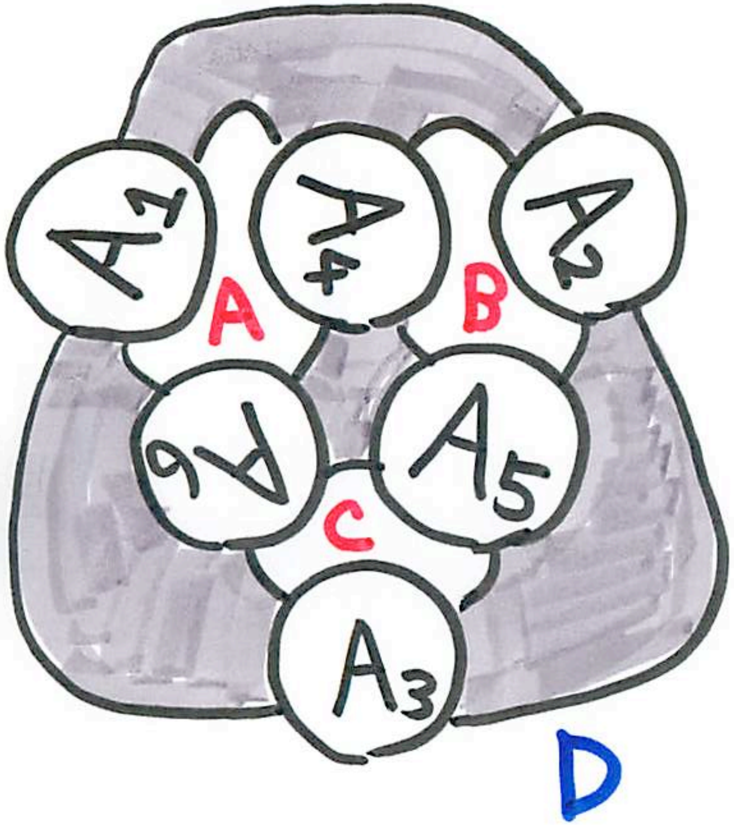
$$W_A \in \left\llbracket \frac{q}{p}, \frac{s}{r} \right\rrbracket_{L^{-1}R}, \quad W_B \in \left\llbracket \frac{q}{p}, \frac{s}{r} \right\rrbracket_{L^{-1}R}$$)

- replacing a tangle with another tangle

$\longrightarrow$ replacing a local coarse relation with another one.

[coarse presentation]

example



Generator

$$W_1, \cdots, W_6, A, B, C, (D = 1)$$

Relation

$$W_1 \in [[m_1, M_1]]_A \quad W_4 \in [[m_4, M_4]]_{B^{-1}A}$$

$$W_2 \in [[m_2, M_2]]_B \quad W_5 \in [[m_5, M_5]]_{B^{-1}C}$$

$$W_3 \in [[m_3, M_3]]_{C^{-1}} \quad W_6 \in [[m_6, M_6]]_{C^{-1}A}$$

$$W_6 W_4 W_1 = 1$$

$$W_4^{-1} W_5^{-1} W_2 = 1$$

$$W_6^{-1} W_3^{-1} W_5 = 1$$

Good point to use coarse presentation

- We are able to treat "complex" diagrams
  (complex diagram might give a simple coarse presentation)
- If we prove NON-LO for one coarse presentation we get a proof of non-left-orderability for (infinitely many) links.

§III. Application

Theorem (I.)

D : link diagram with tangle-strand decomposition

If all local coarse relations are of the form

$W_A \in [\![0, \infty]\!]$

$\hookrightarrow$ $\pi_1(\Sigma_2(D))$ is NON-LO

Corollary. (Boyer-Gordon-Watson, Greene)

L : Alternating link

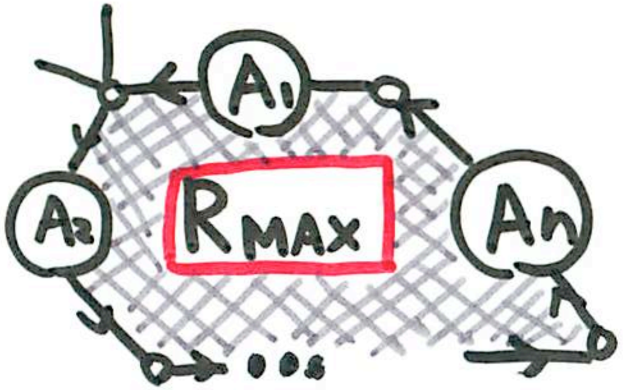
$\Rightarrow$ $\pi_1(\Sigma_2(L))$ is NON-LO

(Remark: $\Sigma_2(L)$ is an L-space)

[Proof of Theorem]

Assume $\pi_1(\Sigma_2(L))$ is LO, and take a left-ordering $<$ of $\pi_1(\Sigma_2(L))$.

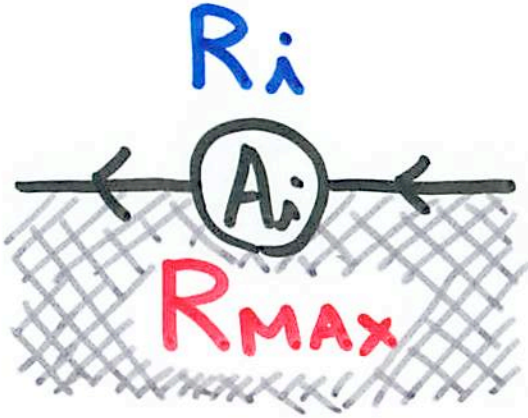
Let R_{MAX} be the $<$-maximal region generator.



- Global cycle relation for ∂R_{MAX}

$$W_{A_1} \cdot W_{A_2} \cdots W_{A_n} = 1 \quad \cdots\cdots \text{①}$$

- Local coarse relation



$$W_{A_i} \in [[0, \infty]]_{R_{MAX}^{-1} R_i}$$

$$R_i < R_{MAX} \Rightarrow R_{MAX}^{-1} R_i < 1$$

So $\underline{W_{A_i} < 1}$ $\cdots$ ②

Then $W_{A_1} \cdot W_{A_2} \cdots W_{A_n} < 1$, contradicts ① $\square$

- In a similar manner, we can prove

$\pi_1(\Sigma_2(L))$ is NON-LO for many other cases.

Forexample:

Theorem (I.)

K : positive knot of genus two

$\pi_1(\Sigma_2(K))$ is NON-LO

(Remark:)

- K might be non-alternating.
- $\Sigma_2(K)$ is an L-space