Shioda's conjecture via Alexander modules

The ultimate goal of these lectures is to prove the following conjecture by T. Shioda: *for any integer* $m \geqslant 1$*, the classes of lines contained in the Fermat surface* Φ_m *span a primitive subgroup in* $\operatorname{Pic}\Phi_m$. In particular (due to Shioda), if m is prime to 6, the classes of lines span $\operatorname{Pic}\Phi_m$.

For proof, we show that the group $H_1(\Phi_m \smallsetminus L_m)$ is torsion free, where L_m is the union of the lines. To this end, we consider the ramified abelian covering $\Phi_m \smallsetminus L_m \to \Phi_1 \smallsetminus L_1$, use Zariski–van Kampen approach to compute the fundamental group $\pi_1(\Phi_1 \smallsetminus (R \cup L_1))$, where R is the ramification locus (the union $R \cup L_1$ is a relatively simple arrangement of 7 lines in the plane Φ_1), and then compute the so-called *Alexander module* (or rather Alexander complex) of this arrangement, showing that its appropriate reduction has no integral torsion.

If time permits, we will also consider the possible generalizations to the so-called *Delsarte surfaces* and prove Shimada's conjecture on cyclic Delsarte surfaces: this generalization involves other reductions of the same Alexander complex. Note that the literal extension of the primitivity statement to *all* Delsarte surfaces fails, as simple examples show.