Invariant ordering of groups and low-dimensional topology I

: ordering of groups and examples

Tetsuya Ito (RIMS)

Branched Coverings, Degenerations, and Related Topics 2015

§ I-1.

Definition

A binary relation $\leq$ on a set X is a partial ordering

- (reflexivity) $a \leq a$ for all $a \in X$
- (Antisymmetry) $a \leq b$ and $b \leq a \Rightarrow a = b$
- (Transitivity) $a \leq b$ and $b \leq c \Rightarrow a \leq c$

$\leq$ is a total ordering (simply ordering)

if $a \leq b$ or $b \leq a$ holds for all $a, b \in X$

(i.e. every pair of elements $a, b \in X$.

Notion of ordering is quite ubiquitous and fundamental

- order on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \dots$
  (inequality in analysis)
- Zorn's lemma
- Asympotics (sufficiently large / small, etc ...)

$\boxed{\text{Object of the series of talks}}$

Total ordering on a group G

$\Downarrow$ related to

- Low-dimensional topology
- One-dimensional dynamics
- Combinatorial / Geometric group theory   and more ...

definition

G : group , $<_G$: total ordering on G

$<_G$ is a left-ordering (resp. right-ordering)

$\overset{\text{def}}{\iff}$ $a <_G b \Rightarrow ga <_G gb$ for all $g, a, b \in G$

(resp. $a <_G b \Rightarrow ag <_G bg$ 〃)

If $<_G$ is both left- and right- ordering,
$<_G$ is a bi-ordering.

- G is LO (left-orderable) $\overset{\text{def}}{\iff}$ G admits a left-ordering
- G is BO (bi-orderable) $\overset{\text{def}}{\iff}$ G admits a bi-ordering

- LO / BO group has various properties.

<u>Lemma</u>

(i) G is LO $\Longrightarrow$ G is torsion-free

(ii) G is LO $\Rightarrow$ Group ring $\mathbb{Z}G$ has no zero-divisor.

(iii) G is BO $\Rightarrow$ G has the unique root property

$$g^n = h^n \Rightarrow g = h \quad \forall g, h \in G$$

($\because$) (i)

$$\cdots <_G g^{-1} <_G 1 <_G g <_G g^2 <_G g^3 <_G \cdots$$

(ii)

$$\left(\sum c_i g_i\right)\left(\sum d_j h_j\right) = \sum (c_i d_j)(g_i h_j)$$

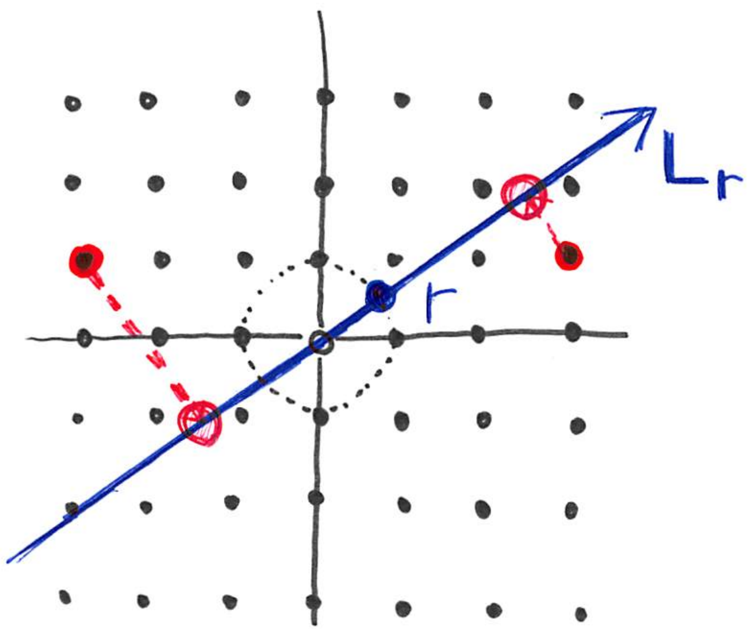
look at $<_G$-$\overset{\text{maximum}}{\text{minimum}}$ term among $\{g_i h_j\}$.

(iii) $g <_G h \Rightarrow g^2 <_G gh <_G h^2$

$$\cdots \Rightarrow g^n <_G h^n$$

§ I-2 Example of ordering (1)

ordering on $\mathbb{Z}^2$



$(-4, 1) <_r (4, 1)$

View $\mathbb{Z}^2 \subset \mathbb{R}^2$

For $r \in S^1 \subset \mathbb{R}^2$, consider the line L_r connecting O and r.

$$\pi_r : \mathbb{R}^2 \longrightarrow L_r \cong \mathbb{R} \quad \text{orthgonal projection}$$

Define

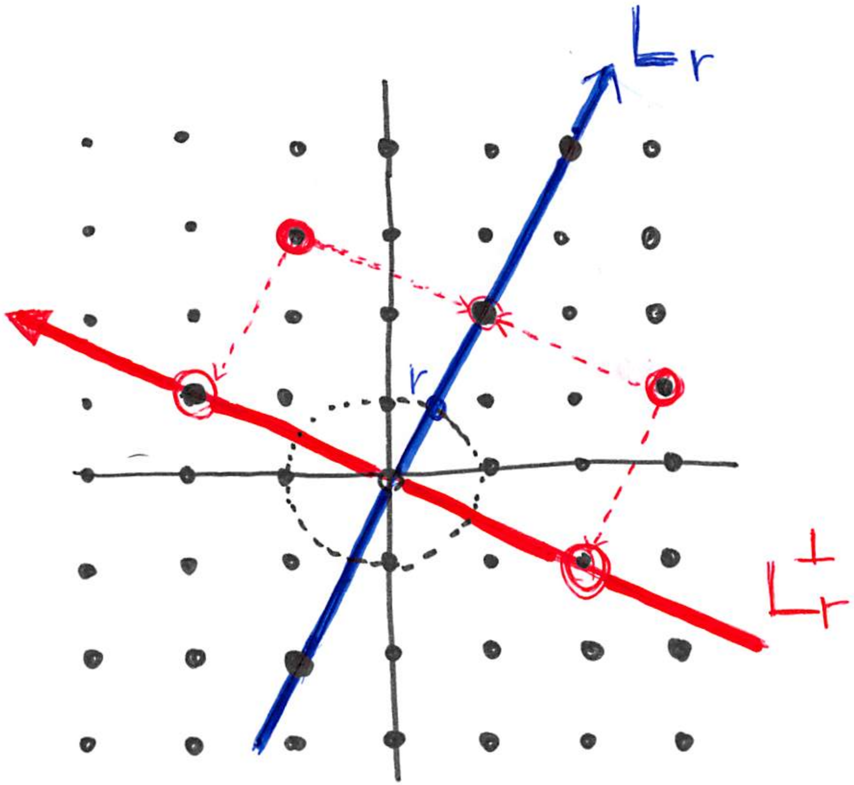
$$a <_r b \overset{\text{def}}{\Longleftrightarrow} \pi_r(a) < \pi_r(b)$$

- If r is an irrational point

  $<_r$ defines BO of $\mathbb{Z}^2$ ($\pi_r|_{\mathbb{Z}^2}$ is injective.)

- If r is a rational point

  $<_r$ is not a total ordering.



$(4,1) <_r (-1,4)$

"Oriented"
$L_r^\perp$: orthgonal line (we have two choices!)

Then modify the definition of $<_r$ as

$$a <_r b$$

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \pi_r(a) < \pi_r(b) \\ \quad \text{or} \\ \pi_r(a) = \pi_r(b) \\ \quad \pi_r^\perp(a) < \pi_r^\perp(b) \end{cases}$$

Conclusion

$\mathbb{Z}^2$ has uncountably many bi-orderings

parametrized by $\begin{cases} r \in S^1 & \text{, irrational} \\ r, \{\pm\} \in S^1 \times \{\pm\} & \text{rational} \end{cases}$

(↳ by introducing a notion of "space of orderings"
we can say that the moduli = space of orderings
is homeomorphic to the Cantor set.)

[Fact] (Linnell '11)

The cardinal of the set of left-orderings is either finite or uncountable.

ordering of F_n

$F_n = \langle x_1, \dots, x_n \rangle$ free group of rank n.

Magnus expansion

Non-commutative formal power series i.e. $X_i X_j \neq X_j X_i$

$$\Theta : F_n \hookrightarrow \mathbb{Z}\langle\langle X_1, \dots, X_n \rangle\rangle$$

$$x_i \longmapsto 1 + X_i$$

$$x_i^{-1} \longmapsto 1 - X_i + X_i^2 - X_i^3 + \cdots \quad \left(= \frac{1}{1+X_i}\right)$$

Write

$$\Theta(g) = 1 + \sum_{i_1, \dots, i_k} \underbrace{C_{i_1, \dots, i_k}(g)}_{\text{coefficient of the Magnus expansion}} X_{i_1} X_{i_2} \cdots X_{i_k}$$

define the Magnus ordering $<_M$ by

$$f <_M g \overset{\text{def}}{\Longleftrightarrow} \{C_{i_1,\cdots,i_k}(f)\} < \{C_{i_1,\cdots,i_k}(g)\}$$

with respect to the lexicographical ordering + degree

<u>example.</u>

$f = x_1 x_2 \quad , \quad g = x_2^{-1} x_1$

$$\Theta(f) = 1 + X_1 + \boxed{X_2} + X_1 X_2$$

$$\Theta(g) = 1 + X_1 + \boxed{(-1) X_2} + (-1) X_2 X_1 + X_2 X_2 + \cdots$$

same. same.

$\Rightarrow \quad f <_M g$

§I.3 Left-ordering and Dynamics

Theorem

G : countable group

G is LO $\iff$ $G \subset \mathrm{Homeo}^+(\mathbb{R})$

i.e. G faithfully acts on $\mathbb{R}$ as orientation-preserving homeomorphisms.

i.e. Construction of ordering of G

$\Updownarrow$ equivalent

Construction of action on $\mathbb{R}$

[Proof]

($\Leftarrow$: Faithful action to ordering)

Take an enumeration $\{q_0, \cdots, q_n, \cdots\}$ of $\mathbb{Q} \subset \mathbb{R}$.

Given $G \curvearrowright \mathbb{R}$, define

$$g < h \iff \begin{cases} g(q_0) = h(q_0), \cdots, g(q_{n-1}) = h(q_{n-1}) \\ \text{and } g(q_n) < h(q_n) \end{cases} \quad \text{for some } n$$

i.e. compare the images of $\{g(q_i)\}$ and $\{h(q_i)\}$

- orientation preserving $\Rightarrow$ $<$ is a left-invariant relation
- $\mathbb{Q} \subset \mathbb{R}$ is dense
  action is homeomorphism $\Rightarrow$ $<$ is a total ordering.

($\Rightarrow$ ordering to faithful action)

Take an enumeration $\{g_0, \cdots\}$ of G

We construct order-preserving injection (as a Set, not Group)

$$i : (G, <_G) \hookrightarrow (\mathbb{R}, <)$$

By (*1) $i(g_0) = 0$

(*2) i is defined on $\{g_0, \cdots, g_{n-1}\}$

$$\Downarrow$$

$$i(g_n) = \begin{cases} i(g_{min}) - 1 & \text{if } g_{min} = \min\{g_0, \cdots, g_{n-1}\} >_G g_n \\ i(g_{Max}) + 1 & \text{if } g_{Max} = \max\{g_0, \cdots, g_{n-1}\} <_G g_n \\ \frac{1}{2}\left(i(g_m) + i(g_M)\right) & \text{if } g_m < g_n < g_M \end{cases}$$

(and no g_i satisfies $g_m < g_i < g_M$)

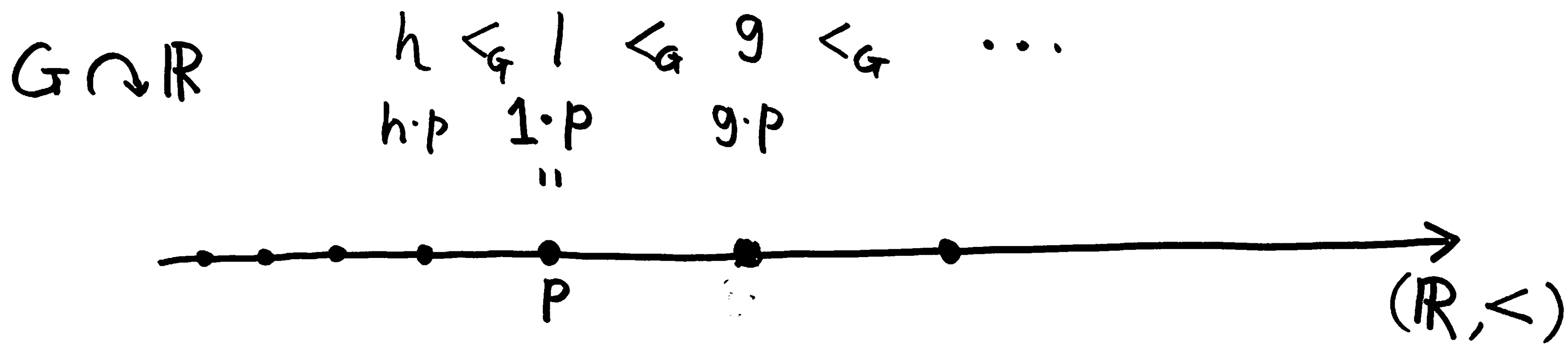
Then $G \curvearrowright G$ extends

$$\begin{array}{ccc} G \curvearrowright & G & \\ & \cap & \\ G \curvearrowright & \mathbb{R} & \end{array}$$

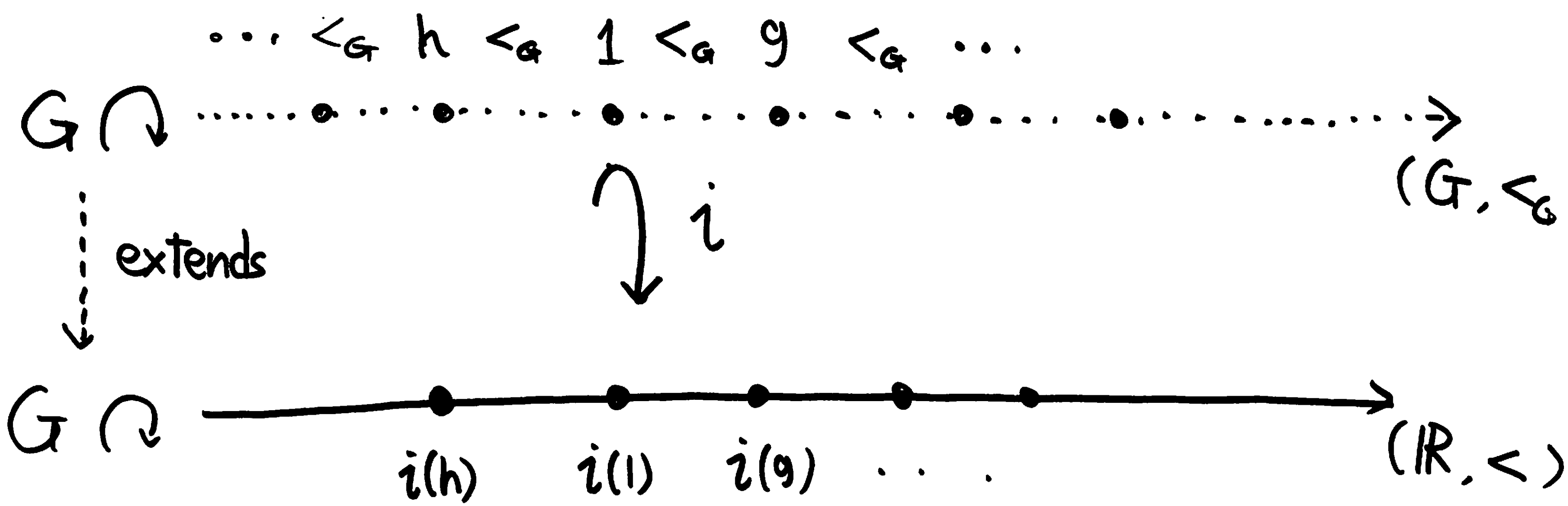
as orientation-preserving homeo.

(Schematic picture)

(⇐)



(⇒)



There are various relations between algebraic property of orderings and dynamical property of action.

example

Theorem (Hölder's theorem)

G admits an Archimedian ordering

$(1 <_G g, g' \Rightarrow {}^\exists N \in \mathbb{Z} \quad g' < g^N)$

$\Leftrightarrow$ G admits a faithful, free action on $\mathbb{R}$

$\Leftrightarrow$ $G \subset \mathbb{R}$ as group

Idea-Sketch of Proof

We construct order-preserving injection $i : G \hookrightarrow \mathbb{R}$
so that it is a homeomorphism.

take $1 \neq g \in G$. we use $\mathbb{Z} = \langle g \rangle \overset{i}{\subset} \mathbb{R}$ as a "measure"

- define : $i(g^n) = n$.
- observe : $g^k = h^\ell \Rightarrow i(h)$ should be defined $i(h) = \dfrac{k}{\ell}$
- define $\rho(h) \in \mathbb{Z}$ so that $g^{\rho(h)} \leq_G h <_G g^{\rho(h)+1}$

  $\rho(h)$ = "approximation" of i

$$i(h) = \lim_{N \to \infty} \frac{\rho(h^n)}{N}$$ is well-defined homomorphism.

§ I-4 Example of ordering (2)

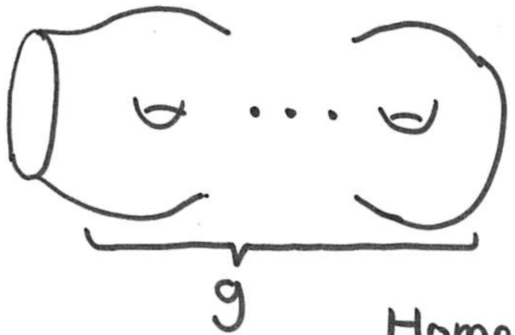
$$\begin{tikzcd}
\widetilde{SL}(2;\mathbb{R}) \arrow[d, dashed, "\text{universal cover}"'] & \subset & \widetilde{\mathrm{Homeo}^+(S^1)} \arrow[d, dashed, "\text{lift to } \widetilde{S^1} = \mathbb{R}"] & \subseteq & \mathrm{Homeo}^+(\mathbb{R}) \\
SL(2;\mathbb{R}) & \subset & \mathrm{Homeo}^+(S^1) & &
\end{tikzcd}$$

($SL(2;\mathbb{R}) \curvearrowright \{\text{line in } \mathbb{R}^2 \text{ passing } 0\} \cong S^1$)

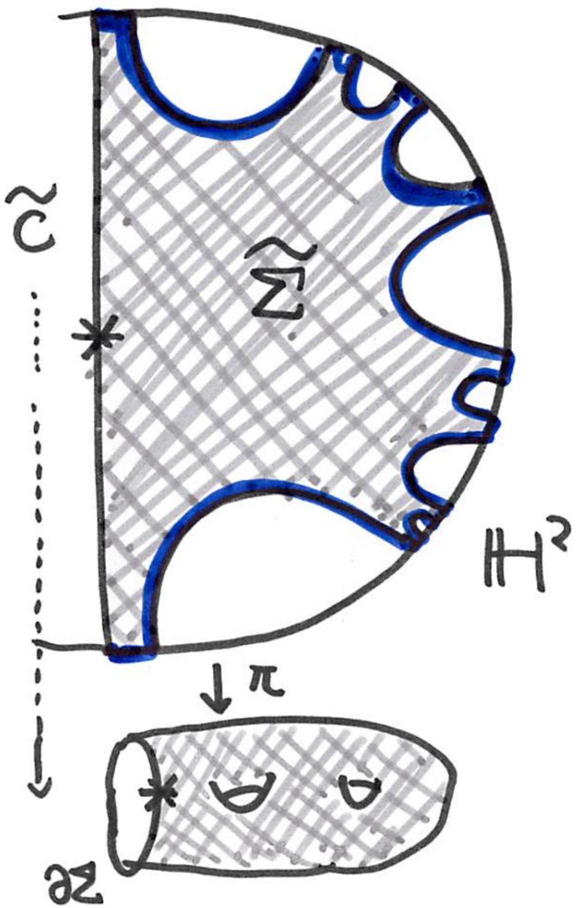
$\hookrightarrow$ Lie group $\widetilde{SL}(2;\mathbb{R})$ is LO.

(Note: Most algebraic group (lattice of) $\not\subset \mathrm{Homeo}^+(S^1)$)

$\Sigma = \Sigma_{g,1} =$



$$MCG(\Sigma_{g,1}) = \left\{ f : \Sigma_{g,1} \overset{\text{Homeo}}{\xrightarrow{\cong}} \Sigma_{g,1} \;\middle|\; f|_{\partial\Sigma} = id \right\} \Big/ \text{isotopy}$$



viewing Σ as hyperbolic surface.

$\tilde{\Sigma} \subset \mathbb{H}^2$: isometric

$\downarrow$

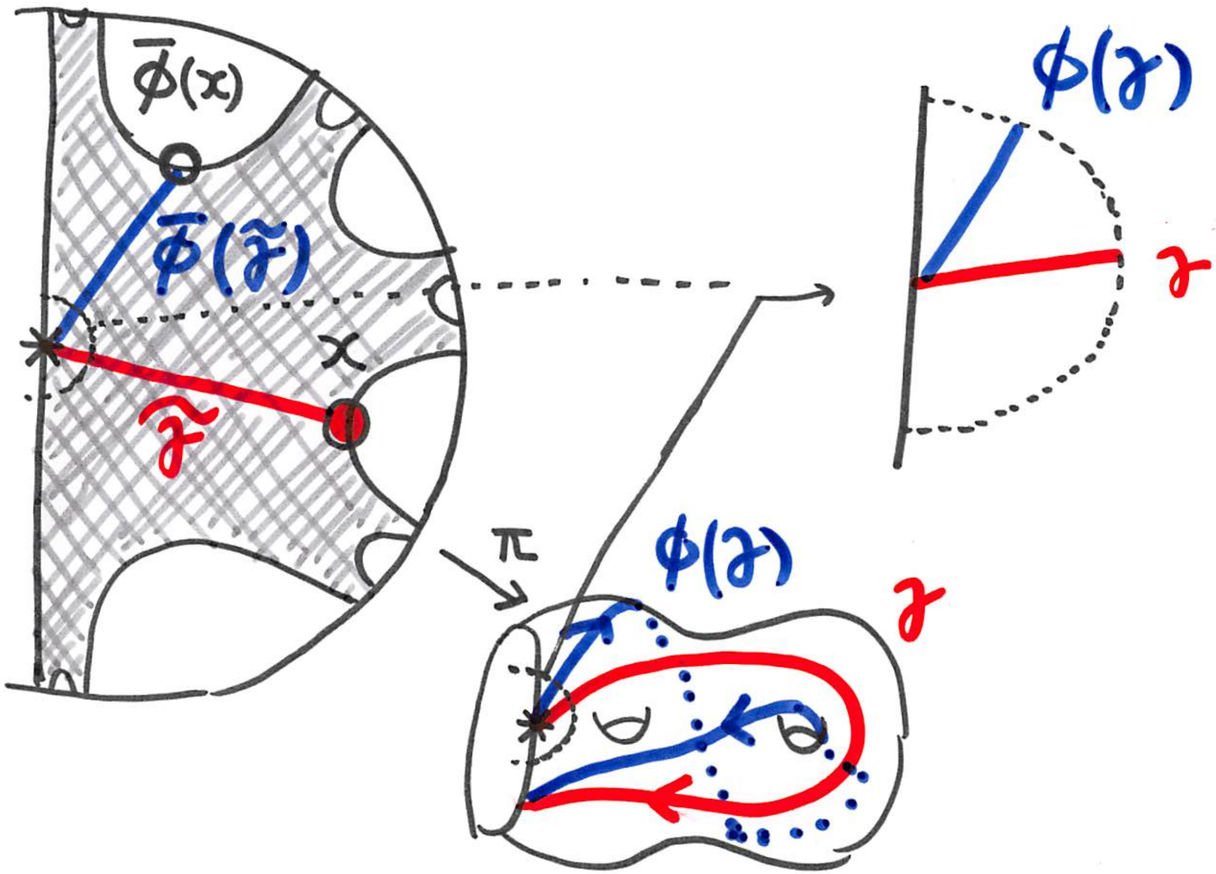
$\overline{\tilde{\Sigma}} \subset \mathbb{H}^2 \cup S^1_\infty$ Compactification

$\phi : \Sigma \to \Sigma$ lifts and extends $\bar{\phi} : \overline{\tilde{\Sigma}} \to \overline{\tilde{\Sigma}}$.

Key Fact. $\overline{\phi}\big|_{\partial\overline{\widetilde{\Sigma}}-\widetilde{c}} = \overline{\psi}\big|_{\partial\overline{\widetilde{\Sigma}}-\widetilde{c}}$ if $\phi \underset{\text{homotopic}}{\sim} \psi$

$\hookrightarrow$ $MCG(\Sigma) \hookrightarrow \mathrm{Homeo}^{+}(\partial\overline{\widetilde{\Sigma}} - \widetilde{c}) = \mathrm{Homeo}^{+}(\mathbb{R})$

$\hookrightarrow$ $MCG(\Sigma)$ is LO



MCG action on $\partial\overline{\widetilde{\Sigma}}$

$\|$

action on germs of geodesics.

Def

A left-ordering from $MCG \curvearrowright \partial\widetilde{\overline{\Sigma}} - \widetilde{C} = \mathbb{R}$ is called (Nielsen-) Thurston type ordering.

example (Dehornoy ordering)

$$\alpha <_D \beta \overset{\text{def}}{\Longleftrightarrow} \alpha^{-1}\beta \text{ is written as a word over } \{\sigma_i, \sigma_{i+1}^{\pm 1}, \cdots, \sigma_{n-1}^{\pm 1}\}$$
with at least one σ_i, for some i

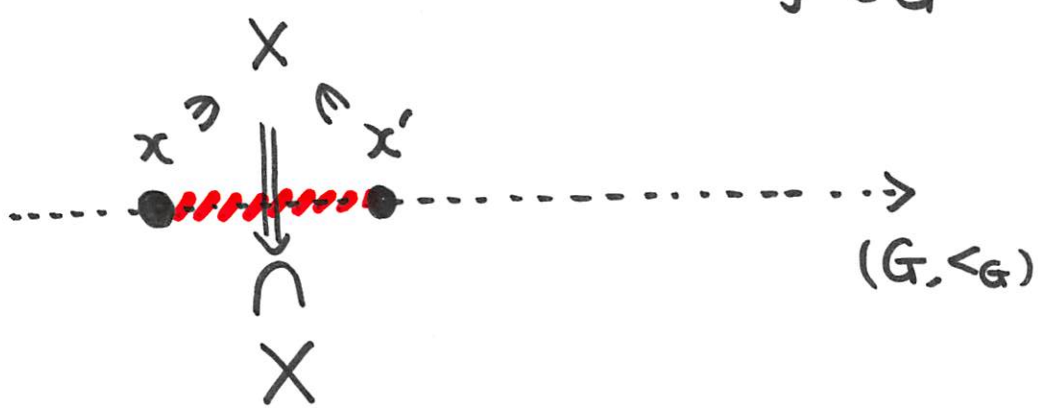
- (Dehornoy '94) $<_D$ is a left-ordering (highly non-trivial!)
- (Fenn-Greene-Rourke-Rolfsen-Wiest '99, Short-Wiest '00) $<_D$ is a special one of Thurston-type ordering.

§ I-5 Algebraic method to construct orderings

① Quotient

$X \subset G$ is convex with respect to an ordering $<_G$

$\overset{\text{def}}{\iff}$ $\quad x \le_G g \le_G x' \quad \begin{matrix} x, x' \in X \\ g \in G \end{matrix} \quad \Rightarrow \quad g \in X$



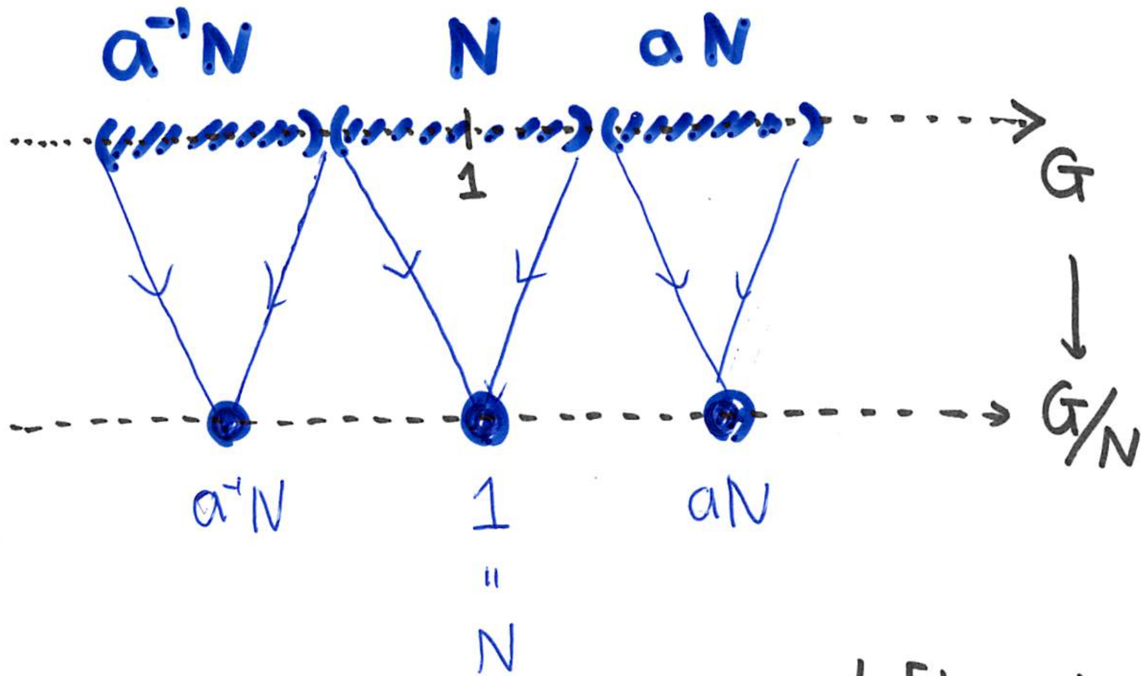
For a convex normal subgroup $N \subset G$

G/N has a left ordering $<_{G/N}$

def

Quotient ordering $<_{G/N}$

$$aN <_{G/N} bN \underset{\text{def}}{\iff} a <_G b \qquad \left(\begin{array}{l} a, b \in G : \\ \quad \text{representative of coset} \end{array}\right)$$



$<_{G/N}$ is well-defined and left-ordering if so is $<_G$.
(bi-)

② Extension

$$1 \to K \hookrightarrow G \xrightarrow{\phi} H \to 1 \quad \text{group extension}$$
$$K = \operatorname{Ker}\phi$$

$\boxed{2\text{-}1}$ Construction of left ordering

$<_K$: left-ordering of K

$<_H$: left-ordering of H

Define
$$g <_G g' \iff \begin{array}{c} \phi(g) <_H \phi(g') \\ \text{or} \\ \phi(g) = \phi(g') \\ 1 <_K g^{-1}g' \end{array}$$

Then $<_G$ is a left-ordering on G.

$\boxed{2\text{-}2}$ Construction of bi-ordering

$<_K$: l-ordering of K

$<_H$: bi-ordering of H

Assume that $<_K$ is G-conjugacy invariant :

$$k <_K k' \Rightarrow gkg^{-1} <_K gk'g^{-1} \quad \text{for all } g \in G,\ k, k' \in K.$$

Then $<_G$ is a bi-ordering on G

[right-invariance]

$$g <_G g'$$

$\phi(g) < \phi(g')$ case $\quad \longrightarrow \quad \phi(gg'') <_H \phi(g'g'')$

$\phi(g) = \phi(g')$ case $\quad \longrightarrow \quad$

$$(gg'')^{-1}(g'g'') = g''^{-1}(g^{-1}g')g'' >_K 1$$

§ I-6 Burus-Hale Theorem

Theorem

G : group

if for every finitely generated subgroup $H \leq G$ admits a surjection $\phi_H : H \twoheadrightarrow L_H$ to a LO group G, G is LO.

<u>example - Corollary</u>

G is locally indicable $\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}H \subset G$ fin. gen. subgroup

$H \twoheadrightarrow \mathbb{Z}$

Locally indicable $\Longrightarrow$ LO

(idea - sketch of proof)

Assume G is not LO.

Take "minimum" non-LO finitely generated subgroup $H \leq G$.

By hypothesis, $\exists \phi_H : H \twoheadrightarrow L_H$ (L_H : LO)

$$1 \longrightarrow K \hookrightarrow H \overset{\phi_H}{\twoheadrightarrow} L_H \longrightarrow 1$$

$$K = \operatorname{Ker} \phi_H$$

"minimality" assumption of H $\Rightarrow$ K is LO

group extension construction $\Rightarrow$ H is LO

$\Downarrow$

Contradiction

Corollary

$$G * H \text{ is LO} \iff \text{Both } G \text{ and } H \text{ are LO}$$

[Remark]

Amalgamated product case

$$G *_A H \text{ is LO} \iff \begin{array}{l} \text{Both } G \text{ and } H \text{ are LO} \\ \quad + \\ \text{additional (complicated)} \\ \text{compatibility conditions} \end{array}$$