Invariant ordering of groups and low-dimensional topology II

: Orderability of 3-manifold groups

§ II-1. Motivating Questions

Geometrization Theorem (Perelman)

$\downarrow$

The topology of 3-manifold M is (in most cases) determined by $\pi_1(M)$

(Geometric) Group theory (of $\pi_1(M)$)

Active recently $\downarrow$ $\uparrow$ (Classical)

(Algebraic/Geometric) Topology (of M)

<u>example</u> ($\Leftarrow$ is easy, but $\Rightarrow$ is non-trivial)

- $\pi_1(M) = \pi_1(N) * \pi_1(N') \iff M = N \# N'$ (Kneser)
- $^\exists \mathbb{Z}_p \triangleleft \pi_1(M) \iff M$ is Seifert fibered
  (Normal cyclic subgroup)
  (Seifert fiberd space conjecture)
- $[\pi_1(M), \pi_1(M)]$ is finitely generated $\iff$ M is a surface bundle over S^1
  (Case $H_1(M) \cong \mathbb{Z}$)
  (Stallings)

and more ...

Motivating question

What property of M is reflected

by orderability/ordering of $\pi_1(M)$?

i.e.

$$\begin{tikzcd} \boxed{\text{ordering of } \pi_1(M)} \arrow[r, bend left, dashed, "?"] & \boxed{\text{Property of } M} \arrow[l, bend left, dashed, "?"] \end{tikzcd}$$

Group Theory Side — Topology side

§II-2 Boyer-Rolfsen-Wiest theorem.

Theorem (Boyer-Rolfsen-Wiest '05)

M : compact, irreducible 3-manifold

$\pi_1(M)$ is LO $\iff$ $^\exists$surjection $\phi : \pi_1(M) \twoheadrightarrow G$ for some LO group G.

This is a refinement of

Theorem (Burns-Hale)

A Group G is LO $\iff$ For every finitely generated subgroup H $^\exists$surjection $\phi_H : H \twoheadrightarrow L_H$ for some LO group L_H.

c.f.

Theorem (Howie '82)

M : compact, irreducible 3-manifold

$\pi_1(M)$ is locally indicable $\Longleftrightarrow$ $^\exists$surjection $\pi_1(M) \twoheadrightarrow \mathbb{Z}$

(i.e. $^\forall H \subset \pi_1(M)$, finitely generated
$^\exists \phi_H : H \twoheadrightarrow \mathbb{Z}$)

In dynamics language, B-R-W Theorem says,

$$\pi_1(M) \curvearrowright \mathbb{R} \text{ faithfully} \Longleftrightarrow \pi_1(M) \curvearrowright \mathbb{R} \text{ Non-trivially}$$

[Proof of B-R-W Theorem]

$H \subset \pi_1(M)$ finitely generated subgroup

$\boxed{\text{Case I}}$ $[\pi_1(M) : H] < \infty$

$[G : \phi(H)] < \infty \Rightarrow \phi_H : H \twoheadrightarrow \phi(H) \subset G$

LO. non-trivial

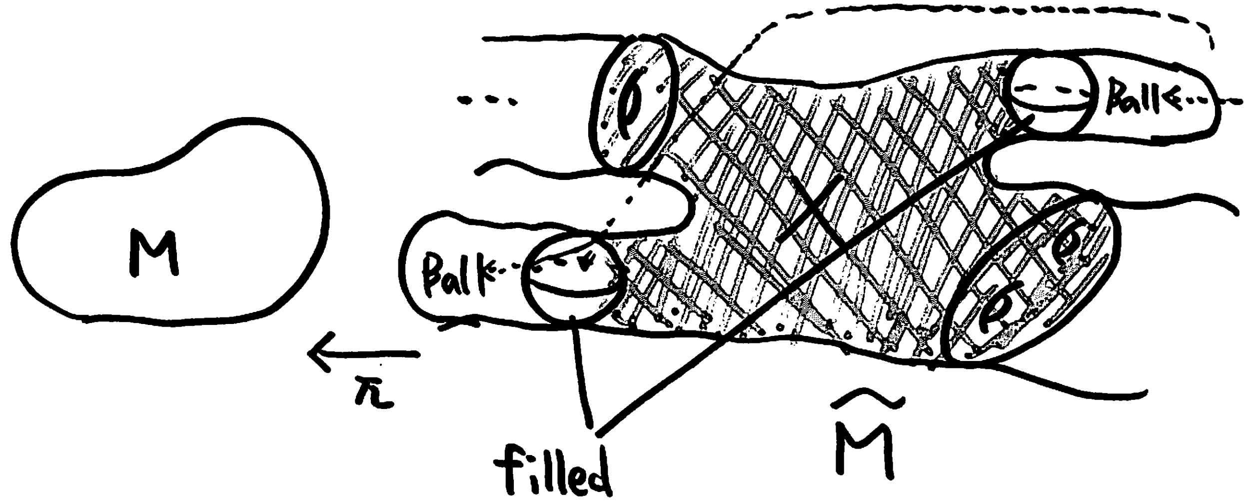
$\boxed{\text{Case II}}$ $[\pi_1(M) : H] = \infty$

$\pi : \widetilde{M} \longrightarrow M$ covering associated to H

$[\pi_1(M) : H] = \infty \Rightarrow \widetilde{M}$ is not compact

<u>Scott compact core theorem</u>

$\Rightarrow {}^{\exists} X \subset \widetilde{M}$ compact submanifold $\pi_1(X) \cong \pi_1(\widetilde{M}) = H$



$\tilde{M}$ is irreducible

$\Downarrow$

X has non-empty boundary.

M is irreducible $\rightsquigarrow$ $\tilde{M}$ is irreducible

so sphere boundary can be filled by balls

(without changing $\pi_1(X)$.)

$\longrightarrow$ X is compact 3-manifold with non-sphere bounday

$\longrightarrow$ $b_1(X) \geqq 1$ $\longrightarrow$ ${}^{\exists}\, \pi_1(X) \twoheadrightarrow \mathbb{Z}$

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Corollary

If M is not $\mathbb{Q}HS$, $\pi_1(M)$ is LO.

Remark.

- B-R-W Theorem (Burns-Hale theorem) only tells abstract existence of left-ordering.
- explicit (more concretely defined, easy to calculate $\cdots$) examples of left-orderings on 3-manifold group are less known. (except several simple cases eg. fibered case)

§II-3 Orderabilty Criterion

By B-R-W Theorem

$\pi_1(M) = G$ is LO $\Leftrightarrow$ ${}^{\exists}\phi : G \longrightarrow \mathrm{Homeo}^+(\mathbb{R})$ Non-Trivial

$\mathrm{Homeo}^+(\mathbb{R})$ is very big and hard to treat algebraically.

$\cup$

$\widetilde{SL}(2;\mathbb{R})$: much familar, easier to treat algebraically.

$\downarrow$ univ. cover

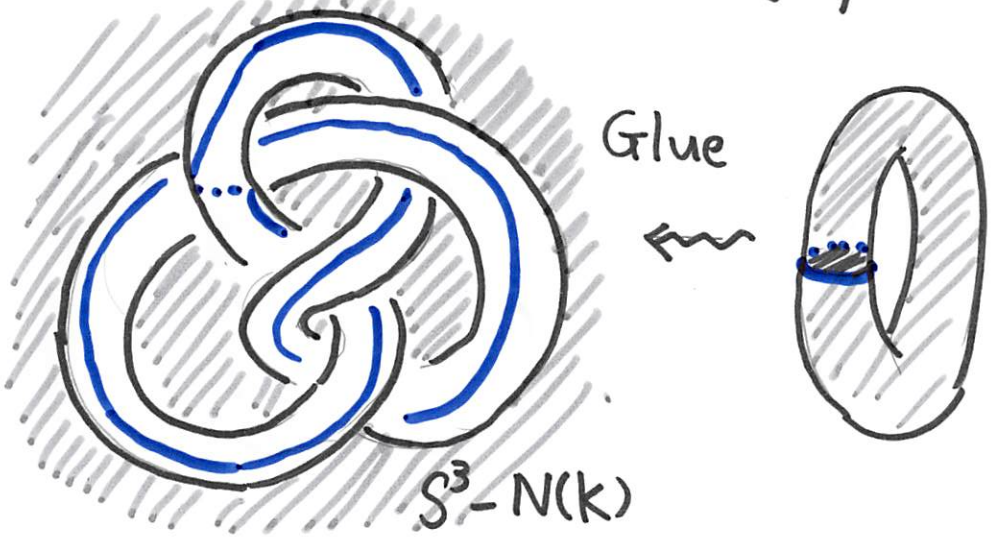
$SL(2;\mathbb{R}) \subset SL(2;\mathbb{C})$: well-studied in 3-dim topology

(Culler-Shalen theory, hyperbolic geometry)

Refined Question.

$\exists$ $\pi_1(M) \longrightarrow \widetilde{SL}(2;\mathbb{R})$ non-trivial ?

(Review of Dehn surgery)



$K \subset S^3$: knot

ℓ: longitude of K

m: meridian of K

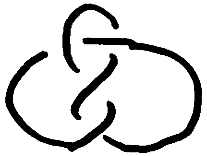
For $r = p/q \in \mathbb{Q} \cup \{\infty = \frac{1}{0}\}$

Slope r Dehn surgery along K

$$M_K(r) = (S^3 - N(K)) \cup (\text{Solid torus})$$

glued so that curve $\ell^q m^p \subset \partial N(K)$ bounds a disc.

<u>example</u> (Boyer-Gordon-Watson '13)

$K =$  (Figure eight knot) $\quad G = \pi_1(S^3 \setminus K)$

$\exists$ 1-parameter family of representation

$$\rho_s : G \longrightarrow SL(2;\mathbb{R}) \qquad \text{defined} \quad s \in [\tfrac{1+\sqrt{5}}{2}, \infty]$$

Consider Dehn surgery $M_K(r)$.

$$\pi_1(M_K(r)) \cong G / m^p \ell^q = 1$$

so

$$\begin{tikzcd} G \arrow[r, "pr"] \arrow[d, "\rho_s"'] \arrow[dr, phantom, "\circlearrowright"] & \pi_1(M_K(r)) \arrow[dl, dashed, "\exists \rho_s'"] \\ SL(2;\mathbb{R}) & \end{tikzcd}$$

$\iff \rho_s(m^p \ell^q) = 1$

: Algebraic equations of s.

By computation $\rho'_s : \pi_1(M_K(r)) \longrightarrow SL(2;\mathbb{R})$ exists

$\Longleftrightarrow \quad r \in [0,4)$

$$\begin{tikzcd}
 & \widetilde{SL}(2;\mathbb{R}) \arrow[d, "\substack{\text{univ.}\\ \text{cover}}"] & \subset \mathrm{Homeo}^+(\mathbb{R}) \\
\pi_1(M_K(r)) \arrow[ur, dashed, "\substack{\exists\\ \text{lift ?}}"] \arrow[r, "\rho'_s"'] & SL(2;\mathbb{R}) & \subset \mathrm{Homeo}^+(S^1)
\end{tikzcd}$$

By computation ρ'_s lifts $\Rightarrow \ {}^{\exists}\widetilde{\rho'_s} : \pi_1(M_K(r)) \longrightarrow \underbrace{\widetilde{SL}(2;\mathbb{R})}_{\text{LO group}}$

$\Rightarrow$ $\boxed{\text{Conclusion}}$

$\pi_1(M_K(r))$ is LO if $r \in [0,4)$

[Summary for $SL(2;\mathbb{R})$-representation method]

Step 1 Find / parametrize representation

$$\rho_s : \pi_1(S^3 - K) \longrightarrow SL(2;\mathbb{R})$$

(This is done by solving algebraic equations)

Step 2. For $p/q = r \in \mathbb{Q}$, find and check

$^\exists s$: parameter $\rho_s(m^p \ell^q) = 1$

(This also done by solving algebraic equations)

Step 3. Check $\rho_s' : \pi_1(M_K(r)) \longrightarrow SL(2;\mathbb{R})$

admits a lift $\widetilde{\rho_s'} : \pi_1(M_K(r)) \longrightarrow \widetilde{SL}(2;\mathbb{R})$

(This require some additional argument/computations)

Hakamata-Teragaito, Tran, (and more) . . .

§ II. 4 Construction of Action on $\mathbb{R}$
— Relation to Foliation

Toy idea — Why foliation?

We want : non-trivial action $\pi_1(M) \curvearrowright \mathbb{R}$

We have : 〃 $\pi_1(M) \curvearrowright \widetilde{M}$ universal cover

In many cases, $\widetilde{M} \cong \mathbb{R}^3$

$\exists$ $\pi_1(M)$-equivariant splitting $\widetilde{M} \cong \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{R}^2$?

Decompose M ($=3$ dim) as (1-dim) $\times$ (2-dim)

(Codim 1) Foliation "should" appear.

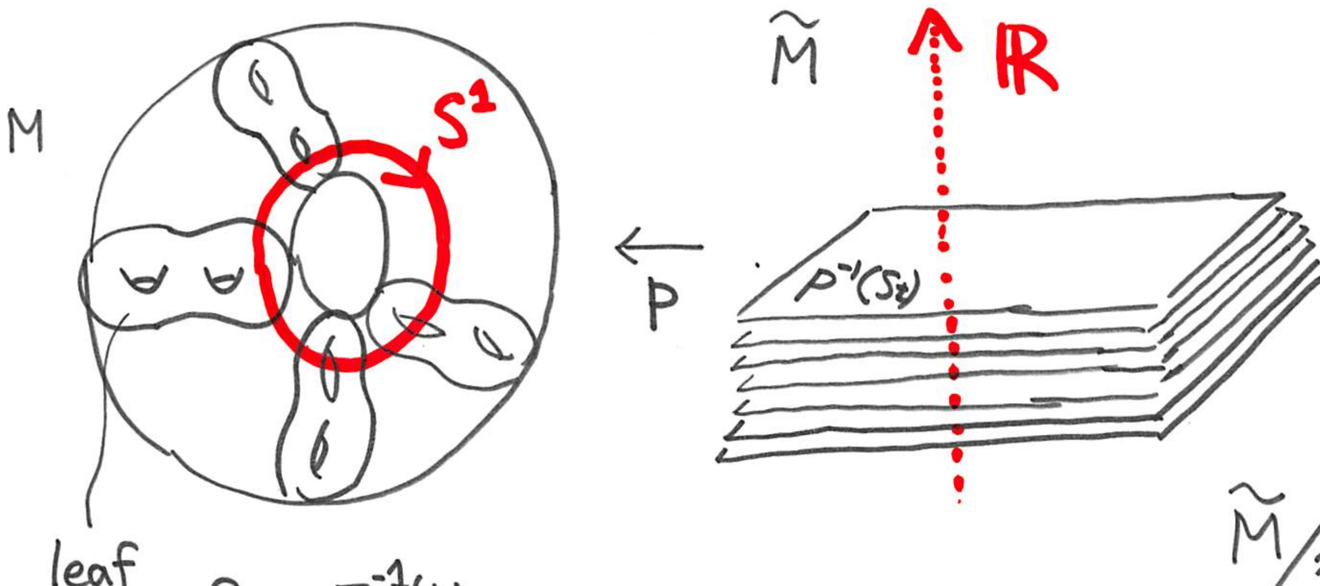
example.

Foliation $\mathcal{F}$ is $\mathbb{R}$-covered

$\overset{\text{def}}{\Longleftrightarrow}$ For a lift $\widetilde{\mathcal{F}}$ (Foliation on the universal covering $\widetilde{M}$)

leaf space $\widetilde{M}/\widetilde{\mathcal{F}} \underset{\text{Homeo}}{\cong} \mathbb{R}$

ex: M is a surface bundle over S^1 : $\pi : M \longrightarrow S^1$, leaf = Fiber



$S_t = \pi^{-1}(t)$

$$\widetilde{\mathcal{F}} = \underset{\left(\overset{\shortparallel}{\widetilde{S_t}}\right)}{\mathbb{R}^2} \times \underset{\left(\overset{\shortparallel}{\widetilde{S^1}}\right)}{\mathbb{R}}$$

$$\widetilde{M}/\widetilde{\mathcal{F}} = \mathbb{R} .$$

$\pi_1(M) \curvearrowright \widetilde{M}$ induces $\pi_1(M) \curvearrowright \widetilde{M}/\widetilde{\mathcal{F}}$

so by B-R-W theorem

Theorem

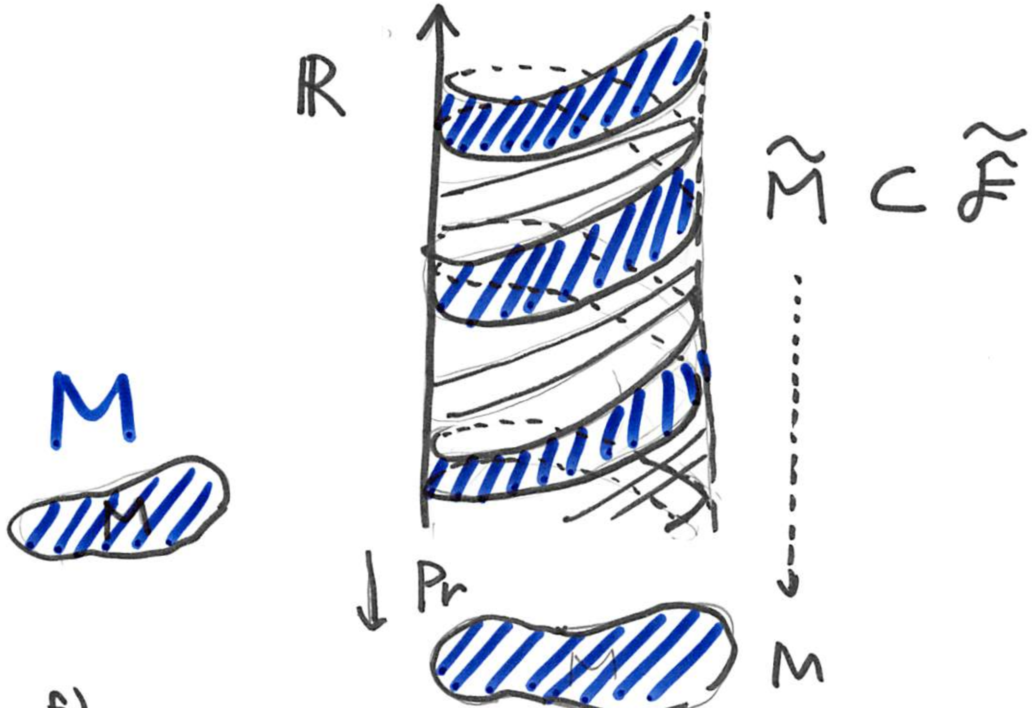
M admits a co-oriented $\mathbb{R}$-covered foliation
$\Rightarrow$ $\pi_1(M)$ is LO

More generally

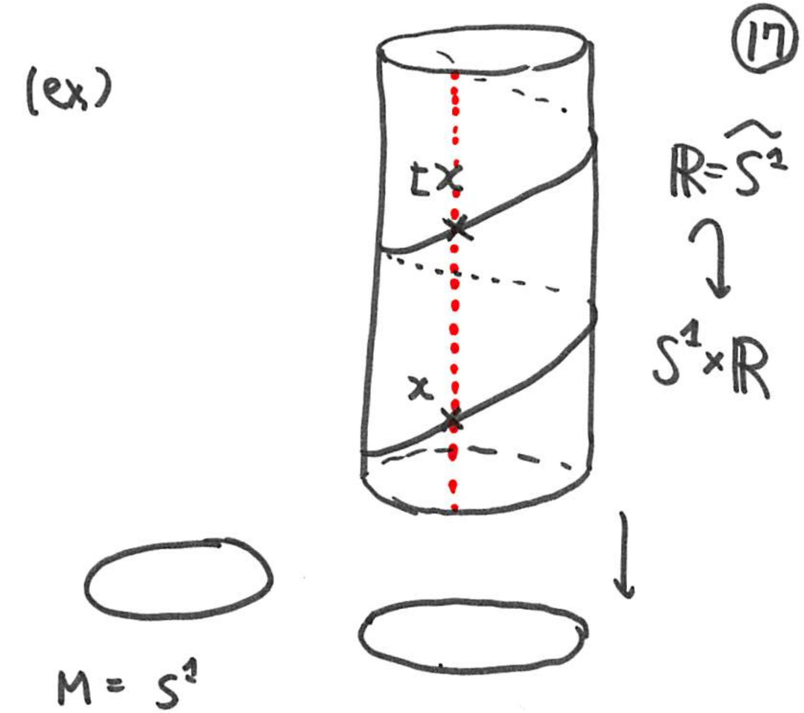
Theorem (Farrell '76 (c.f. Anel-Clay '12))

M : closed (not necessarily 3-dim) manifold

$\pi_1(M)$ is LO $\Leftrightarrow$ $\exists$ $\mathbb{R}$-covered foliation $\mathcal{F}$ on $M \times \mathbb{R}$ s.t.
(i) $pr : L \longrightarrow M$ is covering for each leaf
(ii) $\exists$ leaf of $\mathcal{F}$ $\cong \widetilde{M}$ (so $\widetilde{M} \subset M \times \mathbb{R}$)



(ex)



(Sketch of)
[Proof]

$\Rightarrow$) $\pi_1(M)$ LO $\Rightarrow$ $\pi_1(M) \curvearrowright \mathbb{R}$

combine $\pi_1(M) \curvearrowright \mathbb{R}$ and $\pi_1(M) \curvearrowright \widetilde{M}$ altogether :

$$\widetilde{M} \times_{\pi_1(M)} \mathbb{R} = \widetilde{M} \times \mathbb{R} \big/ (x,t) \sim (g^{-1}x, gt) \cong M \times \mathbb{R}$$

orbit of $\pi_1(M)$ action = leaf .

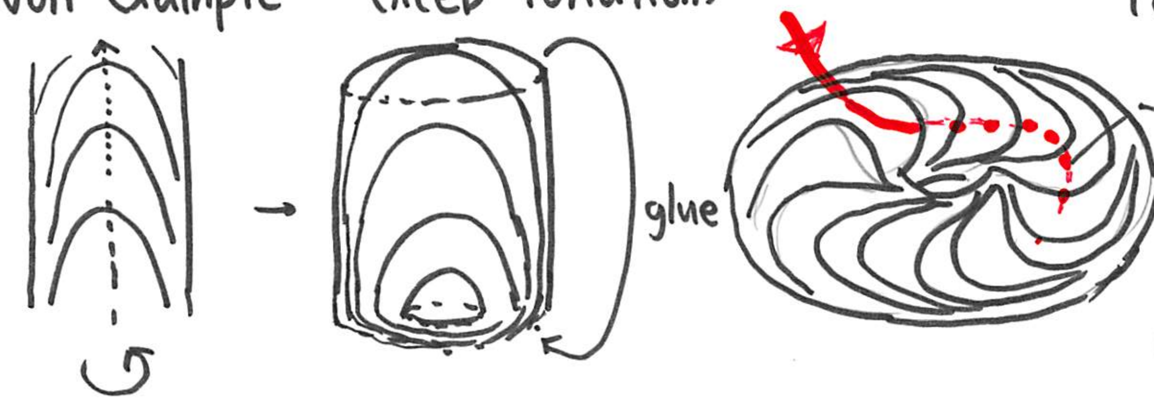
Def

Codim 1 foliation $\mathcal{F}$ of M is taut

$\Leftrightarrow$ $^\exists$ circle $\gamma \subset M$ which intersects every leaf

Non-example (Reeb foliation)

Foliation on $D^2 \times S^1$



transverse arc cannot go outside.

(Reeb component)

Taut foliation is a useful structure in studying 3-manifolds.

- Thurston norm
- Contact structure
- Dehn surgery

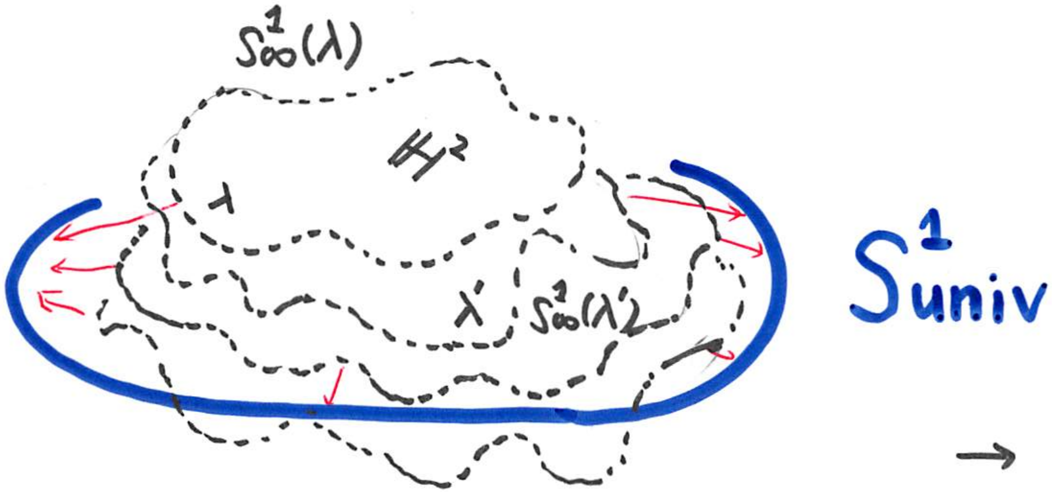
Theorem (Calegari-Dunfield '03)

(Atoroidal) $\mathbb{Z}HS$ M admits a co-oriented taut foliation

$\Rightarrow$ $\pi_1(M)$ is LO

[Sketch of proof]

- equip a metric on $\widetilde{M}$ so that leaf of $\widetilde{\mathcal{F}} \underset{\text{isometric}}{\cong} \mathbb{H}^2$



Taut foliation

$\exists\, S^1_{univ}$ (universal circle)

which "contains" all ideal boundary $S^1_\infty(\lambda)$ for all $\lambda \subset \widetilde{\mathcal{F}}$

$\rightarrow \pi_1(M) \curvearrowright S^1_{univ}$

M is $\mathbb{Z}HS$ $\rightarrow$ $\pi_1(M) \curvearrowright \widetilde{S^1_{univ}} \cong \mathbb{R}$

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- orderability (and their generalization) serves as an obstruction to "good" codim 1 decomposition (foliation, Lamination)

<u>example.</u>

(Roberts, Shareshian, Stein '03)
example of hyperbolic 3-manifolds
without Reebless foliation.
(taut)

(Fenley '07)
example of hyperbolic 3-manifolds
without essential Lamination.

§II-5 Non-orderability criterion

to prove $G = \pi_1(M)$ is not LO,

we try to find "forbidden relations" in G.

$\Rightarrow$ $\{g_1, \dots, g_n\} \subset G$ $\rightarrow$ 2^n-possibility of orderings

$$1 <_G g_i \quad \text{or} \quad 1 >_G g_i$$

For each possibility of orderings we have

$$1 <_G g_i^{\varepsilon_i} \qquad \varepsilon_i \in \{\pm 1\}$$

$$\rightarrow \quad \underbrace{\text{word over } \{g_1^{\varepsilon_1}, \dots, g_n^{\varepsilon_n}\}}_{\neq 1} >_G 1$$

so if we find a word w over $\{g_1^{\varepsilon_1}, \dots, g_n^{\varepsilon_n}\}$ with $w = 1$

this possibility cannot occur.

ex.) $n=1$

$\{g\} \subset G$

$1 <_G g \Rightarrow$ word over $\{g\} >_G 1 \Rightarrow g^n \neq 1 \quad \forall n$

$1 >_G g \Rightarrow$ word over $\{g^{-1}\} >_G 1 \Rightarrow g^{-n} \neq 1 \quad \forall n$

$\Rightarrow$ Torsion-free

ex.) $n=2$

$\{a, b\} \subset G$

if a, b satisfy relations, like

$$\begin{cases} a b^2 a b a^3 = 1 \\ b^{-1} a^5 b^{-2} a b^{-1} = 1 \\ b a^{-1} b a^{-2} b^4 a^{-1} = 1 \end{cases}$$

then G cannot be LO

Systematic example : non-LO property of double branched cover

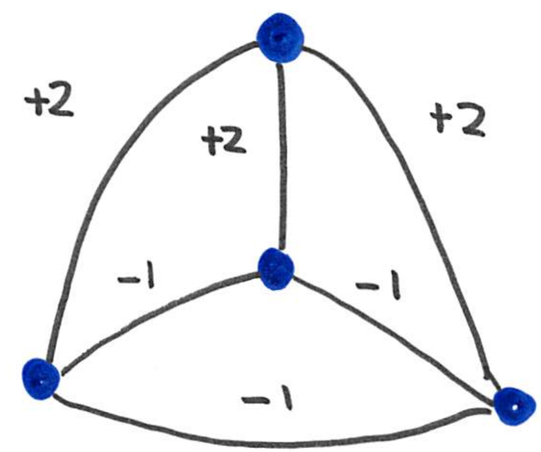
knot → Checker board surface = disc + band

express as a graph

+n =

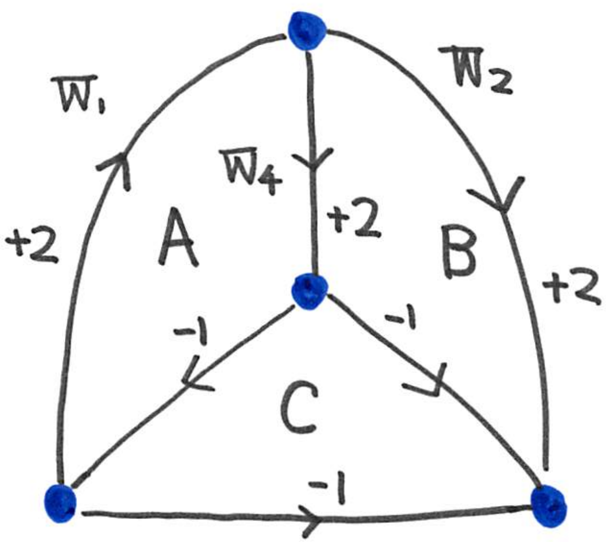
n twist

−n =



From the graph we get a presentation of

$$\pi_1(\Sigma_2(K)) = \pi_1(\text{Double branched covering of } K) \quad \text{(Brunner '97)}$$

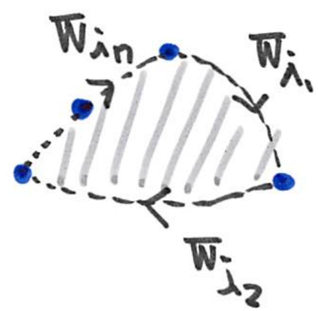


<u>generator</u>

A, B, C : complementary regions

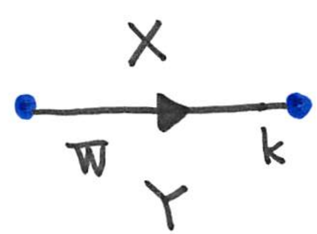
$\varpi_1, \varpi_2, \cdots, \varpi_6$: edge

<u>relation</u>



$\rightsquigarrow \varpi_{i_n} \varpi_{i_{n-1}} \cdots \varpi_{i_1} = 1$

$(\partial(\text{region}) = 1)$



$\rightsquigarrow \varpi = (X^{-1}Y)^k$

Theorem (Boyer-Gordon-Watson, I., Greene)

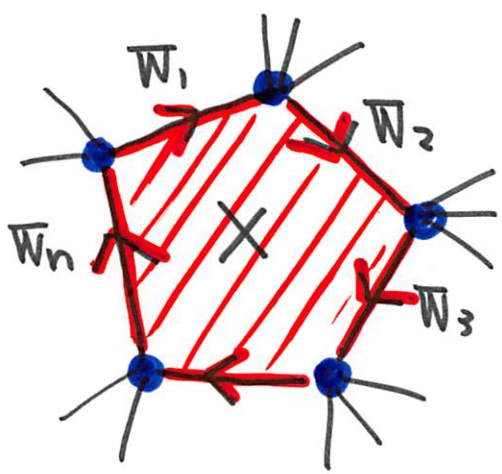
K is alternating $\Rightarrow$ $\pi_1(\Sigma_2(K))$ is not LO

[proof]

K is alternating $\Rightarrow$ all labels of the graph are positive.

Assume $\pi_1(\Sigma_2(K))$ admits a left-ordering $<$.

Take complementary region M which is maximal w.r.t. $<$.



- $w_i = \left([\text{some region}]^{-1} X \right)^{\text{positive number}} > 1$

$\Rightarrow$ $\underbrace{w_n w_{n-1} \cdots w_1}_{\parallel \atop 1} > 1$

contradiction.

Remark.

- Using presentation of $\pi_1(\Sigma_2(K))$ and suitable generalization we can construct many knots with $\pi_1(\Sigma_2(K))$ non LO.

- Deducing a contradiction (by assuming G is LO) often uses a chain of inequality, like

$$a_1 < a_2 < \cdots < a_n < a_1$$

$\downarrow$

sometimes we use unified argument to prove parametrized family of groups and their presentation, like

$$G_\lambda = \left\langle x, z \cdots \;\middle|\; \begin{matrix} x^\lambda = z \\ \cdots \\ \cdots \end{matrix} \right\rangle$$

(For example $\begin{matrix} x^\lambda = z \\ 1 < z \end{matrix}$ $(\lambda > 0)$ $\Rightarrow$ $1 < x$)