Invariant ordering of groups and low-dimensional topology III

: Expected connection of orderings and geometry/topology

§ III-1 : L-space conjecture

Def

A rational homology 3-sphere M is an L-space

$\overset{\text{def}}{\Longleftrightarrow}$ Heegaard Floer homology $\widehat{HF}(M)$ satisfies

$$\operatorname{rank} \widehat{HF}(M) = |H_1(M;\mathbb{Z})|$$

- In general, $\operatorname{rank} \widehat{HF}(M) \geqq |H_1(M;\mathbb{Z})|$

  so L-space is a 3-manifold with the "simplest" $\widehat{HF}$.

- $\{\text{Lens space}\} \subsetneqq \{\text{L-space}\}$

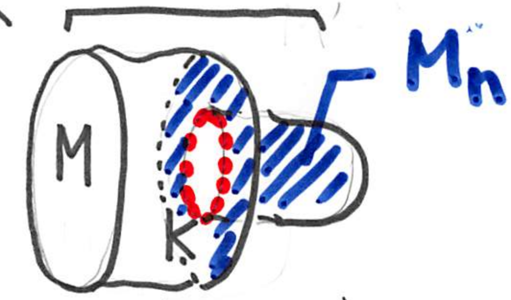
  L-space is a generalization of Lens space.

Typical Construction and example of L-space

M : $\mathbb{Q}HS$ $\quad$ $K \subset M$: knot

Cobordism

$M \longrightarrow M_n$



attach 2-handle along K
(with framing n)

For $n \in \mathbb{Z}$ $\quad$ M_n = n-slope Dehn surgery along K

$\mathbb{Z}$-slope Dehn surgery = attaching 2-handle

↳ $\exists$ "triangle" of cobordisms

$$\begin{tikzcd} M \arrow[rr, "x"'] & & M_n \arrow[dl, "\overline{\upsilon}"] \\ & M_{n+1} \arrow[ul, "\nabla"] & \end{tikzcd}$$

On the other hand, cobordism $M \xrightarrow[x]{} M_n$ induces

$$F_x : \widehat{HF}(M) \longrightarrow \widehat{HF}(M_n)$$

Theorem (Surgery exact triangle, Ozsvath-Szabo '04)

Cobordisms

$$\begin{tikzcd} M \arrow[rr, "X"] & & M_n \arrow[dl, "U"] \\ & M_{n+1} \arrow[ul, "V"] & \end{tikzcd}$$

induces an exact sequence

$$\cdots \longrightarrow \widehat{HF}(M) \xrightarrow{F_X} \widehat{HF}(M_n) \xrightarrow{F_U} \widehat{HF}(M_{n+1}) \xrightarrow{F_V} \widehat{HF}(M) \longrightarrow \cdots$$

In particular,

$$\operatorname{rank} \widehat{HF}(M_{n+1}) \leqq \operatorname{rank} \widehat{HF}(M) + \operatorname{rank} \widehat{HF}(M_n).$$

Since $|H_1(M_{n+1};\mathbb{Z})| = |H_1(M;\mathbb{Z})| + |H_1(M_n;\mathbb{Z})|$

Corollary

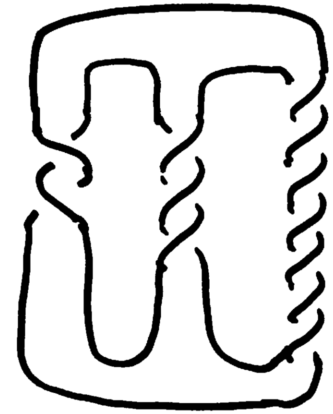
$r \in \mathbb{Q}$.

If M and $M_K(r)$ are L-spaces,

$M_K(s)$ is an L-space for all $s \geqq r$

example

$(-2, 3, 7)$ pretzel knot K



K

$M_K(18) \cong L(18, 5)$

So for $S \geqq 18$

$M_K(S)$ is an L-space

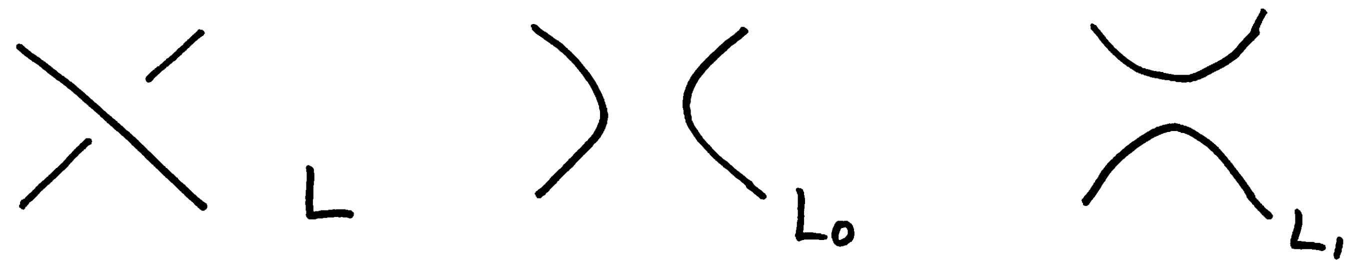
(K is hyperbolic so

$M_K(S)$ is in general, hyperbolic.)

※ L-space can be used to study Lens surgery problem.

Another construction of L-spaces
(another application of surgery exact triangle)

$L, L_0, L_1 \subset S^3$: link related by Skein relation



$\Sigma(L), \Sigma(L_0), \Sigma(L_1)$: Double branched covering

(By Montesinos trick) $\Sigma(L), \Sigma(L_0), \Sigma(L_1)$ are related by integral surgeries

so

$$\begin{tikzcd} \Sigma(L_0) \arrow[rr] & & \Sigma(L_1) \arrow[dl] \\ & \Sigma(L) \arrow[ul] & \end{tikzcd}$$

triangle of cobordism

$\Rightarrow \ \to \widehat{HF}(\Sigma(L_0)) \to \widehat{HF}(\Sigma(L_1)) \to \widehat{HF}(\Sigma(L)) \to \cdots$ exact

Thus, if

- Both $\Sigma(L_0)$ and $\Sigma(L_1)$ are L-spaces
- $|H_1(\Sigma(L);\mathbb{Z})| = |H_1(\Sigma(L_0);\mathbb{Z})| + |H_1(\Sigma(L_1);\mathbb{Z})|$

Then $\Sigma(L)$ is an L-space

Note that

$$|H_1(\Sigma(L);\mathbb{Z})| = \det(L) \quad (= |\Delta_K(-1)|)$$

This motivate to define :

Definition

The set of quasi-alternating link $\mathcal{Q}$ is a set characterized by

(Q1) Unknot $\in \mathcal{Q}$

(Q2) $L_0, L_1 \in \mathcal{Q}$ and $\det(L_0) + \det(L_1) = \det(L)$

$\Rightarrow L \in \mathcal{Q}$

Corollary

If L is a quasi-alternating, $\Sigma(L)$ is an L-space

(Remark)

$$(\text{Alternating links}) \subsetneq Q = \{\text{quasi-alternating links}\}$$

※ For a quasi-alternating knot

$\widehat{HFK}$, Kh (knot Floer homology, Khovanov homology) is thin

i.e. supported in a small diagonal neighboorhood.

↳ quasi-alternating knots are interesting

in knot homologies.

L-space Conjecture (explicitly stated in Boyer-Gordon-Watson '13)

M : irreducible rational homology 3-sphere

$\pi_1(M)$ is NOT LO

$$\Updownarrow$$

M is an L-space

※ There is no (known) direct relation between $\pi_1(M)$ and $\widehat{HF}(M)$

※ L-space is important both for theories and applications.

$\downarrow$

More direct characterization (without using $\widehat{HF}$) is desired.

In many cases L-space conjecture is verified
(some?)

by checking (non-)LO property and $\widehat{HF}$ individually

- Seifert fibered Space (BGW '13)
- Solve 3-manifold (〃 '13)
- Double branched covering of alternating links (〃 , Greene, I. '13)
- Graph manifolds which are $\mathbb{Z}HS$ (Boileau-Boyer, Clay-Lidman-Watson '13)
- Several Dehn surgery along some knots (Clay-Watson, Hakamata-Teragaito, Tran, Motegi-Teragaito and More...)

§ III.2 Why L-space conjecture would be true ?

Refined version of conjecture

M: irreducible $\mathbb{Q}HS$

M is an L-space

$\pi_1(M)$ is NOT LO

M does not have a co-oriented taut foliation

Theorem (Ozsvath-Szabo '04)

An L-space does not have a co-oriented taut foliation

$$\begin{tikzcd} \boxed{\text{Group}} \\ \boxed{\text{3-Manifold}} \arrow[u, "\pi_1"'] \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small] \text{Finite} \subset {}^{\exists}\text{Torsion} \arrow[d, leftrightarrow] & \not\subset & \text{Non-LO} \arrow[d, leftrightarrow, "?"] & \not\subset & \pi_1(\mathbb{Q}HS) \arrow[d, leftrightarrow] \\ S^3\text{-geometry} & \not\subset & \text{L-space} & \not\subset & \mathbb{Q}HS \end{tikzcd}$$

$\pi_1(\text{L-space}) \overset{\text{must have}}{=}$ intermediate between finite (${}^{\exists}$torsion) and $\pi_1(\mathbb{Q}HS)$.

※ Question

What property "characterize" $\pi_1(\mathbb{Q}HS)$ among $\pi_1(\text{3-manifold})$

($\pi_1(M)$ has property $**$ $\Rightarrow$ M is $\mathbb{Q}HS$?)

Important (and possible for non-Heegaard Floer homology specialist) Question / Problem

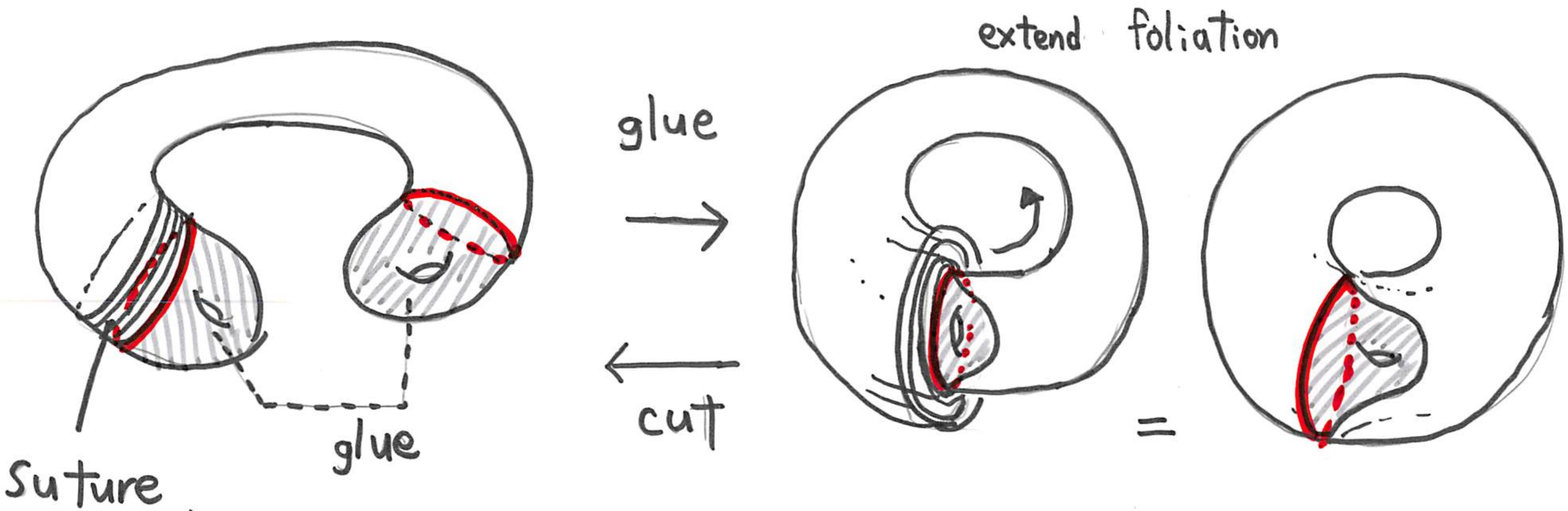
- Consider "triangle" of cobordism (in surgery exact triangle)

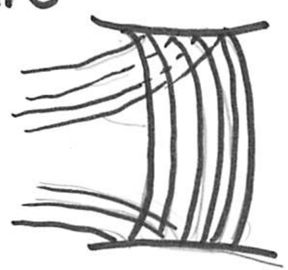
$$\begin{tikzcd} M \arrow[rr, "X"] & & M_0 \arrow[dl, "U"] \\ & M_1 \arrow[ul, "V"] & \end{tikzcd}$$

relate/study LO and non-LO propertie among $\pi_1(M)$, $\pi_1(M_0)$, $\pi_1(M_1)$

- More generally study a relation ship between cobordism $M \xrightarrow{X} N$ and LO-/non LO- of $\pi_1(M)$, $\pi_1(N)$.

- One of the most useful technique to construct taut foliation is sutured manifold theory (cut-and-paste of taut foliation)





Q. $\exists$ relations between sutured decomposition and left-ordering

§III.3 Application to topology/geometry

Question

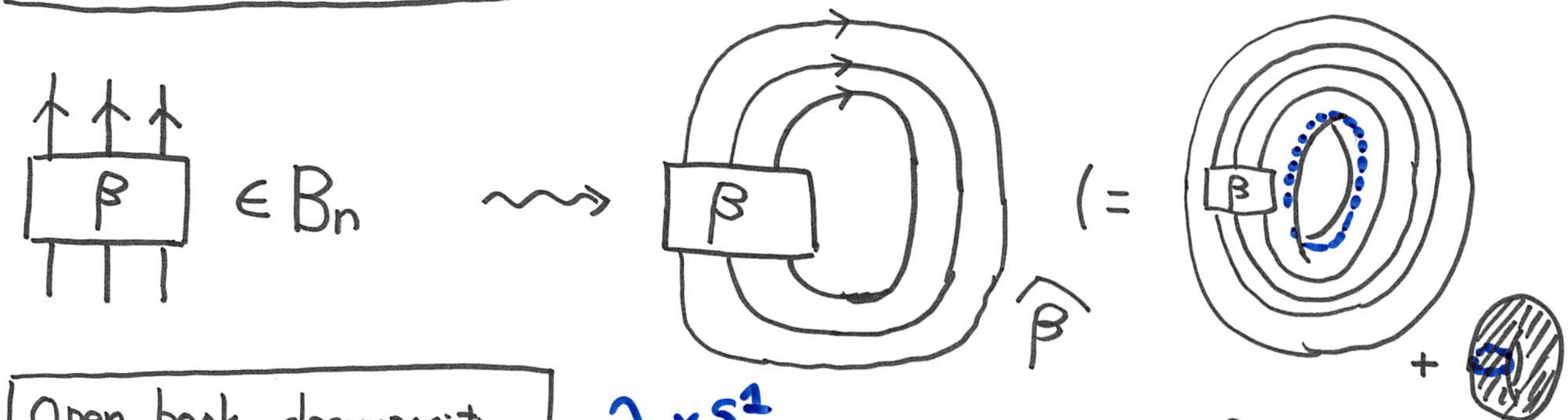
Can we use ordering (Not orderability) to study topology and geometry ?

(Model examples from Riemannian geometry)

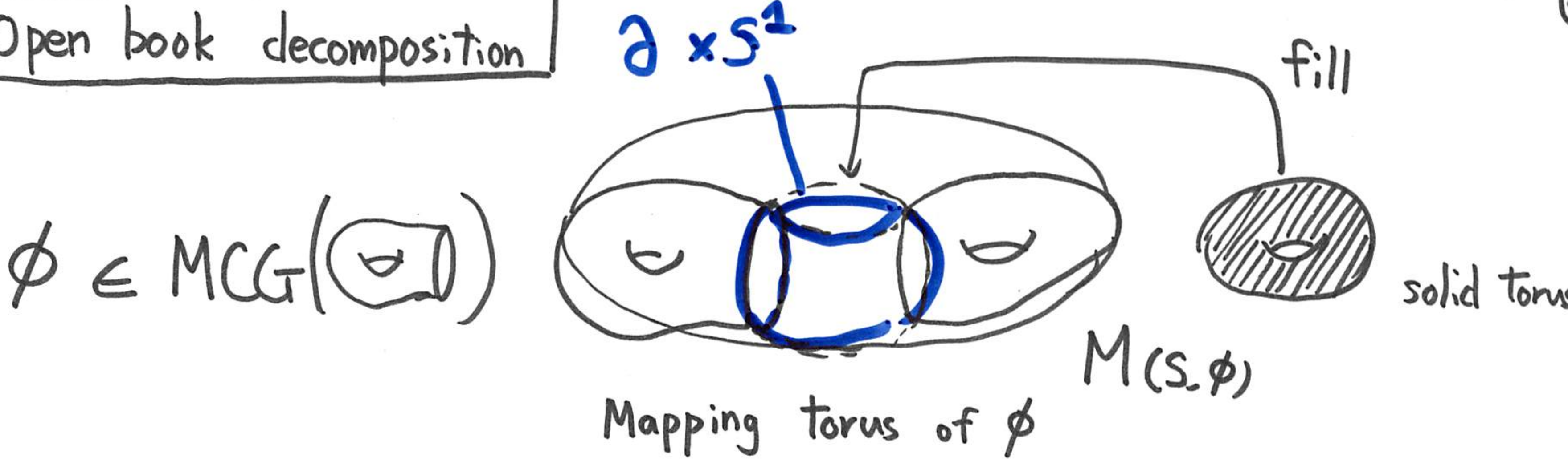
(Ricci-, sectional, ⋯) Curvature is positive/negative, sufficiently large/small, in certain Lange → · Manifold has property ~ · Topology of manifold

Knots, 3-manifolds are expressed by braid group / mapping class group
(by cut-and-paste)

Closed braid (in S^3)



Open book decomposition



Motivated from Riemannian geometry case,

Expectation

$\beta \in B_n$ or $\phi \in MCG(\Sigma)$ is positive/negative, sufficiently large/small, in certain range $\Rightarrow$ $\hat{\beta}$ or $M_{(S,\phi)}$ has property ~

This is indeed true.

Meta-Theorem (I.)

With respect to Thurston-type ordering $\beta \in B_n$ or $\phi \in MCG(\Sigma)$ is sufficiently large/small $\Rightarrow$ Topology/Geometry of $\hat{\beta}$ or $M_{(S,\phi)}$ can be directly read from β, ϕ.

Def

fix Thurston-type ordering of B_n (or $MCG(\Sigma)$)

(i.e. ordering from $MCG \curvearrowright \partial\overline{\widetilde{\Sigma}}$ $(\subset \mathbb{H}^2 \cup S^1_\infty)$)

$T = T_{\partial\Sigma}$ (Dehn twist along $\partial\Sigma$)

(B_n case $T = \Delta^2 = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$)

$$[\phi] \overset{\text{def}}{=} \text{integer} \quad T^{[\phi]} \leq \phi < T^{[\phi]+1}$$

$$c(\phi) \overset{\text{def}}{=} \lim_{N\to\infty} \frac{[\phi^N]}{N} \in \mathbb{R}$$

(• Remark: Recall Hölder's theorem and its proof.)

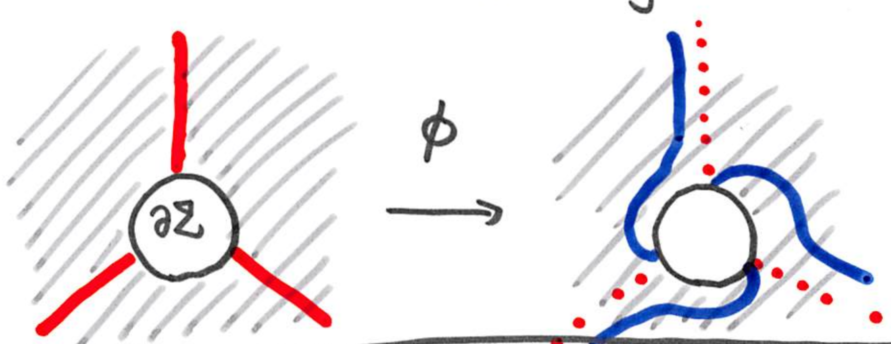
$c : MCG \longrightarrow \mathbb{R}$ is no longer homomorphism, but is a quasi-morphism :

- $| c(\phi\psi) - c(\phi) - c(\psi) | \leqq 1$
- $c(\phi^n) = n\, c(\phi)$

Proposition (I-Kawamuro.)

$c : MCG \longrightarrow \mathbb{R}$ is equal to Fractional Dehn Twist coefficient (introduced by Honda-Kazez-Matic '08)

= "amount of twisting near boundary



$c(\phi) = \frac{1}{3}$

Actually $c : MCG \longrightarrow \mathbb{Q} \subset \mathbb{R}$ and it is numerical approximation of the ordering.

Theorem (I-Kawamuro)

Assume $\phi \in MCG(\Sigma)$ satisfies $|c(\phi)| > 1$

Then

$$\phi \text{ is } \begin{cases} \text{periodic} \\ \text{reducible} \\ \text{pseudo-Anosov} \end{cases} \Longleftrightarrow M_{(S,\phi)} \text{ is } \begin{cases} \text{Seifert-fibered} \\ \text{Toroidal} \\ \text{Hyperbolic} \end{cases}$$

Theorem (I-Kawamuro)

If $|c(\phi)| > 1$, $M_{(S,\phi)}$ is irreducible.

Theorem (I '11)

- $\beta \in B_n \qquad g(\widehat{\beta}) > \beta - 1$
- $c(\alpha), c(\beta)$ ... sufficiently large
  $\Rightarrow \widehat{\alpha} = \widehat{\beta}$ if and only if α and β are conjugate
- $|c(\beta)| > 2$, β is pseudo-Anosov
  $\Rightarrow \widehat{\beta}$ is a hyperbolic knot.

If ordering is large/small

braid group theory $\approx$ knot theory !

Application to Quantum invariant

V : module of quantum group $U_q(\mathfrak{g})$

$$\varphi_V : B_n \longrightarrow GL(V^{\otimes n}) \qquad \text{quantum representation}$$

Quantum invariant of a knot $K = \widehat{\beta}$

$$Q^V(K) = (\text{quantum}) \text{ trace of } \varphi_V(\beta)$$

Big open problem

Does Q^V detect the unknot ?

(ex. Does the Jones polynomial detect the unknot ?

observation: $\varphi_\nabla(\beta) = 1 \Rightarrow Q^\nabla(\widehat{\beta \cdot \sigma_1 \cdots \sigma_{n-1}}) = Q^\nabla(\widehat{\sigma_1 \cdots \sigma_{n-1}})$

$= Q^\nabla(\text{unknot})$

$\downarrow$

Conjecture (Bigelow)

φ_∇ is not faithful $\Rightarrow$ Q^∇ cannot detect the unknot

Theorem (I. '15)

$1 \neq N \triangleleft B_n$: non-trivial normal subgroup

$\#\{c(\beta) \mid \beta \in N\} = \infty$ i.e. N is unbounded w.r.t Thurston-type ordering.

Corollary

Bigelow's conjecture is true

Actually, we have a stronger result

Corollary

If $\varphi_{\nabla} : B_n \longrightarrow GL(\nabla^{\otimes n})$ is not faithful,

- For any $N > 0$ $\exists K$: hyperbolic knot

  $g(K) > N$, $Q^{\nabla}(K) = Q^{\nabla}(\text{unknot})$

- More generally for any knot K

  $\exists$ infinitely many knots $K_1, K_2, \cdots$

  s.t. $Q^{\nabla}(K_i) = Q^{\nabla}(K)$

§ III.4 Comments for other related topics

• Bi-orderability

For BO property, Burns-Hale theorem (B-R-W theorem) fails

[counter example]

$\pi_1(S^3 - \text{trefoil}) = \langle x, y \mid x^2 = y^3 \rangle$ is not BO because $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$

$\cong \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ $\quad (\sigma_1\sigma_2)^3 = (\sigma_2\sigma_1)^3$

but it is locally indicable

$^\forall H \subset \pi_1(S^3 - \text{trefoil})$: finitely generated subgroup

$^\exists H \twoheadrightarrow \mathbb{Z}$ ($\mathbb{Z}$ is BO group !)

↳ In most case, one prove G is BO

by somewhat explicitly constructing bi-ordering.

Fibered knot case

$$1 \longrightarrow F_{2g} \longrightarrow \pi_1(S^3-K) \longrightarrow \mathbb{Z} \longrightarrow 1$$

$\Rightarrow$ $\pi_1(S^3-K)$ is BO $\Leftrightarrow$ $\exists$ bi-ordering on F_{2g} preserved by monodromy

Theorem (Clay-Rolfsen, Perron-Rolfsen)

(i) $\pi_1(S^3-K)$ is BO $\Rightarrow$ $\Delta_K(t)$ has at least one positive real root.

(ii) All roots of $\Delta_K(t)$ are positive real $\Rightarrow$ $\pi_1(S^3-K)$ is BO

(i) + Fact (by Ni) A knot $K \subset S^3$ admits an L-space surgery $\Rightarrow$ K is fibered

Corollary

For a (not necessarily fibered knot) K, if $\pi_1(S^3-K)$ is BO

$\Rightarrow$ $M_K(r)$ is not an L-space for $r \in \mathbb{Q}$.

- Space of orderings

$LO(G) = \{ <_G \mid \text{left-orderings on } G \}$

$LO(G) \curvearrowleft G$ by

$$g\ (<_G \cdot a)\ h \iff ga <_G ha.$$

By equipping certain natural topology, $LO(G) \curvearrowleft G$ is a continuous action.

Question
- Determine topological space $LO(G)$.
- Study an action $LO(G) \curvearrowleft G$.

$LO(G) \curvearrowleft G$ is useful.

Theorem (Witte-Morris '10)
G is amenable and LO $\Rightarrow$ G is locally indicable.