Curve complexes and the DM-compactification of moduli spaces

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Notation

$\Sigma_{g,n}$: an oriented closed surface of genus $g \geqq 2$
with n-points deleted

$\Gamma_{g,n} = \Gamma(\Sigma_{g,n})$: mapping class group of $\Sigma_{g,n}$

$T_{g,n} = T(\Sigma_{g,n})$: Teichmüller space of $\Sigma_{g,n}$

$\dim_{\mathbb{C}} T_{g,n} = 3g - 3 + n$ complex analytic space

$\Gamma_{g,n}$ acts on $T_{g,n}$ complex analytically,
and properly discontinuously.

$M_{g,n} = T_{g,n}/\Gamma_{g,n}$: moduli space

$\overline{M_{g,n}}$: compactification of $M_{g,n}$
(Deligne-Mumford 1969)

The purpose of this talk is

to construct a “natural” orbifold structure on the DM-compactification $\overline{M_{g,n}}$ of $M_{g,n}$, making use of N. V. Ivanov’s “scissored Teichmüller space” $P^{\varepsilon}_{g,n}$.

This construction clarifies the role of W. J. Harvey’s curve complex $\mathcal{C}_{g,n}$ in the compactification of $M_{g,n}$.

(Ivanov [’87] introduced $P^{\varepsilon}_{g,n}$ in his cohomological study of the mapping class group $\Gamma_{g,n}$.)

Basic definitions

We consider a pair (S, w) of a Riemann surface S and an orientation preserving homeomorphism

$$w : S \to \Sigma_{g,n}.$$

Two such pairs (S, w) and (S', w') are equivalent $(S, w) \sim (S', w')$ iff $\exists$ a biholomorphic map $t : S \to S'$ s.t. the following diagram homotopically commutes:

$$\begin{array}{ccc} S & \xrightarrow{w} & \Sigma_{g,n} \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \mathrm{id.}} \\ S' & \xrightarrow[w']{} & \Sigma_{g,n} \end{array}$$

Basic definitions 2

Teichmüller space

$T_{g,n}$ is defined by

$$T_{g,n} \stackrel{\text{def.}}{=} \{(S, w)\}/\sim .$$

The mapping class group $\Gamma_{g,n}$ is defined by

$$\Gamma_{g,n} \stackrel{\text{def.}}{=} \{\text{ori.pres.homeos} : \Sigma_{g,n} \to \Sigma_{g,n}\}/\text{isotopy}.$$

The action of $\Gamma_{g,n}$ on $T_{g,n}$ is defined by

$$[f]_*[S, w] \stackrel{\text{def.}}{=} [S, f \circ w],$$

where $[f] \in \Gamma_{g,n}$ and $[S, w] \in T_{g,n}$.

Length function $L : T_{g,n} \to \mathbb{R}$

$T_{g,n}$ is a complex analytic space (Weil, Ahlfors, 1960) and is a bounded domain (Bers, 1961) of $\dim_{\mathbb{C}} T_{g,n} = 3g - 3 + n$.

Let C be an essential simple closed curve on $\Sigma_{g,n}$ i.e. C is not homotopic to a point nor to a puncture.
For any point $p = [S, w] \in T_{g,n}$, let $l_p(C)$ be the length of the simple closed geodesic $\hat{C}$ on S homotopic to $w^{-1}(C)$.
Define $L : T_{g,n} \to \mathbb{R}$ by

$$L(p) \overset{\text{def.}}{=} \min_{C \subset \Sigma_{g,n}} l_p(C).$$

The scissored Teichmüller space $P^{\varepsilon}_{g,n}$

The length function

$$L : T_{g,n} \to \mathbb{R}$$

is a piecewise real analytic function.
(Fenchel-Nielsen, Abikoff['80])
Let $\varepsilon > 0$ be a sufficiently small number. Then we define $P^{\varepsilon}_{g,n}$ as follows:

$$P^{\varepsilon}_{g,n} \overset{\text{def.}}{=} \{p \in T_{g,n} \mid L(p) \geqq \varepsilon\}.$$

$P^{\varepsilon}_{g,n}$ is a real analytic manifold with corners.

To what extent should ε be small?

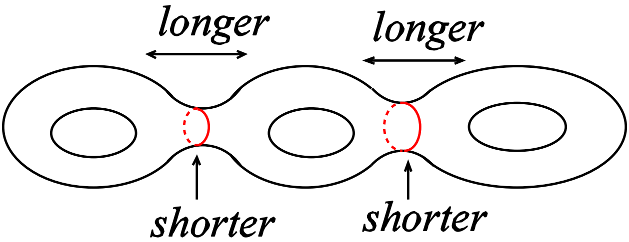
Theorem (L. Keen[1973], W. Abikoff[1980])

There is an universal constant M such that two distinct simple closed geodesics on S are disjoint, if their lengths are $< M$.

The number ε should be taken as $\varepsilon < M$.

Explanation of the Theorem

If the red curves become shorter, transverse curves become longer.



Facets of $P^{\varepsilon}_{g,n}$ (1)

Suppose a point $p_0 = [S_0, w_0]$ is on the boundary $\partial P^{\varepsilon}_{g,n}$, then we have

$$L(p_0) = \varepsilon.$$

Then there exist a finite number of simple closed geodesics

$$\hat{C}_1, \cdots, \hat{C}_k$$

on (S_0, w_0) such that $l_{p_0}(\hat{C}_i) = \varepsilon, i = 1, \ldots, k$. They are disjoint (because $\varepsilon < M$), and

$$k \leqq 3g - 3 + n.$$

Facets of $P^{\varepsilon}_{g,n}$ (2)

Let σ denote the set of pairwise disjoint simple closed curves on $\Sigma_{g,n}$:

$$\sigma = \{C_1, \cdots, C_k\}.$$

Define the facet $F^{\varepsilon}(\sigma)$ corresponding to σ by

$$F^{\varepsilon}(\sigma) := \{p \in P^{\varepsilon}_{g,n} \mid l_p(\hat{C}_i) = \varepsilon, i = 1, \cdots, k\}$$

For $\forall p = [S, w]$ in $F(\sigma)$, we assume that other simple closed geodesics on S have length $> \varepsilon$.
(The point $p_0 = [S_0, w_0]$ in the previous slide is on this facet.)

Facets of $P^{\varepsilon}_{g,n}$ (3)

A facet $F^{\varepsilon}(\sigma)$ is a real analytic manifold homeomorphic to

$$\mathbb{R}^{2(3g-3+n)-k},$$

where $k = \#\sigma$.

Facets of $P^{\varepsilon}_{g,n}$ are analogous to open faces of a finite polyhedron P.

Incidence relation: If $\sigma \subset \sigma'$, then we have

$$\overline{F^{\varepsilon}(\sigma)} \supset F^{\varepsilon}(\sigma').$$

A facet is itself an infinite polyhedron.

Abelian subgroup $\Gamma(\sigma)$

Let σ denote $\{C_1, \cdots, C_k\}$ as before. Let $\tau(C_i)$ be the right handed (negative) Dehn twist about C_i, and define a subgroup $\Gamma(\sigma)$ of $\Gamma_{g,n}$ to be the subgroup generated by

$$\tau(C_i), \quad i = 1, \cdots, k.$$

Since σ is a disjoint union of s.c.c's, the group $\Gamma(\sigma)$ is abelian. More precisely, $\Gamma(\sigma)$ is a free abelian group of rank k

Action of $\Gamma(\sigma)$ on $F^\varepsilon(\sigma)$

Since the action of $\Gamma_{g,n}$ on $T_{g,n}$ preserves the Poincaré metric on Riemann surfaces (hence preserves the length function L), and since

$$\tau(C_i)(C_j) = C_j, \quad i, j = 1, \ldots, k,$$

the twists $\tau(C_i), i = 1, \ldots, k$ preserve $F^\epsilon(\sigma)$.
This action of $\Gamma(\sigma)$ on $F^\varepsilon(\sigma)$ is real analytic and properly discontinuous.

Complex of curves $\mathcal{C}_{g,n}$

W. J. Harvey (1977) introduced an abstract simplicial complex called the complex of curves $\mathcal{C}_{g,n} = \mathcal{C}(\Sigma_{g,n})$:

Definition(Complex of curves)

Vertices of $\mathcal{C}_{g,n}$: isotopy classes of essential simple closed curves on $\Sigma_{g,n}$.
A simplex σ of $\mathcal{C}_{g,n}$: a set of vertices represented by a disjoint union of simple closed curves.

The facets $F^{\varepsilon}(\sigma)$ are in 1:1 correspondece with the simplices σ of $\mathcal{C}_{g,n}$.

Barycentric subdivision of $\mathcal{C}_{g,n}$

Proposition 1

The totality of the facets $\{F^\varepsilon(\sigma)\}_{\sigma\in\mathcal{C}}$ makes a complex (facet complex) analogous to a simplicial complex. The flag complex associated to the facet complex is isomorphic to the barycentric subdivision of the complex of curves $\mathcal{C}(\Sigma_{g,n})$.

Proof: A flag in the facet complex $\overline{F^\varepsilon(\sigma)} \supset \overline{F^\varepsilon(\sigma')} \supset F^\varepsilon(\sigma'')$ corresponds to a flag in the complex of curves $\mathcal{C}$, $\sigma \subset \sigma' \subset \sigma''$. The latter corresponds to a simplex of the barycentric subdivision of $\mathcal{C}(\Sigma_{g,n})$. $\square$

Automorphisms of $\mathcal{C}_{g,n}$

We need the following theorem:

Theorem (Ivanov['97], Korkmaz['99], Luo['00])

With the exceptional cases $\{$of spheres with $\leqq 4$ punctures, tori with $\leqq 2$ punctures and a closed surface of genus $2\}$, the following holds:

$$Aut(\mathcal{C}_{g,n}) = \Gamma^*_{g,n},$$

where $\Gamma^*_{g,n}$ stands for the extended mapping class group (containing orientation reversing homeomorphisms).

Automorphisms of $P^{\varepsilon}_{g,n}$

The scissored Teichmüller space $P^{\varepsilon}_{g,n}$ together with the Teichmüller metric becomes a metric (infinite) polyhedron. Then the folllowing proposition is a corollary to the above theorem.

Proposition 2

With the same exceptions as above, we have the following:

$$Isom_{+}(P^{\varepsilon}_{g,n}) = \Gamma_{g,n}.$$

Proof of Proposition 2

Proof: **An isomorphism of $P^{\varepsilon}_{g,n}$ induces on $\partial P^{\varepsilon}_{g,n}$ an automorphism of the facet complex, thus that of the barycentric subdivion of $\mathcal{C}_{g,n}$, and finally an automorphism of $\mathcal{C}_{g,n}$. The automorphism of $\mathcal{C}_{g,n}$ in turn corresponds (by Ivanov-Korkmaz-Luo's theorem) to an action of the mapping class group $\Gamma_{g,n}$, hence an (orientation preserving) isometry of $T_{g,n}$. □**

Essentialy the same arguments are found in A. Papadopoulos ['08] and K. Ohshika ['11].

The subgroup of $\Gamma_{g,n}$ preserving $F^{\varepsilon}(\sigma)$

Proposition 3.

The subgroup of $\Gamma_{g,n}$ which preserves a facet $F^{\varepsilon}(\sigma)$ is precisely $N\Gamma(\sigma)$, the normalizer of $\Gamma(\sigma)$ in $\Gamma_{g,n}$.

Proof: If a mapping class $[f] \in \Gamma_{g,n}$ preserves $F^{\varepsilon}(\sigma)$, then $[f]$ induces on $\Sigma_{g,n}$ a permutation of $\sigma = \{C_1, \cdots, C_k\}$, and *vice versa*. **Such mapping classes make the normilizer $N\Gamma(\sigma)$ of $\Gamma(\sigma)$.** □

"Fringe" $FR^{\varepsilon}(\sigma)$ bounded by $F^{\varepsilon}(\sigma)$

The fringe $FR^{\varepsilon}(\sigma)$ is defined by

$$FR^{\varepsilon}(\sigma) = \bigcup_{0<\delta<\varepsilon} F^{\delta}(\sigma).$$

Then we have

Cor. to Proposition 3

The subgroup of $\Gamma_{g,n}$ which preserves the fringe $FR^{\varepsilon}(\sigma)$ is the normalizer $N\Gamma(\sigma)$. The action of $N\Gamma(\sigma)$ on $FR^{\varepsilon}(\sigma)$ is properly discontinuous.

Proof: $FR^{\varepsilon}(\sigma)$ is foliated by the facets $F^{\delta}(\sigma)$, and Corollary holds for each leaf $F^{\delta}(\sigma)$. □

Augmented fringe $\overline{FR^{\varepsilon}}(\sigma)$

Define the augmented fringe as follows

Definition: Augmented fringe

$$\overline{FR^{\varepsilon}}(\sigma) = \bigcup_{0 \leqq \delta < \varepsilon} F^{\delta}(\sigma) \ \left(= FR^{\varepsilon} \cup F^{0}(\sigma)\right).$$

Then $N\Gamma(\sigma)$ acts on $\overline{FR^{\varepsilon}}(\sigma)$ continuosly, but not properly discontinuously. (The action of $\Gamma(\sigma)$ fixes the added ideal boundary $F^{0}(\sigma)$.)

Augmented Teichmüller space $\overline{T}_{g,n}$

The ideal boundary $F^0(\sigma)$ parametrizes the nodal surfaces obtained by pinching the curves in σ to points.

Abikoff ['77] attached to $T_{g,n}$ all ideal boundaries, and considered the augmented Teichmüller space

$$\overline{T}_{g,n} = T_{g,n} \cup \bigcup_{\sigma \in \mathcal{C}} F^0(\sigma).$$

Yamada [04] identified $\overline{T}_{g,n}$ with the Weil-Petersson completion of $T_{g,n}$, and proved the geodesic convexity of the ideal boundaries $F^0(\sigma)$.

Well known fact

The quotient space of $\overline{T}_{g,n}$ under the action of $\Gamma_{g,n}$ is the compactified moduli space $\overline{M_{g,n}}$.

Note that the union of the augmented fringes $\bigcup_{\sigma \in \mathcal{C}} \overline{FR^{\varepsilon}}(\sigma)$ gives an open neighborhood of singular divisors when divided out by the action of $\Gamma_{g,n}$.

A defect of the fringes

To analyse the orbifold structure of $\overline{M_{g,n}}$, the fringes $\overline{FR^{\varepsilon}}(\sigma)$ are inadequete, because they are pairwise disjoint:

$$\overline{FR^{\varepsilon}}(\sigma) \cap \overline{FR^{\varepsilon}}(\sigma') = \emptyset, \quad \text{if} \quad \sigma \neq \sigma'.$$

(Recall that facets are something like open faces of a polyhedron.)
Namely the fringes do not make an open covering of the singular divisors $\bigcup_{\sigma \in \mathcal{C}} F^{0}(\sigma)$.
To remedy the deficiency, we introduce controlled deformation spaces.

Bers' deformation spaces

Let $\sigma \in \mathcal{C}$ be any simplex $\sigma = \{C_1, \cdots, C_k\} \in \mathcal{C}$.
Let $\Sigma_{g,n}(\sigma)$ denote the surface with nodes obtained from $\Sigma_{g,n}$ by pinching each C_i to a point.
Bers [1974] introduced the deformation space $D(\sigma)$ associated with $\Sigma_{g,n}(\sigma)$.
The following theorem is well-known:

Proposition 4

$D(\sigma)$ is homeomorphic to $(T_{g,n}/\Gamma(\sigma)) \cup F^0(\sigma)$.

Complex analytic structure on $D(\sigma)$

Bers announced in 70's that $D(\sigma)$ is a bounded domain, but without proof.
Recently Hubbard and Koch [2014] gave a proof.

Theorem (Hubbard and Koch)

The deformation space $D(\sigma)$ has a complex structure.

I am still trying to understand the details of their arguments, but conceptually the proof is clear: The core part $F^0(\sigma)$ is Teichmüller space of a nodal surface $\Sigma_{g,n}(\sigma)$, and the transverse direction corresponds to the "plumbing coordinates"(cf. Masur[1976]).

The groups $W(\sigma)$

Define

$$W(\sigma) = N\Gamma(\sigma)/\Gamma(\sigma).$$

The groups $W(\sigma)$ are not generally finite groups, but they seem to have certain similarities with the Weyl groups.

Proposition 5

(i) $W(\sigma)$ is the mapping class group of the surface with nodes $\Sigma_{g,n}(\sigma)$.
(ii) $W(\sigma)$ acts on $D(\sigma)$ holomorphically and properly discontinuously.

A Remark

When σ is a maximal simplex of $\mathcal{C}(\Sigma_{g,n})$, the group $W(\sigma)$ is finite. It appeared in Harvey's paper [1979] as the automorphism group of the maximal partition graph K_σ.

In this case, the facet $F^\varepsilon(\sigma)$ (together with the Weil-Petersson metric) is a Lagrangean submanifold of $T_{g,n}$. $F^\varepsilon(\sigma)$ is homeomorphic to $\mathbb{R}^{3g-3+n}$ on which $\Gamma(\sigma)$ acts as translations.

This is exactly the situation of crystallographic groups. Appearance of "Symplectic crystallographic groups" in Teichmüller Theory!

(Terminology "symplectic crystallographic groups" is due to S. Yamada.)

Harvey's paper[1981]

Harvey considers the cuspidal boundary structure

$$\partial T_{g,n} = \bigcup_{\sigma \in \mathcal{C}} T_\sigma \times \mathbb{R}^{\#(\sigma)}$$

and attaced it to the TeichmÏler space $T_{g,n}$. He claims that $T_{g,n} \cup \partial T_{g,n}$ is a real analytic manifold with corners on which $\Gamma_{g,n}$ acts properly discontinuously. (He called this construction "blowing up").

His explanation is vague. Our polyhedron $P^{\varepsilon}_{g,n}$ realizes his idea inside the Teichmüller space more rigourously.

Controlled deformation spces

Let M be the constant of Keen and Abikoff, and we take an ε with $\varepsilon < M$). We insert $6g - 6 + 2n$ numbers between ε and M:
$\varepsilon < \varepsilon_1 < \eta_1 < \cdots < \varepsilon_{3g-3+n} < \eta_{3g-3+n} < M$.
Let $\hat{\varepsilon}$ denote this sequence.
We define the controlled deformation space $D_{\hat{\varepsilon}}(\sigma)$ as follows (σ being $\{C_1, \cdots, C_k\}$)

Definition of $D_{\hat{\varepsilon}}(\sigma)$

$$D_{\hat{\varepsilon}}(\sigma) = \{p = [S, w] \in D(\sigma) \mid l_p(\hat{C}_i) < \varepsilon_k, \\ i = 1, \ldots, k, \text{and other simple closed} \\ \text{geodesics on } S \text{ are longer than } \eta_k.\}$$

Why do we need the controlled deformation spaces?

Because $D(\sigma)$ do not naturally descend to $\overline{M_{g,n}}$. But the controlled deformation spaces $D_{\hat{\varepsilon}}(\sigma)$ do.

Proposition 5

(i) $D_{\hat{\varepsilon}}(\sigma)$ is a bounded domain of $\mathbb{C}^{3g-3+n}$.
(ii) The group $W(\sigma)$ acts on $D_{\hat{\varepsilon}}(\sigma)$ complex analytically and properly discontinuously.
(iii) $D_{\hat{\varepsilon}}(\sigma)/W(\sigma)$ is an open subset of $\overline{M_{g,n}}$.
(iv) $D_{\hat{\varepsilon}}(\sigma)/W(\sigma)$ contains the "main part"of the quotient of the augmented fringe $\overline{FR^{\varepsilon}}(\sigma)/W(\sigma)$
(v) The family $\{D_{\hat{\varepsilon}}(\sigma)/W(\sigma)\}_{\sigma\in\mathcal{C}}$ is an open covering of the singular divisors $\bigcup_{\sigma\in\mathcal{C}} F^0(\sigma)/W(\sigma)$.

Main theorem

Summarizing the above, we have

Theorem (M. IRMA lectures, [2012])

The family $\{(D_{\hat{\varepsilon}}(\sigma), W(\sigma))\}_{\sigma \in \mathcal{C}}$ gives the orbifold charts around the singular divisors in $\overline{M_{g,n}}$.

Remark. **If $\sigma' = f(\sigma)$ by a mapping class $f \in \Gamma_{g,n}$, we consider that $(D_{\hat{\varepsilon}}(\sigma), W(\sigma))$ and $(D_{\hat{\varepsilon}}(\sigma'), W(\sigma'))$ are identical charts.**

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