Mapping Class Group Action on the Space of Geodesic Rays of a Punctured Hyperbolic Surface

Branched Coverings, Degenerations and Related Topics 2015

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Plan

(I) Background

Surface bundles vs Heegaard decompositions

via branched covering

– "monodromy group" of a Heegaard decomposition –

(II) Motivation

McShane's identity and its variation

– the role of the "monodromy group"

(III) Main Theorem

Idea of proof

Further problem

Surface bundles vs

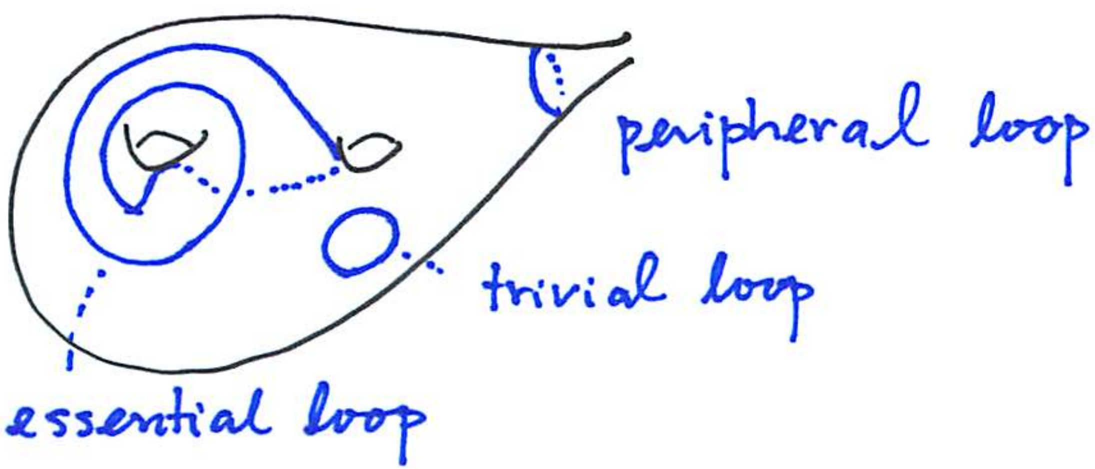
Heegaard decompositions

Notation

Σ : compact conn surface
possibly with boundary and puncture

$$\mathrm{MCG}(\Sigma) = \{\varphi : \Sigma \xrightarrow[\cong]{} \Sigma \text{ homeo}\}/\text{isotopy}$$
$$\cup$$
$\mathrm{MCG}^+(\Sigma)$: ori-pres. subgroup

$$\mathcal{S} := \{\text{essential simple loops in } \Sigma\}/\text{isotopy}$$



Surface bundle over S^1

$$M_\phi := \Sigma \times \mathbb{R} \Big/ (x,t) \sim (\phi(x), t+1)$$

$$1 \to \pi_1(\Sigma) \to \pi_1(M_\phi) \to \mathbb{Z} \to 1$$

Monodromy group $\langle \phi \rangle \subset MCG(\Sigma)$

Fact

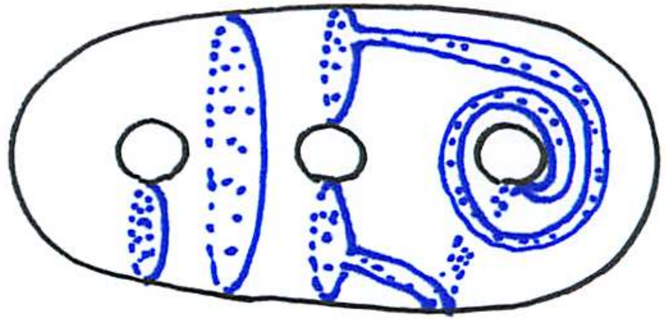
- The action of $\langle \phi \rangle$ on $\mathcal{S}$ preserves "the homotopy class in M_ϕ" of (simple) loops in Σ

  i.e. for any simple loop $\alpha \subset \Sigma = \Sigma \times 0 \subset M_\phi$

  α and $\phi(\alpha)$ are homotopic in M_ϕ
- Moreover for $\alpha, \beta \in \mathcal{S}$,

  $\alpha \sim \beta$ homotopic in $M_\phi \Leftrightarrow \beta = \phi^n(\alpha)$ for some $n \in \mathbb{Z}$

Heegaard decomposition



$$M = V_1 \underset{\Sigma}{\cup} V_2$$

V_i = handlebody with $\partial V_i = \Sigma$

$Z_i = \{\text{meridians of } V_i\} \subset \mathcal{S}$

$\qquad\qquad \|$

simple loop in Σ which bounds a disk in V_i

$$\pi_1(M) \cong \pi_1(\Sigma) \big/ \langle\!\langle Z_1, Z_2 \rangle\!\rangle$$

"Monodromy group of the H-decomposition $M = V_1 \underset{\Sigma}{\cup} V_2$"

$\Gamma := \langle \Gamma_1, \Gamma_2 \rangle \subset MCG(\Sigma)$, where

$$\Gamma_i := MCG_0(V_i) = \{\varphi : V_i \xrightarrow{\cong} V_i \text{, st } \varphi \sim 1_{V_i} \text{ homotopic}\}$$

$$\subset MCG(\Sigma)$$

Fact The action of Γ on $\mathcal{S}$ preserves

"the homotopy class in $M = V_1 \underset{\Sigma}{\cup} V_2$" of simple loops in Σ.

i.e. $\forall \alpha \in \mathcal{S}$, $\forall \gamma \in \Gamma$, $\alpha \sim \gamma(\alpha)$ homotopic in M.

In particular, any element of $\Gamma \cdot (\Delta_1 \cup \Delta_2)$ is null-homotopic in M.

Question (Minsky)

When does the converse hold?

[Lee - S]

The converse holds for the 2-bridge decompositions of hyperbolic 2-bridge links, except for the Whitehead link.

[Ohshika - S]

Partial positive answer for Heegaard splittings of "high Hempel distance" and of "bounded combinatorics".

A relation between surface bundles and H-decompositions

My old observation

Every closed orientable 3-manifold of H-genus g has a Σ_g-bundle M_ϕ as a double branched covering.

$$\{\text{Heegaard decomposition of genus } g\} \rightsquigarrow \{\Sigma_g\text{-bundle}\}$$

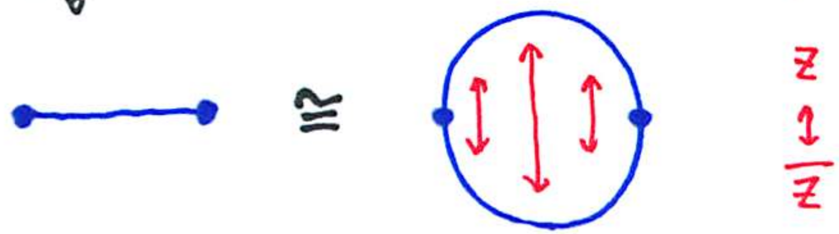
$$\text{"monodromy group"} \rightsquigarrow \text{"monodromy"}$$

[Brooks - Montesinos]

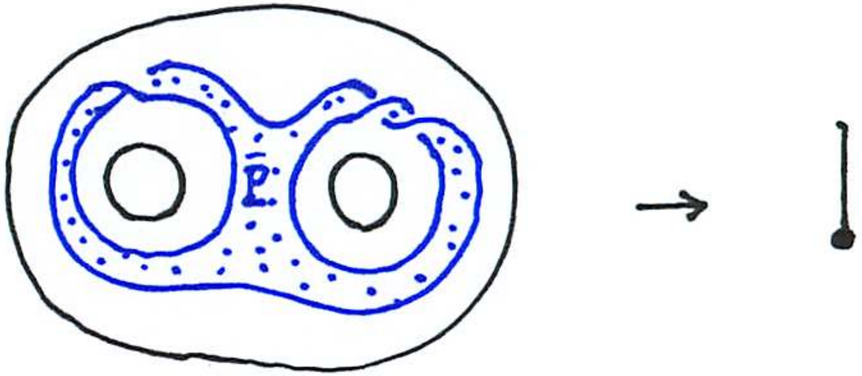
On can choose ϕ to be pseudo-Anosov, and hence M_ϕ to be hyperbolic.

(Idea)

Heegaard decomposition = "Σ_g-bundle" over the orbifold



• V_g is a "Σ_g-bundle" over 



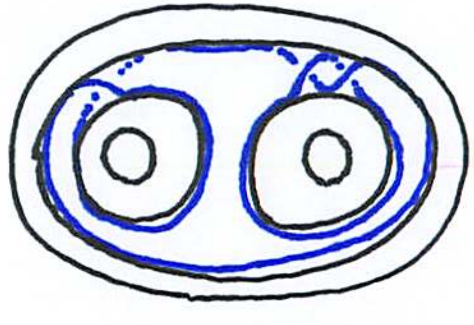
"hight function"

• Double branched covering $\tilde{V}_g$ of V_g branched over ∂P

$$= \text{"}V_g \text{ cut along } P\text{"} \cup \text{"}V_g \text{ cut along } P\text{"}$$

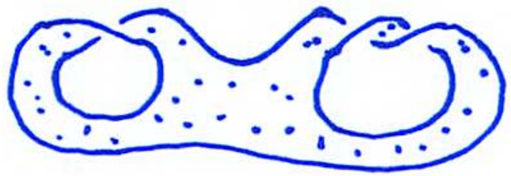
$$\cong \Sigma_g \times I \cup \Sigma_g \times I$$

$$\cong \Sigma_g \times [-1, 1]$$



• Covering transformation : $(x, t) \longmapsto (h(x), -t)$

where h is an orientation-reversing involution of Σ_g

st $\Sigma_g / h = P =$ 

- Conversely for $h : \Sigma_g \to \Sigma_g$ ori-rev. involution with $\mathrm{Fix}\, h \neq \emptyset$, the mapping cylinder $C(h) := \Sigma_g \times [0,1] / (x,0) \sim (h(x),0) \cong V_g$

We may regard $h \in MCG_0(V_g)$

i.e. h extends to a homeo $V_g \xrightarrow[\cong]{} V_g$ homotopic to 1_{V_g}

Consider $\tau_h : \Sigma_g \times [-1, 1] \to \Sigma_g \times [-1, 1]$

$$(x, t) \longmapsto (h(x), -t)$$

Then $\Sigma_g \times [-1,1] \longrightarrow \Sigma_g \times [-1,1] / \tau_h \cong C(h) \cong V_g$

is a double branched covering

with monodromy τ_h.

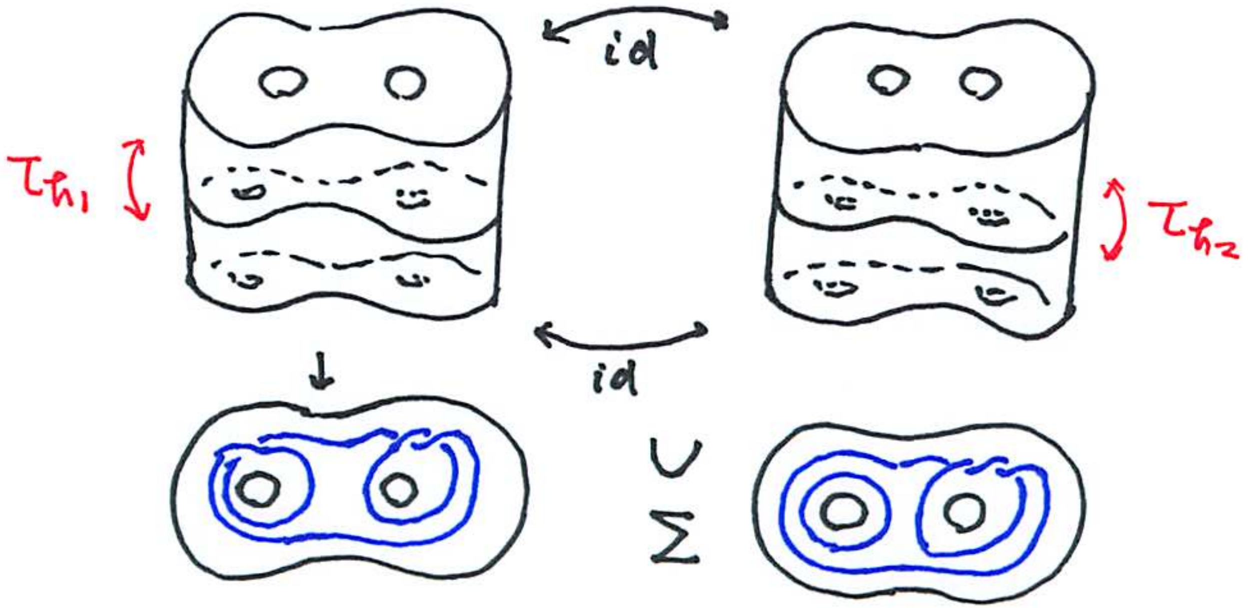
$$M = V_g^{(1)} \underset{\Sigma_g}{\cup} V_g^{(2)}$$ Heegaard decomposition

$$= C(h_1) \underset{\Sigma_g}{\cup} C(h_2)$$ h_i : ori-rev involution of Σ_g st $\mathrm{Fix}\, h_i \neq \emptyset$

$$= \left(\Sigma_g \times [-1, 1] / \tau_{h_1} \right) \cup \left(\Sigma_g \times [-1, 1] / \tau_{h_2} \right)$$

$$= \left(\Sigma_g \times [-1, 1] \underset{\partial}{\cup} \Sigma_g \times [-1, 1] \right) / (\tau_{h_1} \cup \tau_{h_2})$$

$$= M_\phi / \tau$$ where $\phi = h_1 h_2 \in MCG(\Sigma)$



$= M_\phi \quad (\phi = h_1 h_2)$

$\downarrow$ branched covering

$= M$

Summary

$M = V_1 \underset{\Sigma}{\cup} V_2$ Heegaard splitting

$\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ "monodromy group"

$$\Gamma_i = MCG_0(V_i) \subset MCG(\Sigma)$$

$(h_1, h_2) \in \Gamma_1^- \times \Gamma_2^-$ (pair of ori-rev elements)

Then the Σ-bundle M_ϕ with $\phi = h_1 h_2$
is a double branched covering of M.

[Bowditch - Ohshika - S]

For a Heegaard decomposition $M = V_1 \underset{\Sigma}{\cup} V_2$ of high Hempel distance, we have $\Gamma = \Gamma_1 * \Gamma_2$.

Example

For a 2-bridge decomposition with Hempel distance $\geqq 2$, $\Gamma = \Gamma_1 * \Gamma_2 \cong D_\infty * D_\infty$ modulo hyper-elliptic action.

Question

(1) Is Hempel distance $\geqq 2$ enough in [B-O-S] ?

(2) How can we measure the relative position of Γ_1 and Γ_2 in $MCG(\Sigma)$?

(3) Characterize the set $\{ \phi = h_1 h_2 \mid h_i \in \Gamma_i \text{ order } 2 \}$.
Does it consists of only Pseudo-Anosov's if H-d $\geqq 3$?

Motivation

McShane's identity and its variation

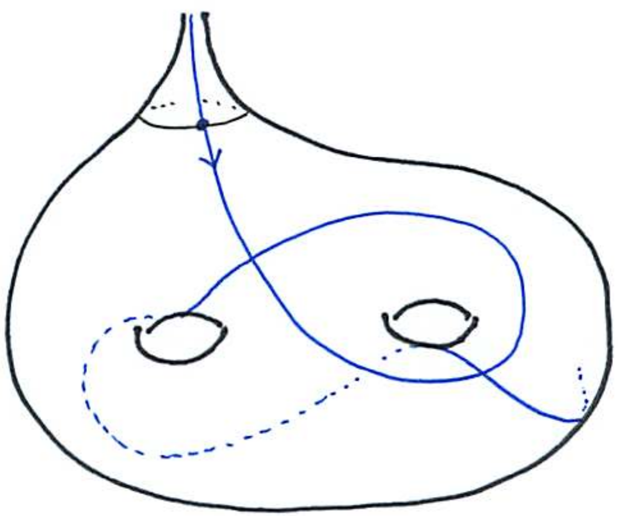
– the role of the monodromy group –

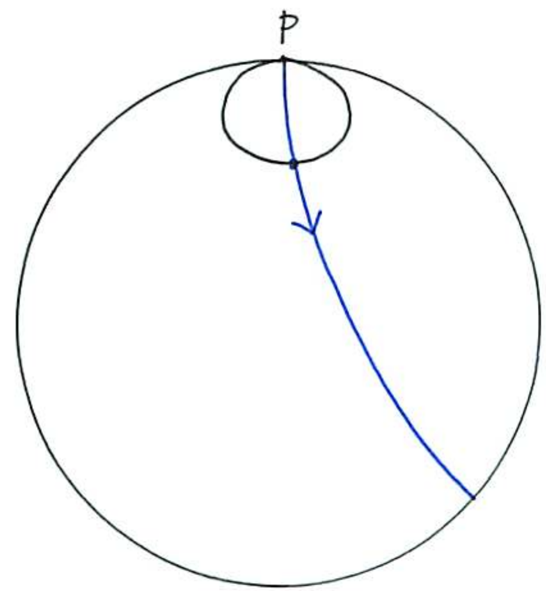
$\Sigma = \mathbb{H}^2/\Gamma$: once-punctured hyperbolic surface of finite area

$\mathcal{G} := \{$ geodesic rays emanating from the puncture $\}$

$\cong$ horo-cycle around the puncture

$\cong \partial\mathbb{H}^2 - \{p\}/\Gamma_p$ $\qquad$ p : parabolic fixed point





$MCG(\Sigma) = \pi_0 \mathrm{Diff}(\Sigma)$ mapping class group of Σ

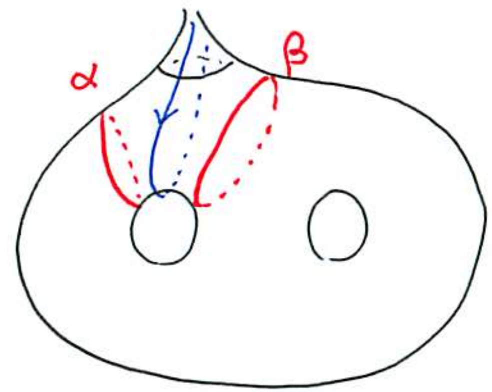
Fact $MCG(\Sigma)$ acts on $\mathcal{G}$.

Problem How does the action of $MCG(\Sigma)$ on $\mathcal{G}$ look like ?

Motivation McShane's identity and its variations

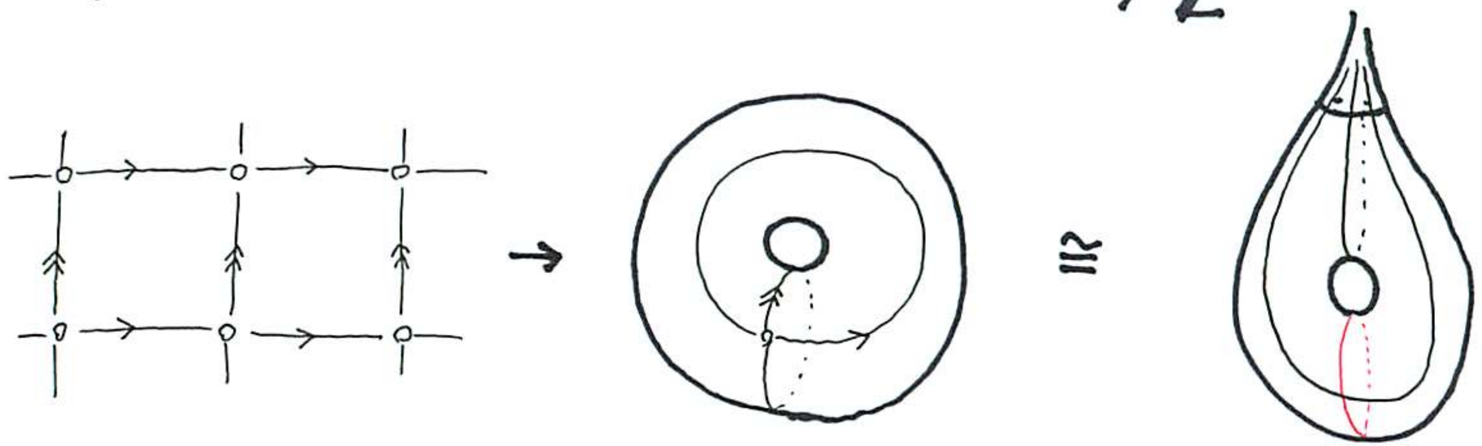
[McShane]

$$\sum_{\delta} \frac{1}{1 + \exp \frac{1}{2}(\ell(\alpha) + \ell(\beta))} = 1$$



where δ runs over proper essential simple arcs joining the puncture to itself.

Once-punctured torus $T \cong \mathbb{R}^2 - \mathbb{Z}^2 / \mathbb{Z}^2$

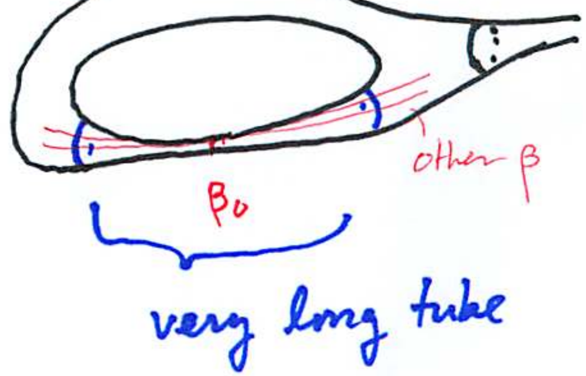


$\mathcal{S} = \{\text{essential simple loop}\}/\sim \;\cong \{\text{line in } \mathbb{R}^2 - \mathbb{Z}^2 \text{ of rational slope}\}$
$\cong \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$

McShane's identity For a complete hyp T,

$$\sum_{\beta} \frac{1}{1+e^{l(\beta)}} = \frac{1}{2}$$

If some loop β_0 is very short



$$\text{L.H.S} \approx \frac{1}{1+e^0} + \frac{1}{1+e^\infty} + \frac{1}{1+e^\infty} = \frac{1}{2}$$

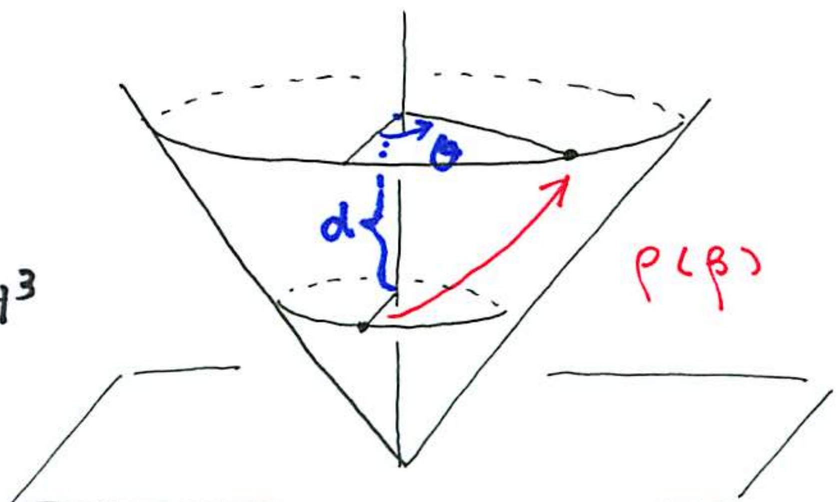
Bowditch's variation for hyperbolic punctured torus bundles

- $MCG^+(T) \cong SL(2,\mathbb{Z}) \ni A$
- $M_A := T \times \mathbb{R} \big/ (x,t) \sim (Ax, t+1)$

  admits a complete hyperbolic structure iff $|\mathrm{Tr}\, A| \geqq 3$
- $\rho : \pi_1(M_A) \hookrightarrow \mathrm{Isom}^+ \mathbb{H}^3 = PSL(2,\mathbb{C})$ discrete faithful
- The complex translation length $l(\rho(\beta))$ of $\rho(\beta) \in \mathrm{Isom}^+ \mathbb{H}^3$

  is defined by $d + i\theta \in \mathbb{C} \big/ 2\pi i \mathbb{Z}$

$\mathbb{H}^3$

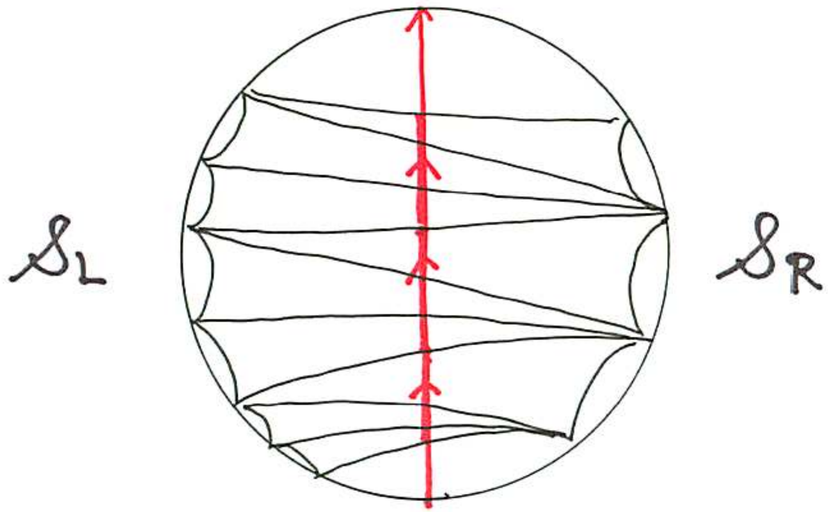


$l(\rho(\beta))$

$=$ (translation length "d")

$+ i$ (rotation angle "θ")

The monodromy group $\langle A \rangle$ acts on $\mathcal{S} \subset \mathcal{PML}(T) = \partial \mathbb{H}^2$

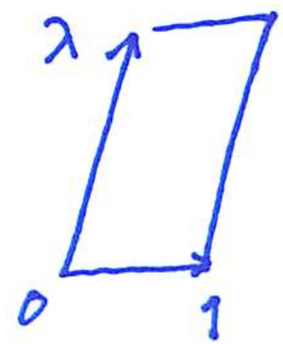


Axis A divides $\mathcal{S}$ into $\mathcal{S}_L \sqcup \mathcal{S}_R$

$$\mathcal{S}/\langle A \rangle = \mathcal{S}_L/\langle A \rangle \sqcup \mathcal{S}_R/\langle A \rangle$$

[Bowditch] The cusp shape of M_A is given by

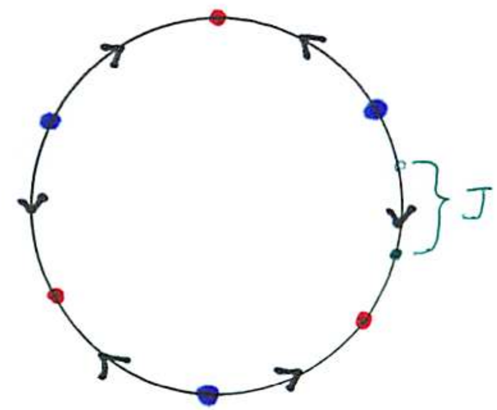
$$\lambda = \sum_{\beta \in \mathcal{S}_L/\langle \phi \rangle} \frac{1}{1 + e^{\ell(\rho(\beta))}}$$



Variation for hyperbolic punctured surface bundles

- $\phi : \Sigma \to \Sigma$: pseudo-Anosov homeo
- $M_\phi = \Sigma \times \mathbb{R} \big/ (x,t) \sim (\phi(x), t+1)$: Σ- bundle over S^1

$$\cong \mathbb{H}^3 / G \qquad \rho : \pi_1(M_\phi) \underset{\cong}{\to} G < PSL(2.\mathbb{C})$$

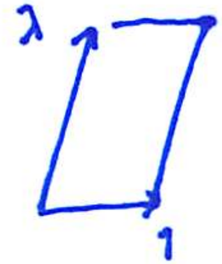


- After taking a power
  the action of ϕ on $\mathcal{G} \cong S^1$ looks like :
- J : a "fundamental interval" for $\phi \curvearrowright \mathcal{G}$

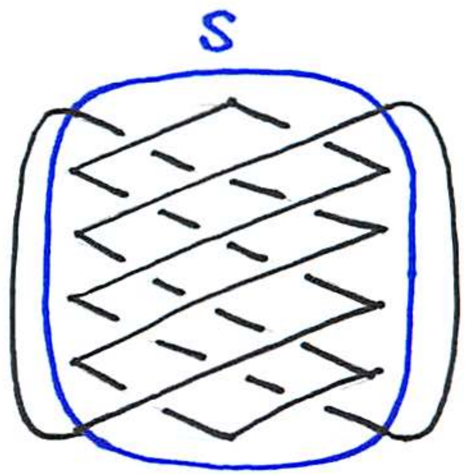
[Bowditch, Akiyoshi - Miyachi - S]

$$\sum_{\delta \in J} \frac{1}{1+\exp\frac{1}{2}\left(\ell_\rho(\alpha)+\ell_\rho(\beta)\right)} = \text{the cusp shape of } \partial M_\phi$$

$\ell_\rho(\alpha)$: complex translation length



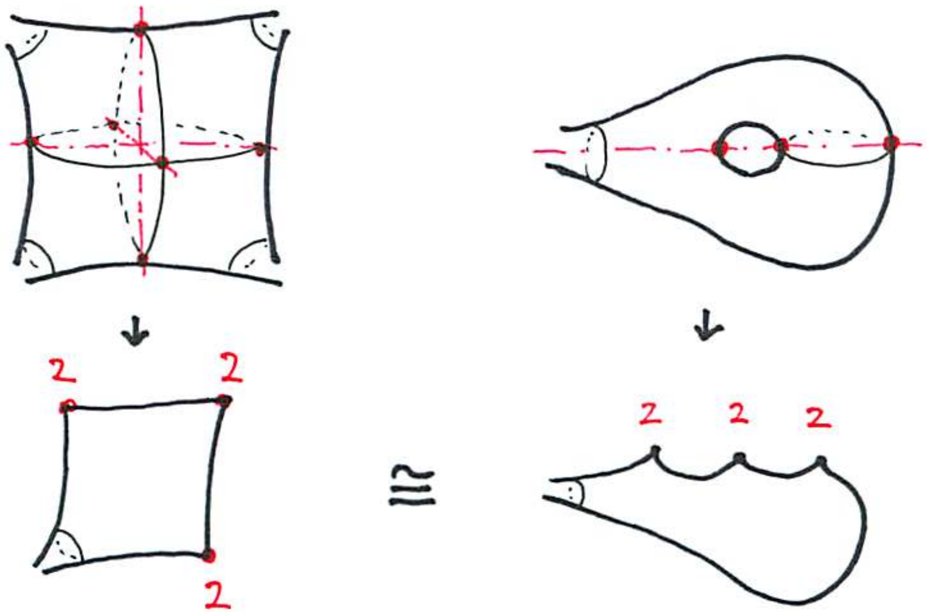
Variation for 2-bridge links



$(S^3, K(\frac{2}{5}))$

- $(S^3, K(r)) = (B^3, t(\infty)) \underset{S}{\cup} (B^3, K(r))$

- 4-punctured sphere S is commensurable with a once-punctured torus T



- The holonomy representation $\rho : \pi_1(S^3 - K) \to PSL(2, \mathbb{C})$

  induces a rep $\rho : \pi_1(T) \to PSL(2, \mathbb{C})$

- $MCG(T) \cong GL(2,\mathbb{Z})$

$$\cup$$

$$G_\infty := \left\{ f \in MCG(T) \;\middle|\; \begin{array}{l} f : S \to S \text{ extends to a homeo of } (B^3, t(\infty)) \\ \text{st } f_* = Id \in Out(\pi_1(B^3 - t(\infty))) \end{array} \right\}$$

$$G_r := \left\{ f \in MCG(T) \;\middle|\; \begin{array}{l} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (B^3, t(r)) \\ \cdots\cdots\cdots\cdots Out(\pi_1(B^3 - t(r))) \end{array} \right\}$$

- $G = \langle G_\infty, G_r \rangle < MCG(T)$

G action on simple loops on S preserves the homotopy class of loops in $S^3 - K(r)$

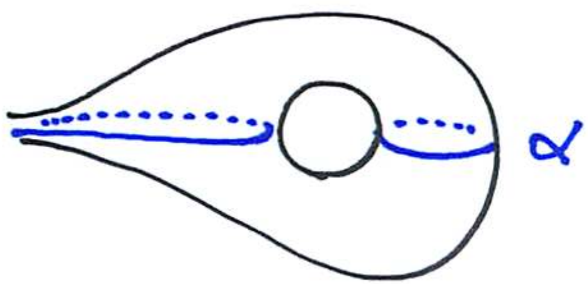
[Lee - S]

$\exists\ J \subset \mathcal{G} \cong S^1$ a "fundamental interval"

for the action of $G \curvearrowright \mathcal{G}$, st

$$\sum_{\alpha \in \mathring{J}} \frac{1}{1+\exp(l_\rho(\alpha))} + \frac{1}{2} \sum_{\alpha \in \partial J} \frac{1}{1+\exp(l_\rho(\alpha))}$$

$=$ Cusp shape of $S^3 - K(r)$



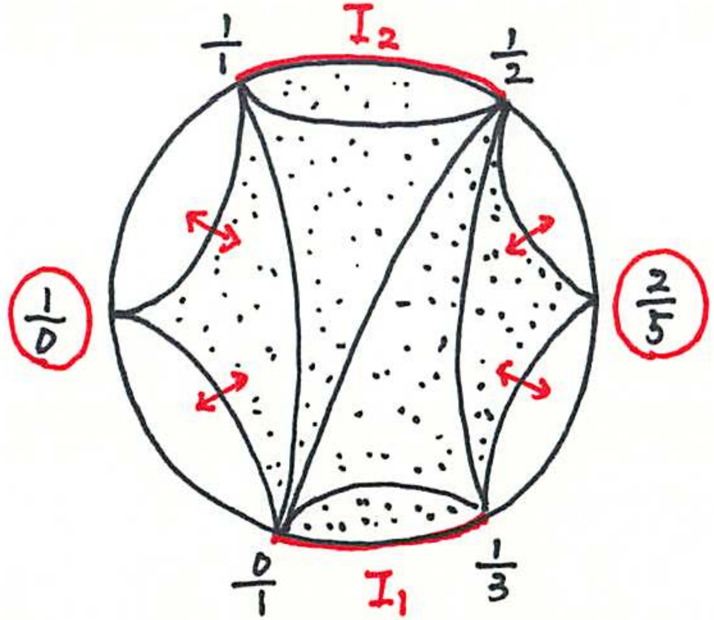
Similarly

$$\mathrm{Aut}(D)$$

$$\cup$$

$$\Gamma_r := \Phi(\pi_o(B^3, t(r))) = \left\langle \begin{array}{l} \text{reflections in the edges of } D \\ \text{with an endpoint } r \end{array} \right\rangle$$

Consider $\hat{\Gamma}_r := \langle \Gamma_\infty, \Gamma_r \rangle < \mathrm{Aut}(D)$



- The limit set $\Lambda(\hat{\Gamma}_r) =$ closure of $\hat{\Gamma}_r \{\infty, r\}$
- $I_1 \cup I_2$ is a fundamental domain of the action of $\hat{\Gamma}_r$ on the domain of discontinuity $\Omega(\hat{\Gamma}_r) := \partial \mathbb{H}^2 - \Lambda(\hat{\Gamma}_r)$

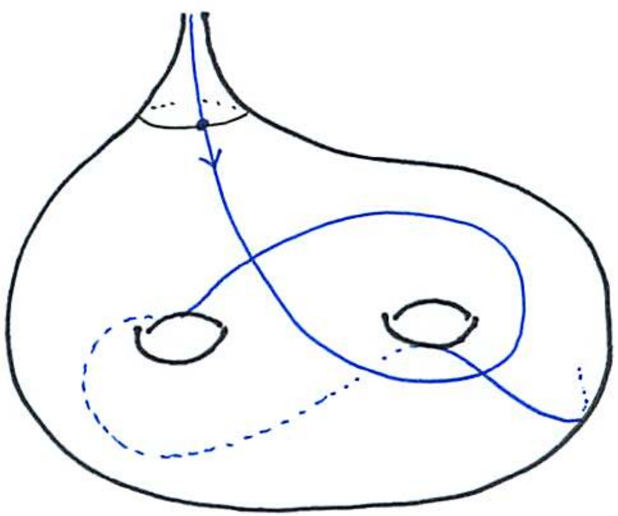
Main Theorem

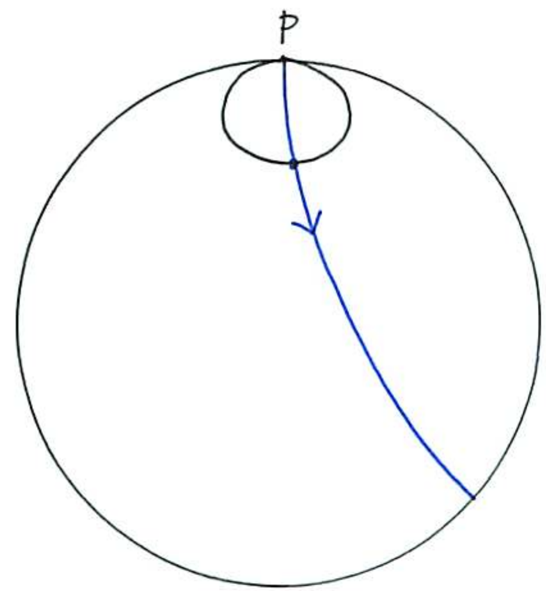
$\Sigma = \mathbb{H}^2/\Gamma$: once-punctured hyperbolic surface of finite area

$\mathcal{G} := \{$ geodesic rays emanating from the puncture $\}$

$\cong$ horo-cycle around the puncture

$\cong \partial\mathbb{H}^2 - \{p\}/\Gamma_p$ $\qquad$ p : parabolic fixed point





Theorem (Bowditch-s)

The non-wandering set of $MCG(\Sigma) \curvearrowright \mathcal{G}$ has measure 0.

Note

- For $G \curvearrowright X$, $x \in X$ is wandering
  $\Leftrightarrow \exists\, U$: nbd of x st $U \cap gU = \emptyset \quad {}^{\forall} g \in G \setminus \{1\}$
- $\mathcal{G} \cong$ "horo-circle of $\Sigma = \mathbb{H}^2/\Gamma$" $\cong \mathbb{R}/\mathbb{Z}$
  So $\mathcal{G}$ inherits the Lebesgue measure of $\mathbb{R}$.
  But, the measure on $\mathcal{G}$ depends on the hyperbolic structure of Σ.

(Idea of Proof)

1. Almost all geodesic rays "fill" Σ.

$g \ni \gamma \rightsquigarrow$ finite or infinite sequence

$\lambda_1(\gamma), \lambda_2(\gamma), \ldots\ldots$

of "simple geodesic arcs"

For almost all γ, $\bigcup_i \lambda_i(\gamma)$ is filling

ie $\forall$ simple loop in Σ intersects $\bigcup_i \lambda_i(\gamma)$

2. If $\varphi \in MCG(\Sigma)$ preserves a filling arc system, then $\varphi = 1$.

Notation

$\Sigma = \mathbb{H}^2/\Gamma$ punctured hyperbolic surface of finite area

$C := \partial\mathbb{H}^2 \ni x, y \rightsquigarrow [x, y]$: oriented geodesic in $\mathbb{H}^2$

$\Pi :=$ { parabolic fixed points of Γ } $\subset C$

$p \in \Pi$ fix, $x \in C \setminus \{p\}$

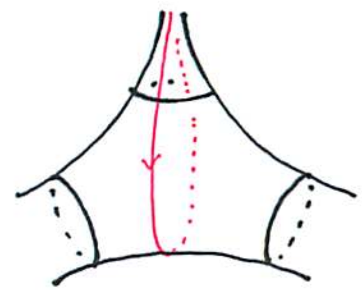
$[p, x]$: simple $\Leftrightarrow$ $\Sigma \supset \pi[p, x]$ is a simple geodesic ray

$[p, x]$: simple proper arc

$\Leftrightarrow$ $\Sigma \supset \pi[p, x]$ is a simple proper arc

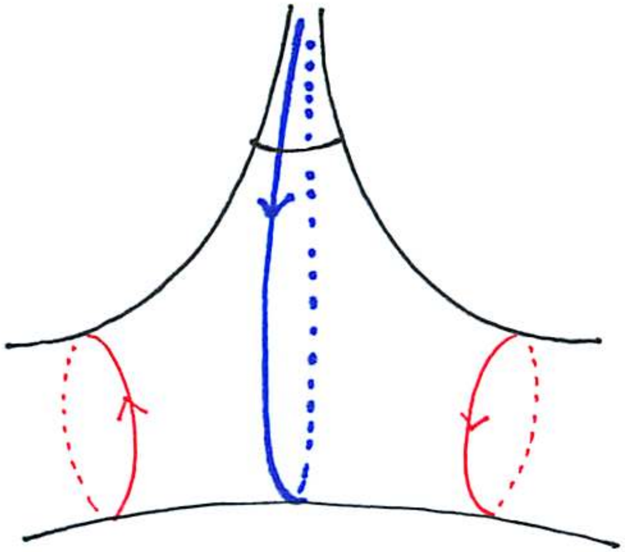
Here $\pi : \mathbb{H}^2 \to \Sigma$ is a projection

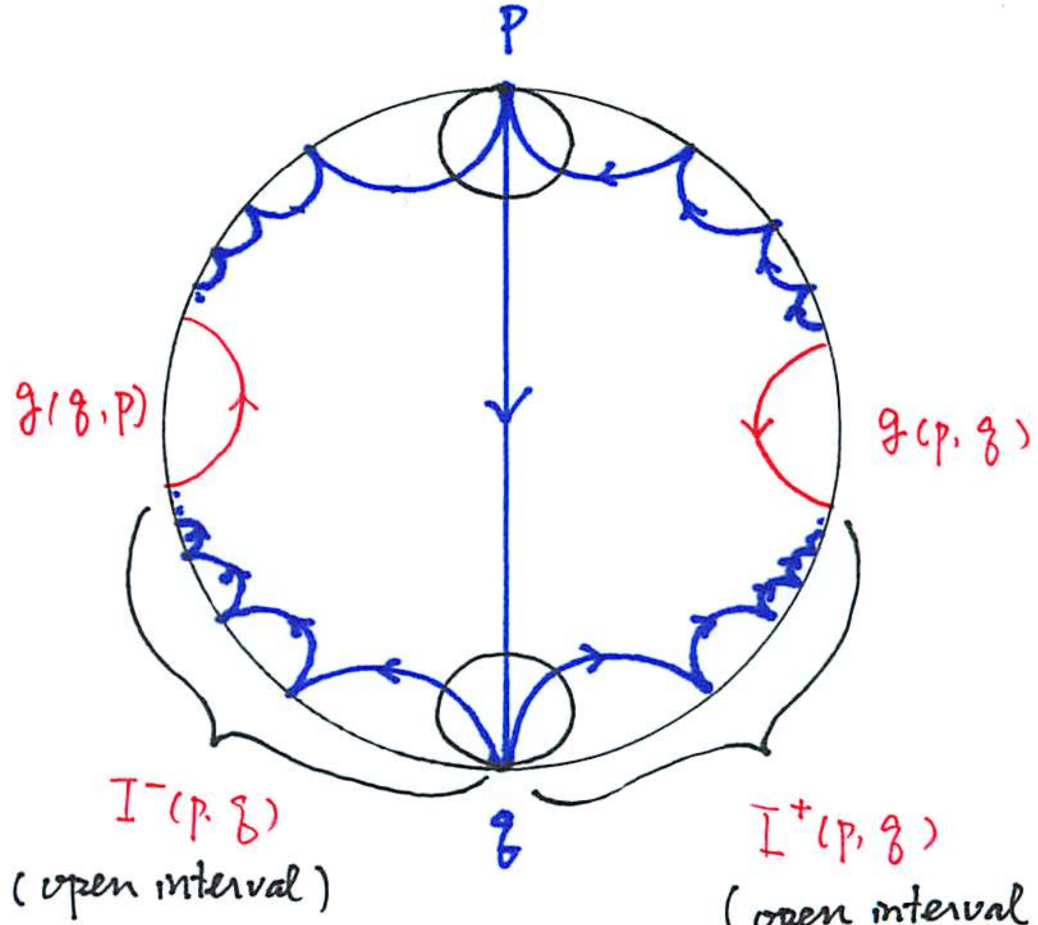
$\Delta(p) :=$ { $q \in \Pi$ | $[p, q]$ is a simple proper arc }



simple proper arc

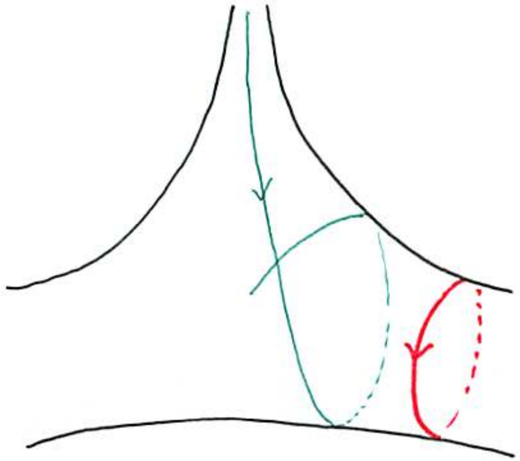
McShane's gap : $q \in \Delta(p)$, $[p, q]$ simple

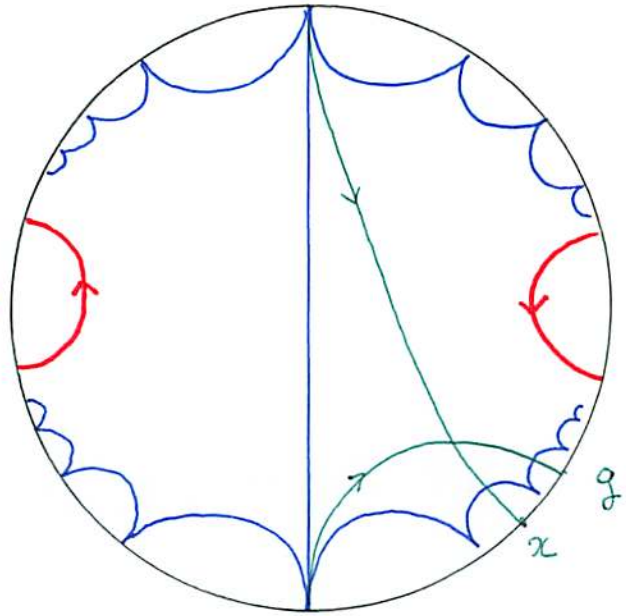




$I^-(p, q) \sqcup I^+(p, q)$: gap at $[p, q]$

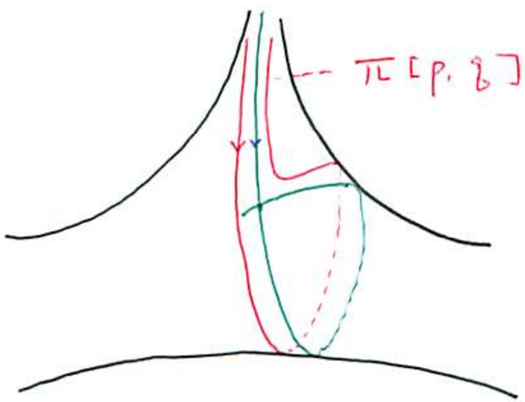
Fact $\forall x \in I^{\pm}(p, q)$, $[p, x]$: non-simple



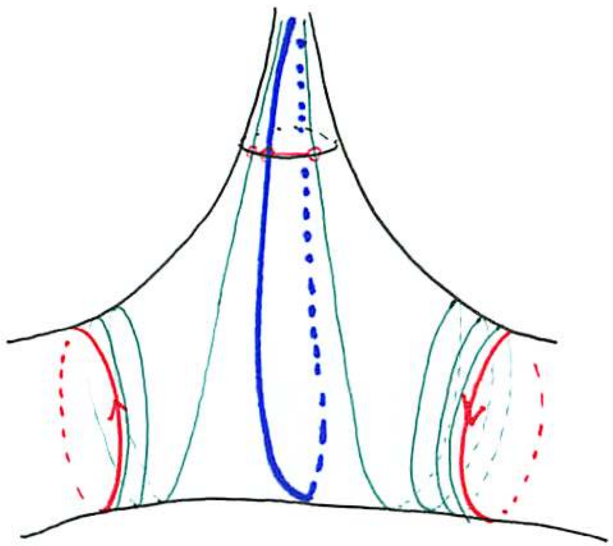


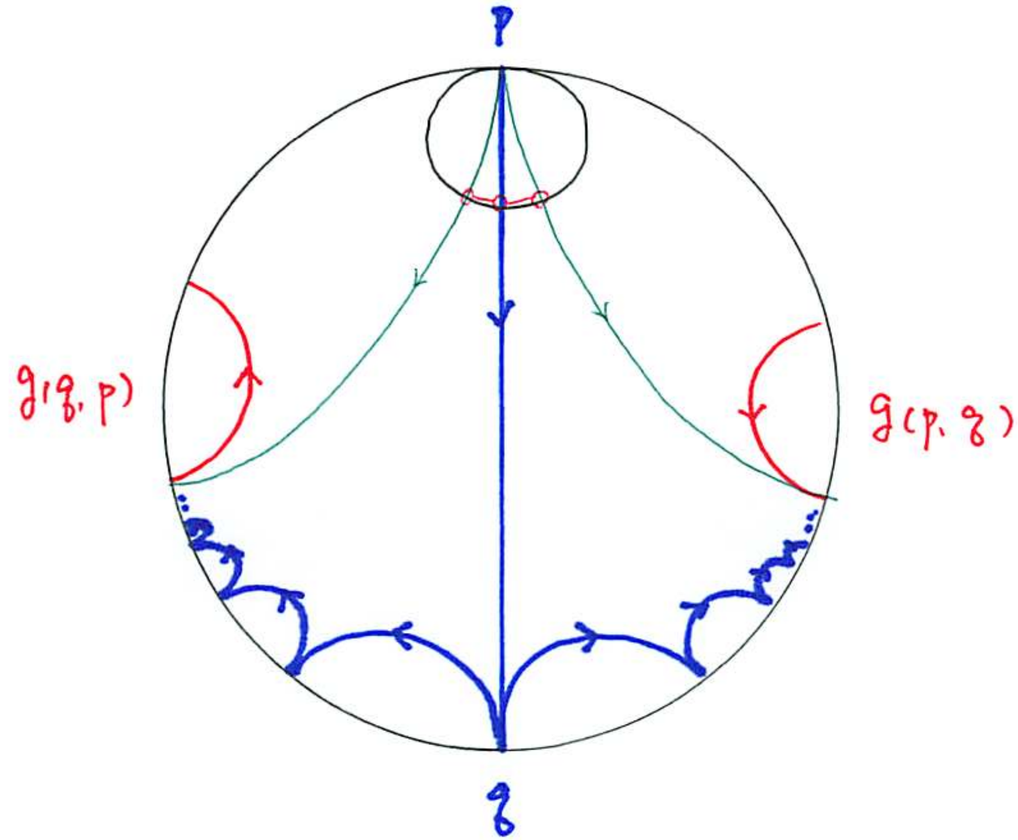
Fact If $[p, x]$ is non-simple, then $\exists!\, q \in \Delta(p)$, $\exists!\, \varepsilon \in \{\pm\}$

st $x \in I^{\varepsilon}(p, q)$



Fact





In the horocircle of length 1, the length of the gap is :

$$\frac{1}{1+\exp\frac{1}{2}\left(\ell(\alpha)+\ell(\beta)\right)}, \quad \text{where } \alpha = g(p,q), \ \beta = g(q,p)$$

Def $\mathcal{I}(p) := \bigsqcup_{\gamma \in \Delta(p)} \left(I^-(p,\gamma) \sqcup I^+(p,\gamma) \right) \subset C \setminus \{p\}$

$\mathcal{R}(p) := C \setminus \left(\{p\} \sqcup \Delta(p) \sqcup \mathcal{I}(p) \right)$

[McShane]

$$\tilde{\mathcal{Q}} := C \setminus \{p\} = \underset{\substack{\updownarrow \\ \text{simple arc}}}{\Delta(p)} \sqcup \underset{\substack{\updownarrow \\ \text{infinite} \\ \text{simple ray}}}{\mathcal{R}(p)} \sqcup \underset{\substack{\updownarrow \\ \text{non-simple}}}{\mathcal{I}(p)}$$

$\Delta(p) \sqcup \mathcal{R}(p)$ has measure 0 in $C \setminus \{p\} \cong \mathbb{R}$

$\mathcal{R}(p) \cong$ Cantor set

McShane's identity

$$1 = \Bigl| \underbrace{(C \setminus \{p\}) / \Gamma_p}_{\substack{\uparrow \\ \text{horocycle} \\ \text{of length } 1}} \Bigr| \underset{\substack{\uparrow \\ \mu(\Delta(p) \sqcup \mathcal{R}(p)) = 0}}{=} \sum_{\gamma \in \Delta(p)/\Gamma_p} \left| I^-(p,\gamma) \sqcup I^+(p,\gamma) \right| = \sum \frac{1}{1 + \exp \frac{1}{2}(\ell(\alpha) + \ell(\beta))}$$

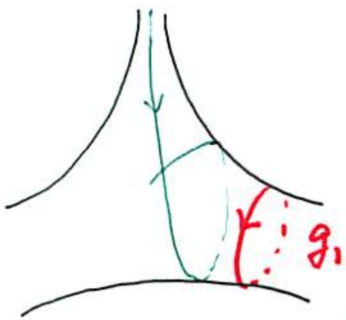
For $x \in l(p)$ (ie $[p, x]$ non-simple), we associate

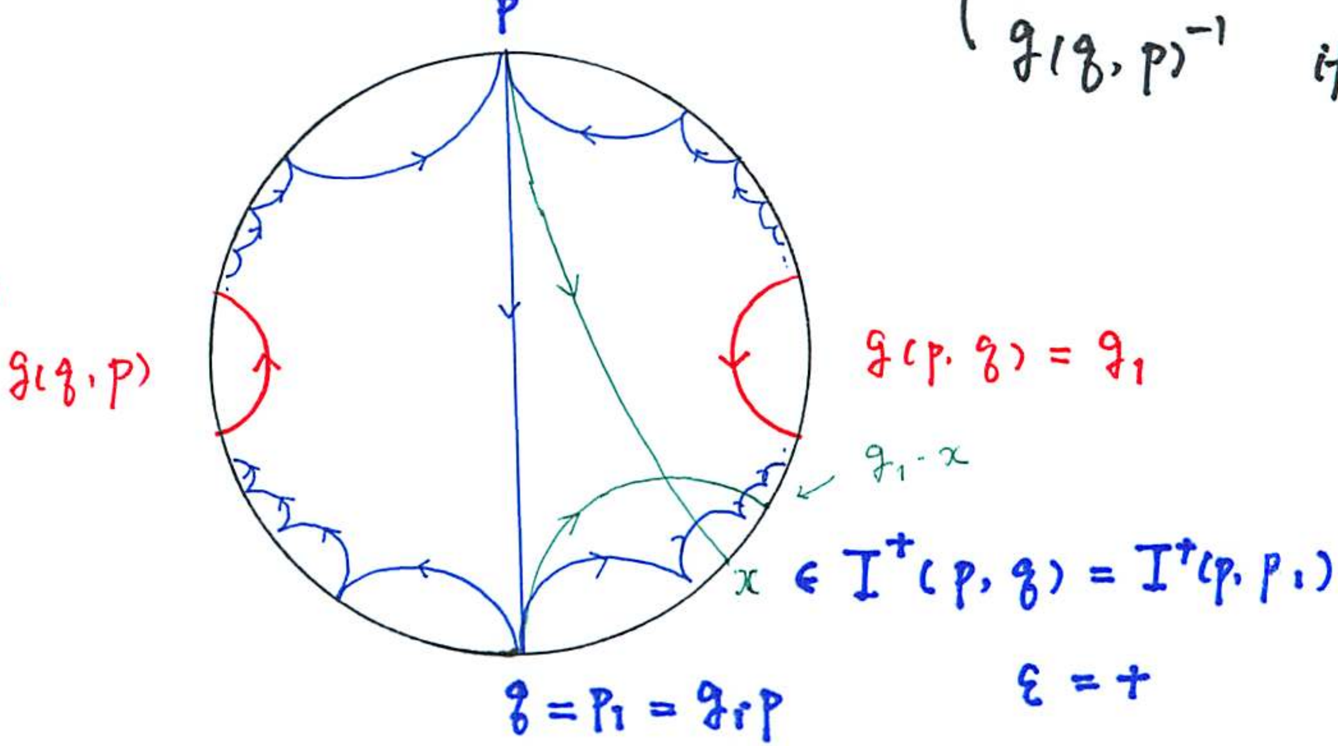
$$p_i = p_i(x) \in \Pi, \quad \varepsilon_i = \varepsilon_i(x) \in \{\pm\}, \quad g_i = g_i(x) \in \Gamma$$

as follows:

Step 1. Since $[p, x]$ non-simple, $\exists! q \in \Delta(p)$ $\exists! \varepsilon \in \{\pm\}$ st $x \in I^{\varepsilon}(p, q)$

Set $p_1 := q$, $\varepsilon_1 := \varepsilon$, $g_1 := \begin{cases} g(p, q) & \text{if } \varepsilon = + \\ g(q, p)^{-1} & \text{if } \varepsilon = - \end{cases}$





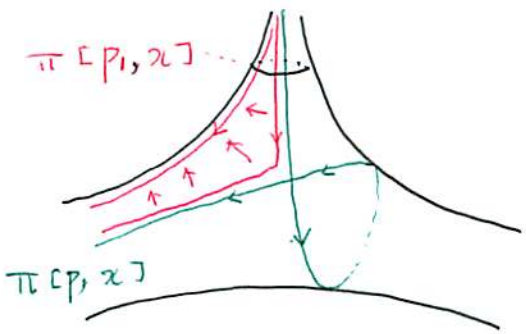
Step 2 Consider the short cut $[p_1, x]$ of $[p, x]$.

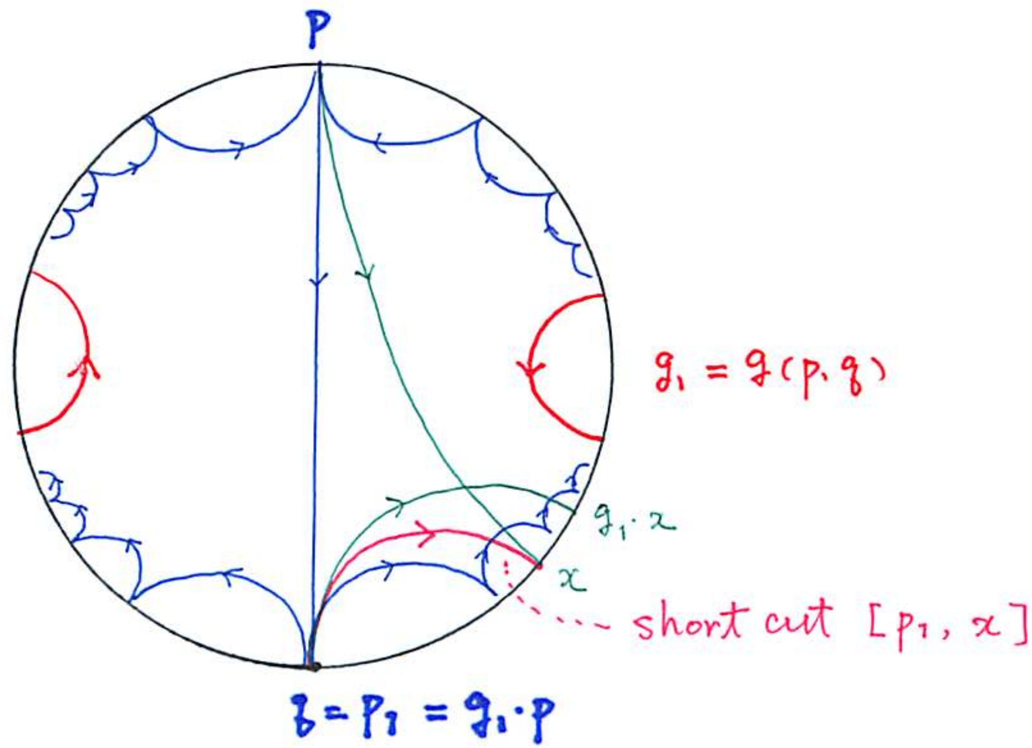
(a) If $x \in \Delta(p_1) \sqcup R(p_1)$ ie $[p_1, x]$ simple, then stop.

(b) If $x \in I(p_1) = \mathbb{C} \setminus (\{p_1\} \sqcup \Delta(p_1) \sqcup R(p_1))$, ie $[p_1, x]$ non-simple

then $\exists!\, q_1 \in \Delta(p_1)$, $\exists\, \varepsilon \in \{\pm\}$, st $x \in I^{\varepsilon}(p_1, q_1)$

Set $p_2 := q_1$, $\varepsilon_2 = \varepsilon$, $g_2 := \begin{cases} g(p_1, q_1) & \text{if } \varepsilon = + \\ g(q_1, p_1)^{-1} & \text{if } \varepsilon = - \end{cases}$





Lemma If $x \in \Pi \setminus \{p\}$, then the sequences are finite.

$(\because)$ · If $x \in \Pi \setminus \{p\}$, then $\pi[p_i, x]$ is a proper arc in Σ for every i.

· The self intersection number of $\pi[p_i . x]$ is monotone decreasing.

Def & Lemma $R := \bigcup_{p \in \Pi} (\Delta(p) \sqcup R(p))$ has measure 0 in C.

$(\because)$ $\Delta(p) \cup R(p)$ has measure 0 in $C \setminus \{p\}$, so in C.

Since Π is countable, $R = \bigcup_{p \in \Pi} (\Delta(p) \sqcup R(p))$ has measure 0.

Observe for $x \in C$

$x \notin R \iff [p. x]$ non-simple for $\forall p \in \Pi$

$\Rightarrow$ The sequences p_i, ε_i, g_i are infinite

Def For $x \in C \setminus R$, $\lambda_i = \lambda_i(x) := \pi[p_{i-1}, p_i]$ simple arc in Σ.

(λ_i) is eventually filling

$\Leftrightarrow$ $\exists n \in \mathbb{N}$, $\bigcup_{i=1}^{n} \lambda_i$ is filling

i.e. $\forall$ essential loop in Σ intersects $\bigcup_{i=1}^{n} \lambda_i$.

Lemma For $x \in C \setminus R$, if (λ_i) is eventually filling, then $[x] \in (C \setminus \{p\}) / \Gamma_p = \mathcal{G}$ is a wandering point of $MCG(\Sigma) \curvearrowright \mathcal{G}$.

(Proof) Suppose $(\lambda_i)_{i=1}^{n} = (\lambda_i(x))_{i=1}^{n}$ is filling. Set $U = \bigcap_{i=1}^{n} I^{\varepsilon_i}(p_{i-1}, p_i)$.

Then for any $y \in U$, $\lambda_i(y) = \lambda_i$ $(1 \le i \le n)$.

Suppose $\phi \in MCG(\Sigma)$ satifies $\phi([U]) \cap [U] \neq \phi$ in $\mathcal{G}$.

Then by the above observation, $\phi(\lambda_i)$ is isotopic to λ_i $(1 \le i \le n)$.

Since $(\lambda_i)_{i=1}^{n}$ is filling, ϕ is isotopic to 1_Σ.

Hence $[x] \in \mathcal{G}$ is wandering w.r.t. $MCG(\Sigma) \curvearrowright \mathcal{G}$.

Def For an essential simple loop $\alpha \subset \Sigma$,

$X(\alpha)$: the component of $X \setminus \alpha$ containg the puncture of Σ.

$G(\alpha) = \pi_1(X(\alpha))$, where $\Gamma_p < G(\alpha) < \Gamma$

$\mathcal{L} := \bigcup \{ \Lambda(G(\alpha)) \mid \alpha \subset \Sigma \text{ ess. simple loop} \}$

Note $\mathcal{L}$ has measure 0, and so $\mathcal{L} \cup R$ has measure 0.

Thus the theorem is a consequence of the following:

Prop For ${}^\forall x \in \mathbb{C} \setminus (R \cup \mathcal{L})$, (λ_i) is eventually filling.

(Proof) Suppose $(\lambda_i) = (\lambda_i(x))$ is not eventually filling for some $x \in \mathbb{C} \setminus R \cup \mathcal{L}$.

Then $\exists \alpha$: ess simple loop st $\lambda_i \subset X(\alpha)$ for ${}^\forall i$.

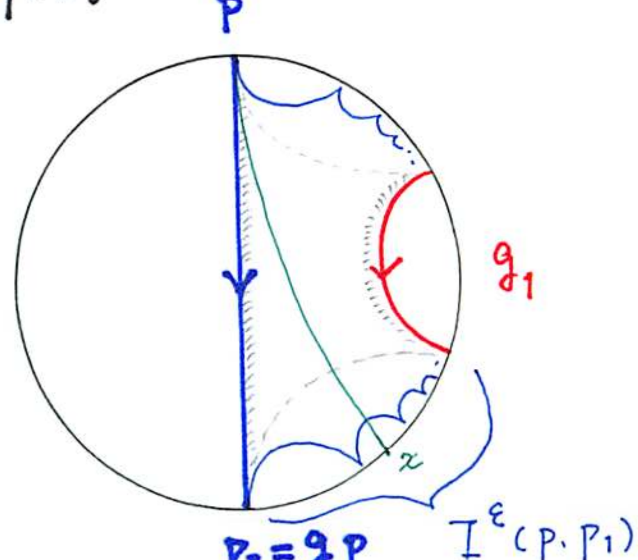
Since $x \notin \mathcal{L}$, $x \notin \Lambda(G(\alpha)) \cong$ Cantor set.

Let J be the component of $\Omega(G(\alpha)) = \mathbb{C} - \Lambda(G(\alpha))$, st $x \in J$.

Recall the sequences

$$(p_i) \subset \Pi, \quad (\varepsilon_i) \subset \{\pm\}, \quad (g_i) \subset \Gamma \quad \text{for } x \in J \subset \Omega(G(\alpha)).$$

Claim 1 If $g_1 \in G(\alpha)$, then $J \subset I^{\varepsilon}(p, p_1)$.



(Proof) Since $p \in \Lambda(G(\alpha))$,

$p_1 = g_1 \cdot p \in \Lambda(G(\alpha))$ if $g_1 \in G(\alpha)$.

So, $[p, p_1] \subset \mathcal{C}(\Lambda(G(\alpha)))$.

Also $\operatorname{axis}(g_1) \subset \mathcal{C}(\Lambda(G(\alpha)))$.

Since $x \in I^{\varepsilon}(p, p_1)$, the component J of $\Omega(G(\alpha))$ containg x, must be contained in $I^{\varepsilon}(p, p_1)$.

Cor If $g_1 \in G(\alpha)$, then $g_1(y) = g_1 = g_1(x)$ for $\forall y \in J$.

Claim 2 $\exists\, g_i \notin G(\alpha)$

(Proof) Assume $g_i \in G(\alpha)$ for $\forall i \in \mathbb{N}$.

Then $p_i = g_i \cdot g_{i-1} \cdots g_1 \cdot p \in G(\alpha) \cdot p \subset \Lambda(G(\alpha))$.

So, we can apply Claim 1 and its Cor to conclude

$g_i(y) = g_i = g_i(x)$ (for $\forall i$) for $\forall y \in J$.

In particular $(g_i(y))_i$ is infinite for $\forall y \in J$.

But, there is $y \in \Pi \cap J$, for which $(g_i(y))_i$ is finite. ※

(Final step of Proof of Prop.)

By Claim 2, we may assume $g_1, \cdots, g_{i-1} \in G(\alpha)$ and $g_i \notin G(\alpha)$.

Then $p_{i-1} \in \Lambda(G(\alpha))$, but $p_i \notin \Lambda(G(\alpha))$.

Thus $[p_{i-1}, p_i]$ intersects $\partial e(\Lambda(G(\alpha)))$.

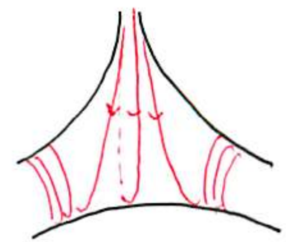
This contradicts the assumption that $\lambda_i = \pi[p_{i-1}, p_i] \subset X(\alpha)$ $\square$

The space of simple geodesic rays

$$\mathcal{SG} := \mathcal{G} \big/ (\text{McShane's gap})$$

$$= \{\text{Simple geodesic rays}\} \big/ \sim$$

$$\cong \bigsqcup S^1$$



Problem

- Let (B^3, t) be a trivial tangle and $\Sigma = \partial B^3 - t$. Does the action of $MCG_0(B^3, t)$ on $\mathcal{SG}$ have a non-empty domain of discontinuity?
- For a bridge decomposition $(S^3, K) = (B^3, t_1) \cup (B^3, t_2)$, does the action of $\langle MCG_0(B^3, t_i)\ (i = 1, 2)\rangle$ on $\mathcal{SG}$ have a non-empty domain of discontinuity?

ご清聴

ありがとうございました