Cyclic branched covers of knots and $L$-spaces

Masakazu Teragaito

Hiroshima University

Branched coverings, degenerations, and related topics 2015
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2. Cyclic branched covers

3. Quasi-alternating links and $Q$-polynomials
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3. Quasi-alternating links and $Q$-polynomials
$L$–space (Ozsváth-Szabó 2005)

**Definition**

A rational homology 3-sphere $Y$ is an $L$–space if its Heegaard Floer homology $\widehat{HF}(Y)$ is a free abelian group with rank equal to $|H_1(Y; \mathbb{Z})|$.

- $S^3$, Poincaré homology sphere
- lens spaces
- elliptic manifolds
- double branched covers over non-split alternating knots/links

* In general, $\text{rank} \, \widehat{HF}(Y) \geq |H_1(Y; \mathbb{Z})|$ for any rational homology sphere $Y$. 
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**L–space conjecture**

It is an open problem to find a non-Heegaard Floer characterization of \(L\)–spaces.

**Conjecture (Boyer-Gordon-Watson 2011)**

Let \(Y\) be an irreducible rational homology sphere. Then \(Y\) is an \(L\)–space if and only if \(\pi_1(Y)\) is not left-orderable.

**Left-orderable**

A non-trivial group \(G\) is **left-orderable** if \(G\) admits a total order such that

\[
a < b \implies ga < gb \quad \text{for any } g, a, b \in G
\]

They confirmed the conjecture for Seifert fibered manifolds, Sol-manifolds, etc.
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Cyclic branched covers

Theorem

Let $L$ be a non-split alternating link in $S^3$. Then,

- $\Sigma_2(L)$ is an $L$–space; [Ozsváth-Szabó]
- $\pi_1 \Sigma_2(L)$ is not left-orderable. [Ito, Greene, BGW]

Examples.

\[
\begin{align*}
\Sigma_2(L) & = L(2,1) \\
\pi_1 \Sigma_2(L) & = \mathbb{Z}_2 \\
\Sigma_2(K) & = L(5,2) \\
\pi_1 \Sigma_2(K) & = \mathbb{Z}_5
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Basic problem

Problem
Which cyclic branched cover of a knot or link is an \( L \)-space?

- Frontal attack: calculate \( \widehat{HF}(\Sigma_d(L)) \). [Levine, Grigsby]
- Restrict to \( \Sigma_2(L) \).
- Fix \( L \) or a class, and study \( \Sigma_d(L) \) (\( d \geq 2 \)).
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Simple questions

Questions

1. Is there a knot/link all of whose cyclic branched covers are $L$–spaces?
2. Is there a knot/link none of whose cyclic branched covers are $L$–spaces?

Answers.

1. Yes.
2. Yes.
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Answer to Question 1

Think $\mathbb{Z}_2 \oplus \mathbb{Z}_d$-cover of $A \cup k$. ($A$ and $k$ are interchangeable.)

All cyclic branched covers of the figure-eight knot are $L$–spaces.

This trick is applicable for 2-bridge knots $C[2b_1, 2b_2, \ldots, 2b_n]$, where all $b_i > 0$. 
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Think $\mathbb{Z}_2 \oplus \mathbb{Z}_d$-cover of $A \cup k$. ($A$ and $k$ are interchangeable.)
Answer to Question 2

(3, 7)-torus knot

Gordon-Lidman 2014

For \((p, q)\)-torus knot \(K\),

\[\Sigma_d(K) \text{ is an } L\text{-space} \iff \pi_1 \Sigma_d(K) \text{ is finite}\]
**Answer to Question 2**

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Further question

**Corollary**

\[ \Sigma_d(\text{trefoil}) \text{ is an } L \text{-space } \iff d \leq 5 \]

**Question [Gordon-Lidman]**

For a knot or link \( L \), if there exists \( d \geq 2 \) such that \( \pi_1 \Sigma_d(L) \) is left-orderable, then is \( \pi_1 \Sigma_e(L) \) left-orderable for any \( e \geq d \)?

**Another evidence [Hu]**

For a 2-bridge knot \( S(p, q) \) with \( p \equiv 3 \pmod{4} \), there exists \( N \) such that \( \pi_1 \Sigma_d(S(p, q)) \) is left-orderable for any \( d \geq N \).
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Example

Let $K$ be the 2-bridge knot $5_2 = S(7, 2)$.

- If $d \geq 9$, then $\pi_1 \Sigma_d(K)$ is left-orderable. \cite{Hu}
- $\Sigma_d(K)$ is an $L$–space for $d = 2, 3, 4, 5$. \cite{Peters, Te, Hori}
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Select a class

Problem
Which cyclic branched cover of a knot is an $L$–space?

- torus knot: Solved.
- cable knot
- doubled knot
- alternating knot
- 2-bridge knot
Select a class

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Which cyclic branched cover of a **knot** is an \( L \)-space?

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Alternating knot

Let $K$ be an alternating knot.

Ozsváth-Szabó 2005

$\Sigma_2(K)$ is an $L$–space.

3-fold covers

Is $\Sigma_3(K)$ an $L$–space?

No! $\rightarrow$ (2, 7)-torus knot
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Genus one alternating knot

Theorem

Let $K$ be a genus 1 alternating knot. Then,

- $\Sigma_3(K)$ is an $L$–space. \hfill [Te]
- $\pi_1 \Sigma_3(K)$ is not left-orderable. \hfill [GL]
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**Theorem**

Let $K$ be a genus 1 alternating knot. Then,

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Two-bridge knot

- $\Sigma_2$ is a lens space, so an $L$–space.
- For $C[2b_1, 2b_2, \ldots, 2b_n]$ with $b_i > 0$, all $\Sigma_d$ is an $L$–space.
- In general, hard to handle.
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Genus one 2-bridge knot

There are two types.

\[ C[2m, 2n] \quad \text{and} \quad C[2m, -2n] \]
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For \( K = C[2m, 2n] \), \( m, n > 0 \), \( \Sigma_d(K) \) is an L–space for any \( d \geq 2 \).

Theorem

For \( K = C[2m, -2n] \), \( m, n > 0 \), \( \Sigma_d(K) \) is an L–space for

- \( d = 3 \)  
- \( d = 4 \)  
- \( d = 5 \)

[Peters]  
[Te]  
[Hori]

Conjecture

For \( K = C[2m, -2n] \), \( m, n > 0 \), \( \Sigma_d(K) \) is not an L–space if \( d \geq 6 \).
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**Theorem**

For $K = C[2m, 2n]$, $m, n > 0$, $\Sigma_d(K)$ is an $L$–space for any $d \geq 2$.

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[Peter]

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Idea

$5_2 = C'[2, -4]$. 
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3-fold cover of $K$
Idea

- Is this alternating?
- No! (This is $9_{49}$.)
- But, it is quasi-alternating.

Ozsváth-Szabó

If $L$ is quasi-alternating, then $\Sigma_2(L)$ is an $L$–space.
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Quasi-alternating link

A quasi-alternating link (QA) is defined recursively.

- The unknot is QA.
- If a diagram of a link $L$ contains a QA-crossing, then $L$ is QA. Here, a crossing is QA if two resolution $L_\infty$, $L_0$ satisfy
  - $L_\infty$ and $L_0$ are QA.
  - $\text{det } L = \text{det } L_\infty + \text{det } L_0$

In particular, any alternating knot or non-split alternating link is QA.
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\[
\begin{array}{c}
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\includegraphics[width=0.3\textwidth]{link_diagram.png}
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In particular, any alternating knot or non-split alternating link is QA.
Example 1

\[ 8_{21} \]

\[ \text{det}=15 \quad \rightarrow \quad \text{det}=2 \]

\[ \text{det}=13 \quad \rightarrow \quad \text{det}=2 \]
Example 2

\begin{align*}
\text{det}=25 & \quad \Rightarrow \quad \text{det}=5 \\
\text{det}=20 & \quad \Rightarrow \quad \text{det}=5
\end{align*}
$Q$-polynomial

For an unoriented link, the $Q$-polynomial $Q_L \in \mathbb{Z}[x, x^{-1}]$ is defined as follows.

- For the unknot $U$, $Q_U = 1$.
- $Q_{L_+} + Q_{L_-} = x(Q_{L_\infty} + Q_{L_\infty})$

\[ 
\begin{array}{cccc}
\times & \times & \times & \times \\
L_+ & L_- & L_\infty & L_0 \\
\end{array}
\]
Basic problem

Problem

Determine whether a given link is QA or not.

Properties of QA-links

- $\Sigma_2(L)$ is an $L$-space.
- $\Sigma_2(L)$ bounds $H_1$-torsion free, negative-definite 4-manifold.
- Homologically thin (knot Floer, reduced Khovanov, reduced odd Khovanov)
  i.e. supported on a single diagonal
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Qazaqzeh-Chbili’s work (2014)

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*If a link $L$ is QA, then*

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**$K = 8_{19}$**

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If a link $L$ is QA, then either,

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1. Figure-eight knot $K$ is alternating, so QA. Since $\deg Q_K = 3$, $\det K = 5$, the above evaluation is optimal.
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For non-alternating knots $12_{n025}, 12_{n093}, 12_{n0115}, 12_{n0138}, 12_{n0199}, 12_{n0355}, 12_{n0374},$

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Qazaqzeh-Chbili

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Greene (Heegaard Floer Theory)

For a QA link,

\begin{array}{|c|c|c|c|}
\hline
\text{det} & 1 & 2 & 3 \\
\hline
\text{knot/link} & \text{unknot} & \text{Hopf link} & \text{trefoil} \\
\hline
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For a QA link \( L \),

1. if \( \det L = 4 \), then \( L \) is the \((2, \pm4)\)-torus link or \( \deg Q_L \leq 2 \);
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Induction on $\det L$

For a QA link $L$, think QA resolutions $L_\infty$ and $L_0$.

If neither $L_\infty$ nor $L_0$ is a $(2, n)$-torus link,

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References

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