Cyclic branched covers of knots and L-spaces

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Branched coverings, degenerations, and related topics 2015

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L–space (Ozsváth-Szabó 2005)

Definition

A rational homology 3-sphere Y is an *L–space* if its Heegaard Floer homlogy $\widehat{HF}(Y)$ is a free abelian group with rank equal to $|H_1(Y;\mathbb{Z})|$.

- S^3, Poincaré homology sphere
- lens spaces
- elliptic manifolds
- double branched covers over non-split alternating knots/links

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L–space conjecture

It is an open problem to find a non-Heegaard Floer characterization of L–spaces.

Conjecture (Boyer-Gordon-Watson 2011)

Let Y be an irreducible rational homology sphere.
Then Y is an L–space if and only if $\pi_1(Y)$ is not left-orderable.

Left-orderable

A non-trivial group G is left-orderable if G admits a total order such that

$$a < b \Longrightarrow ga < gb \quad \text{for any } g, a, b \in G$$

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Cyclic branched covers

Theorem

Let L be a non-split alternating link in S^3. Then,

- $\Sigma_2(L)$ *is an* L*–space;* [Ozsváth-Szabó]
- $\pi_1\Sigma_2(L)$ *is not left-orderable.* [Ito, Greene, BGW]

Examples.

$$\Sigma_2(L) = L(2,1) \qquad \Sigma_2(K) = L(5,2)$$
$$\pi_1\Sigma_2(L) = \mathbb{Z}_2 \qquad \pi_1\Sigma_2(K) = \mathbb{Z}_5$$

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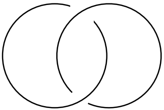

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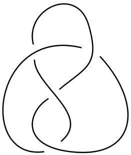

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Basic problem

Problem

Which cyclic branched cover of a knot or link is an L–space?

- Frontal attack: calculate $\widehat{HF}(\Sigma_d(L))$. [Levine, Grigsby]
- Restrict to $\Sigma_2(L)$.
- Fix L or a class, and study $\Sigma_d(L)$ $(d \geq 2)$.

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1. Is there a knot/link all of whose cyclic branched covers are L–spaces?
2. Is there a knot/link none of whose cyclic branched covers are L–spaces?

Answers.

1. Yes.
2. Yes.

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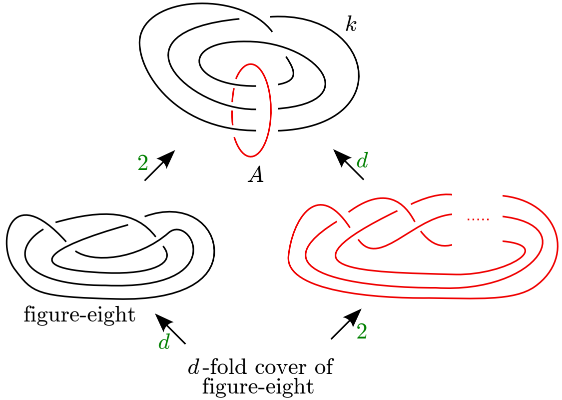

Think $\mathbb{Z}_2 \oplus \mathbb{Z}_d$-cover of $A \cup k$.
(A and k are interchangeable.)

All cyclic branched covers of the figure-eight knot are L–spaces.

This trick is applicable for 2-bridge knots $C[2b_1, 2b_2, \ldots, 2b_n]$, where all $b_i > 0$.

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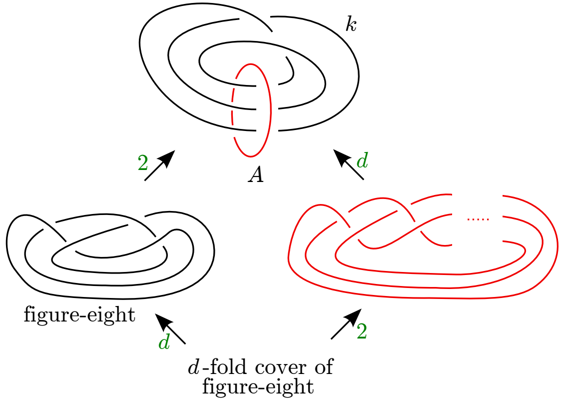

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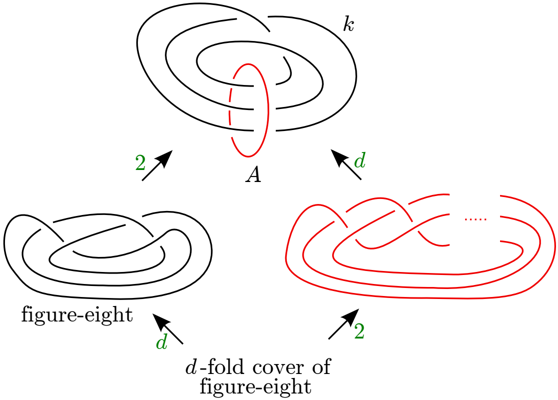

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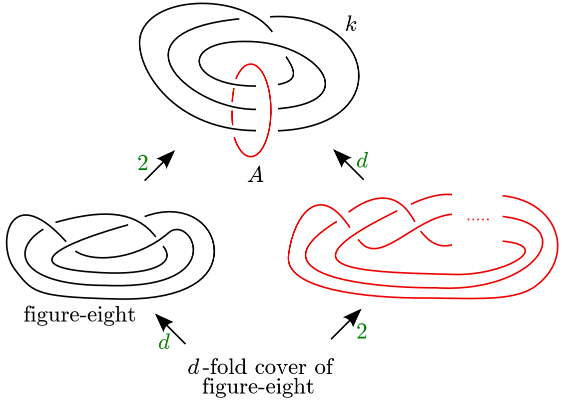

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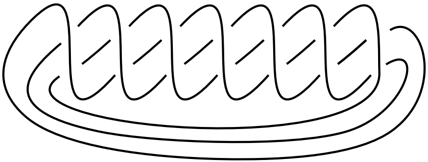
$(3,7)$-torus knot



Gordon-Lidman 2014

For (p,q)-torus knot K,

$$\Sigma_d(K) \text{ is an } L\text{–space} \iff \pi_1 \Sigma_d(K) \text{ is finite}$$

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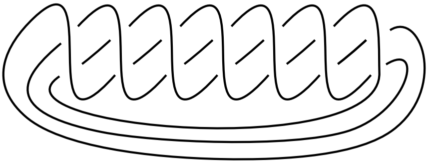

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Further question

Corollary

Σ_d(trefoil) is an L–space $\Longleftrightarrow d \leq 5$

Question [Gordon-Lidman]

For a knot or link L, if there exists $d \geq 2$ such that $\pi_1 \Sigma_d(L)$ is left-orderable, then is $\pi_1 \Sigma_e(L)$ left-orderable for any $e \geq d$?

Another evidence [Hu]

For a 2-bridge knot $S(p, q)$ with $p \equiv 3 \pmod 4$, there exists N such that $\pi_1 \Sigma_d(S(p, q))$ is left-orderable for any $d \geq N$.

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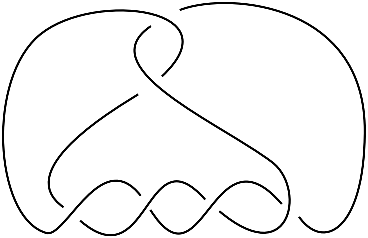

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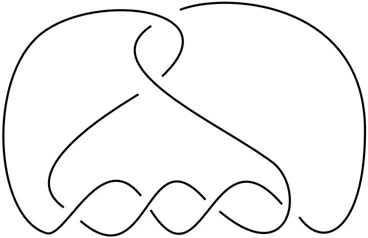

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Which cyclic branched cover of a knot is an L–space?

- torus knot: Solved.
- cable knot
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- alternating knot
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Is $\Sigma_3(K)$ an L–space?

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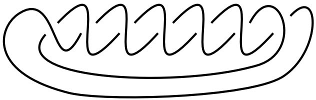

Genus one alternating knot

Theorem

Let K be a genus 1 alternating knot. Then,

- $\Sigma_3(K)$ *is an L–space.* [Te]
- $\pi_1\Sigma_3(K)$ *is not left-orderable.* [GL]

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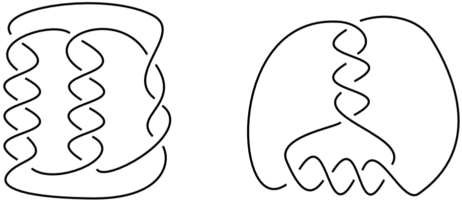

Two-bridge knot

- Σ_2 is a lens space, so an L–space.
- For $C[2b_1, 2b_2, \ldots, 2b_n]$ with $b_i > 0$, all Σ_d is an L–space.
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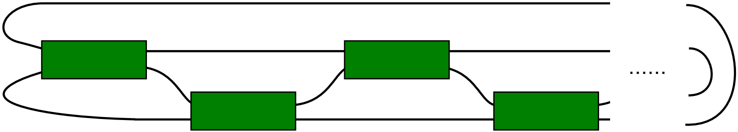

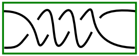

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$C[2m, 2n]$ $C[2m, -2n]$

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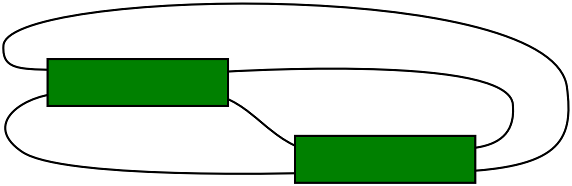

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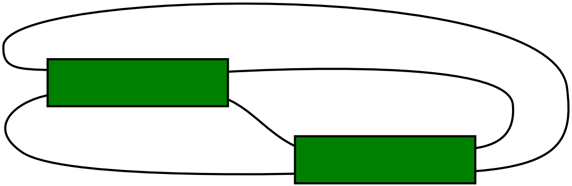

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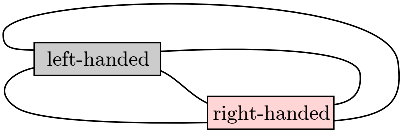

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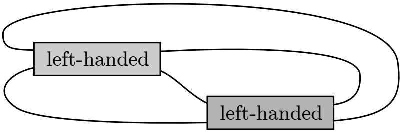

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- $d = 3$ [Peters]
- $d = 4$ [Te]
- $d = 5$ [Hori]

Conjecture

For $K = C[2m, -2n]$, $m, n > 0$, $\Sigma_d(K)$ is not an L–space if $d \geq 6$.

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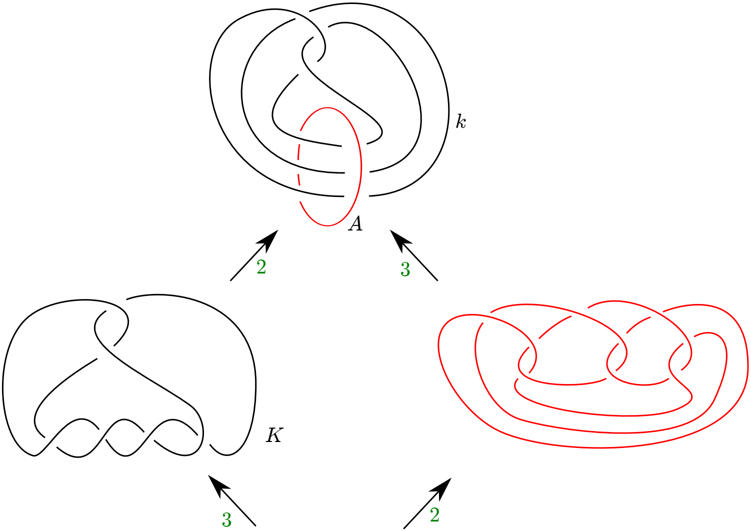

3-fold cover of K

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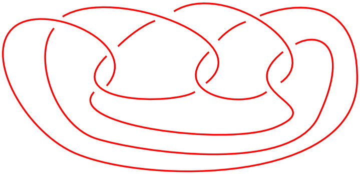

- Is this alternating?
- No! (This is 9_{49}.)
- But, it is quasi-alternating.

Ozsváth-Szabó

If L is quasi-alternating, then $\Sigma_2(L)$ is an L–space.

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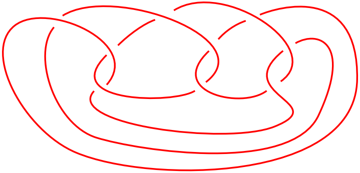

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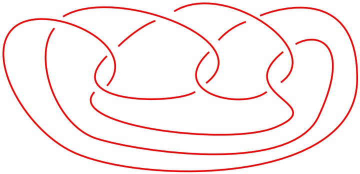

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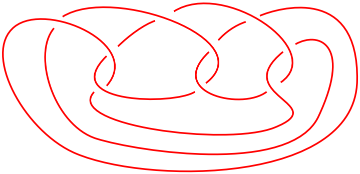

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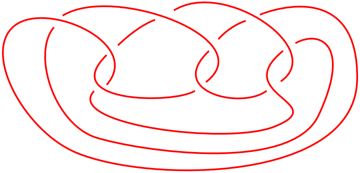

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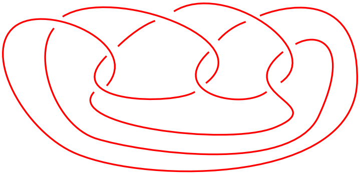

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Quasi-alternating link

A quasi-alternating link (QA) is defined recursively.

- The unknot is QA.
- If a diagram of a link L contains a QA-crossing, then L is QA. Here, a crossing is QA if two resolution L_∞, L_0 satisfy
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  - $\det L = \det L_\infty + \det L_0$

In particular, any alternating knot or non-split alternating link is QA.

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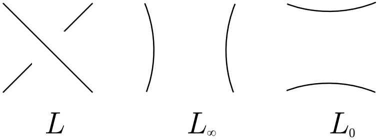

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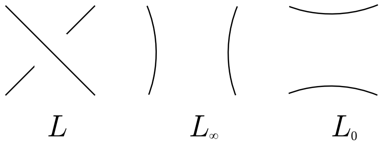

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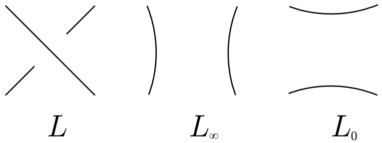

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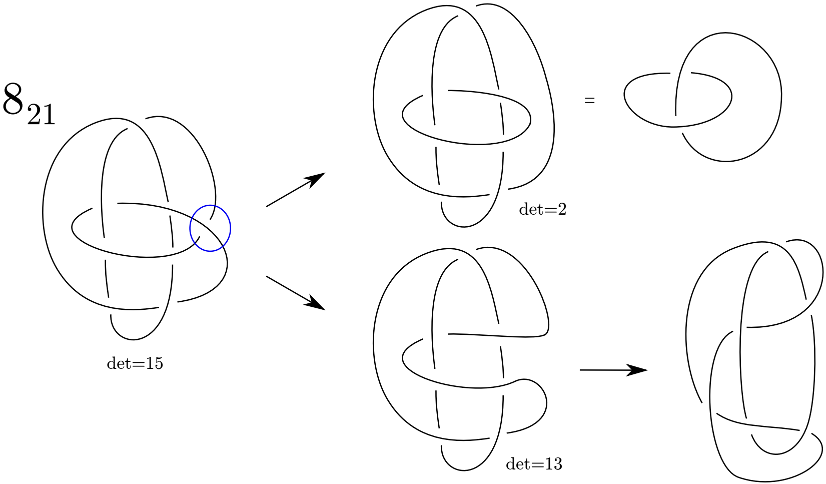

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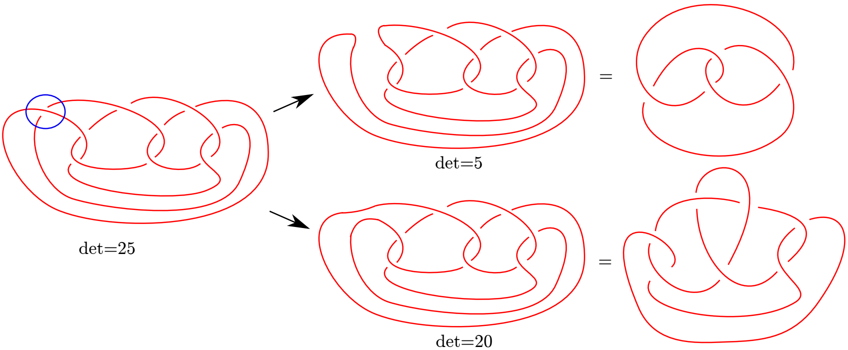
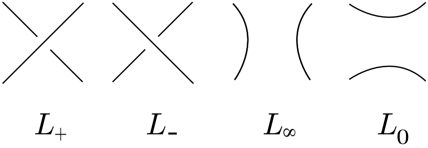


Q-polynomial

For an unoriented link, the Q-polynomial $Q_L \in \mathbb{Z}[x, x^{-1}]$ is defined as follows.

- For the unknot U, $Q_U = 1$.
- $Q_{L_+} + Q_{L_-} = x(Q_{L_\infty} + Q_{L_\infty})$



Basic problem

Problem

Determine whether a given link is QA or not.

Properties of QA-links

- $\Sigma_2(L)$ is an L–space.
- $\Sigma_2(L)$ bounds H_1-torsion free, negative-definite 4-manifold.
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Theorem

If a link L is QA, then

$$\deg Q_L \leq \det L - 1$$

* K. Qazaqzeh and N. Chbili, "A new obstruction of quasi-alternating links", arXiv:1406.0279.

$K = 8_{19}$

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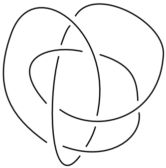

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If a link L is QA, then either,

1. L *is a* $(2, n)$*-torus link* $(n \neq 0)$ *and* $\deg Q_L = \det L - 1$;
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Remark

1. Figure-eight knot K is alternating, so QA.
   Since $\deg Q_K = 3, \quad \det K = 5$, the above evaluation is optimal.
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For a QA link,

det	1	2	3
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For a QA link L,

1. *if $\det L = 4$, then L is the $(2, \pm 4)$-torus link or $\deg Q_L \leq 2$;*
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Induction on $\det L$

For a QA link L, think QA resolutions L_∞ and L_0.

If neither L_∞ nor L_0 is a $(2, n)$-torus link,

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