カスプ特異点とその類似物の構成

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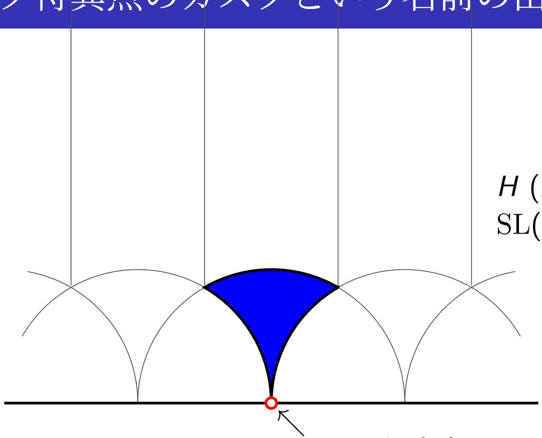
March 2, 2015

カスプ特異点のカスプという名前の由来

H (上半平面 $\simeq$ 円板) への $\mathrm{SL}(2, \mathbf{Z})$ の作用の基本領域

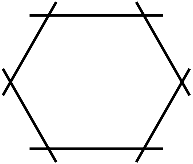


Hilbert modular cusp : H^n の全実代数体の離散群による商空間の有限個の点によるコンパクト化に現れる特異点

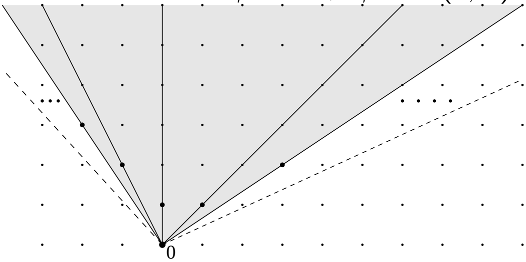
2 次元カスプ特異点 : 特異点解消の例外集合が有理曲線の輪

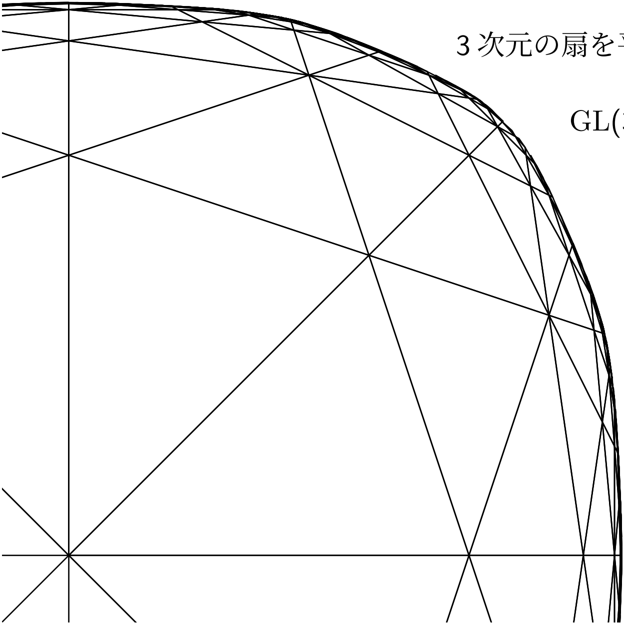
右下図のような扇から構成される

$\gamma^{\mathbf{Z}}$-不変, $\gamma \in \mathrm{SL}(2, \mathbf{Z})$



$\longleftarrow$





3 次元の扇を平面で切断した図

$GL(3, \mathbf{Z})$ の部分群が作用

$N = \mathbf{Z}^n, \ \ N_{\mathbf{R}} = N \otimes \mathbf{R} \ \ (\simeq \mathbf{R}^n)$

カスプ特異点は次の条件をみたす $N_{\mathbf{R}}$ の扇 Σ と $\mathrm{GL}(N)$ の部分群 Γ から構成される。

1. Σ : Γ-不変 $\quad (\gamma\sigma \in \Sigma \ \text{ for } \ \forall\gamma \in \Gamma, \forall\sigma \in \Sigma)$
2. Σ/Γ : 有限集合
3. $C := |\Sigma| \setminus \{0\}$: 開強凸錐 $\quad (|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma)$
4. $\gamma\sigma \neq \sigma \ \text{ for } \ \forall\gamma \neq 1, \ \forall\sigma \neq \{0\}$

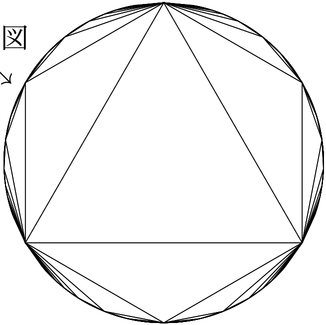
$\Longrightarrow \ \exists(V, p)$: カスプ特異点 s.t. $V \setminus \{p\} \simeq (\mathbf{R}^n + \sqrt{-1}C)/\mathbf{Z}^n \cdot \Gamma$

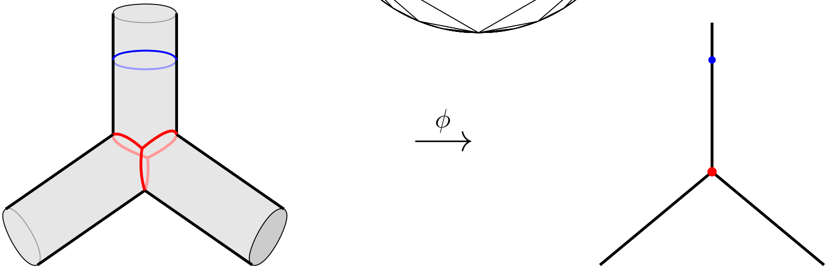
3. と 4. の条件を以下のように弱くしても特異点が構成できる。

3. 開強凸錐 $\Longrightarrow$ 強凸錐, $\quad$ 4. $\sigma \neq \{0\} \Longrightarrow \sigma \not\subset \partial C$

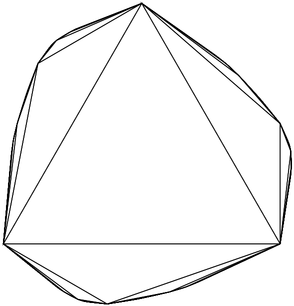
条件 3. と 4. を弱くした例 1

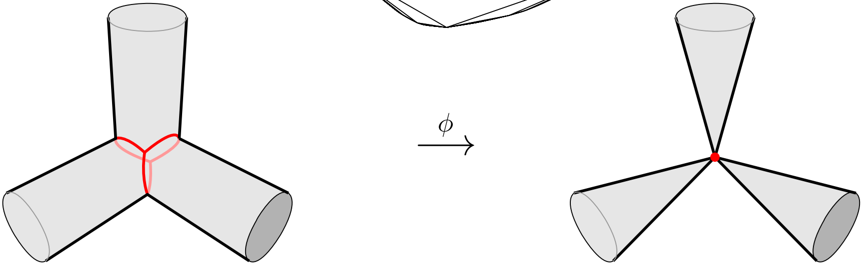
3 次元の扇を平面で切断した図



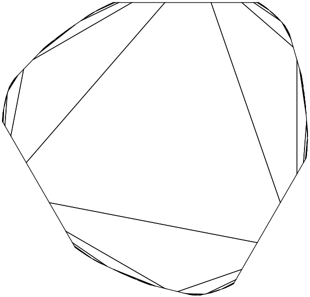


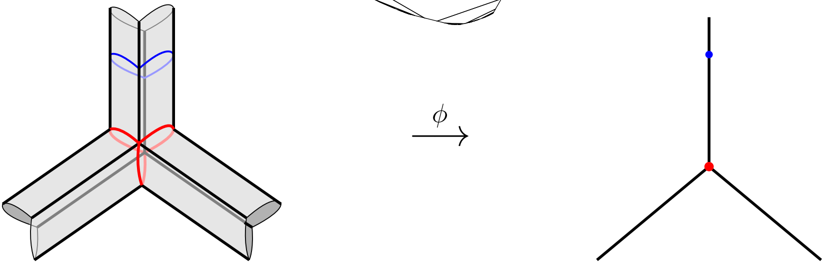
条件 3. と 4. を弱くした例 2





条件 3. と 4. を弱くした例 3





扇の構成

σ : n 次元有理強凸多面錐 $\subset N_{\mathbf{R}}$ （N の有限個の元で張られ、直線を含まない）

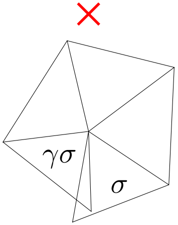
$I_\sigma = \{\sigma$ の $n-1$ 次元面 $\}$

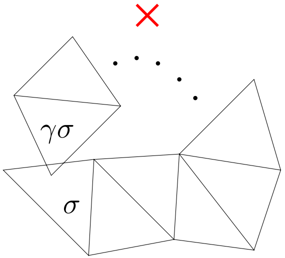
$\bigcup$

$J_\sigma \ni \tau \xrightarrow{\rho} \gamma = \rho(\tau) \in \mathrm{GL}(N)$ s.t. $\gamma x = x$ for $\forall x \in \tau$,
$\gamma\sigma \cap \sigma = \tau$, $(\gamma)^2 = 1$

$\Gamma(\rho) := \langle \rho(\tau) \mid \tau \in J_\sigma \rangle \subset \mathrm{GL}(N)$

$\Sigma(\sigma, \rho) := \{\text{faces of } \gamma\sigma \mid \gamma \in \Gamma(\rho)\}$





$\Sigma(\sigma, \rho)$ が扇となるための条件

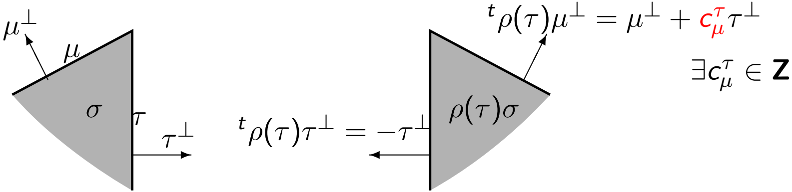
Theorem 3.1

σ, ρ が以下の条件 *I, II, III* をみたせば、$\Sigma(\sigma, \rho)$ は扇であり、
$|\Sigma(\sigma, \rho)| = \bigcup_{\sigma \in \Sigma(\sigma,\rho)} \sigma$ は強凸錐である。

$M = \mathrm{Hom}(N, \mathbf{Z})$, $\langle\ ,\ \rangle : M \times N \to \mathbf{Z}$, $\langle x, \gamma y\rangle = \langle {}^t\gamma x, y\rangle$, ${}^t\gamma \in \mathrm{GL}(M)$

For $\tau \in I_\sigma$, $\exists 1\ \tau^\perp$: primitive $\in M$ s.t. $\langle \tau^\perp, y\rangle = 0$ for $\forall y \in \tau$
$\langle \tau^\perp, y\rangle \geq 0$ for $\forall y \in \sigma$

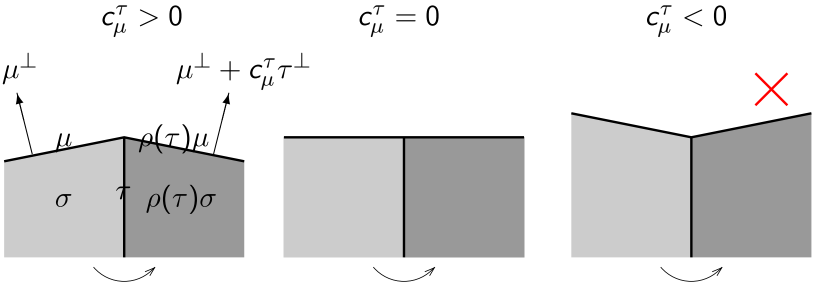
$\Longrightarrow$ For $\tau \in J_\sigma$, $\mu \in I_\sigma$



I. $c^\tau_\mu \geq 0$ for $\forall \tau \in J_\sigma$ and $\forall \mu \in I_\sigma$ with $\dim \tau \cap \mu = n - 2$

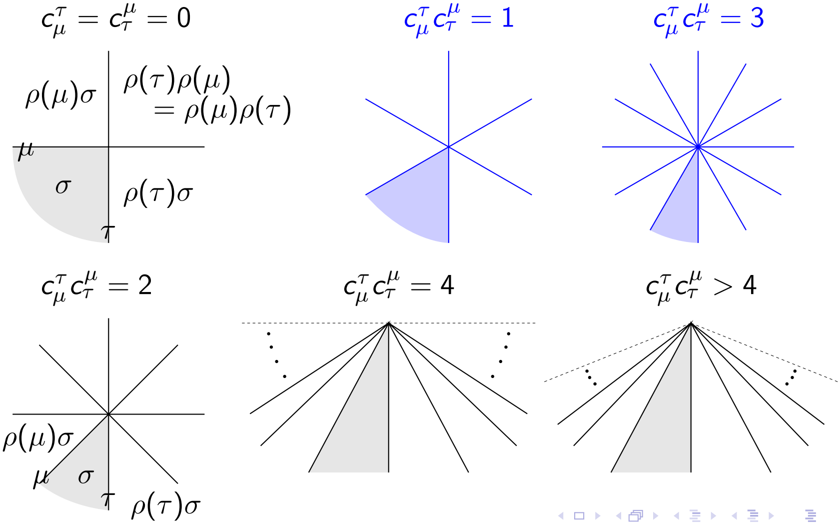
条件 I

I. $c_\mu^\tau \geq 0$ for $\forall \tau \in J_\sigma$ and $\forall \mu \in I_\sigma$ with $\dim \tau \cap \mu = n-2$



条件 II

II. $c^\tau_\mu = c^\mu_\tau = 0, \;\; c^\tau_\mu c^\mu_\tau = 2 \;$ or $\; c^\tau_\mu c^\mu_\tau \geq 4 \;$ for $\; \forall \tau, \mu \in J_\sigma$ with $\dim \tau \cap \mu = n-2$



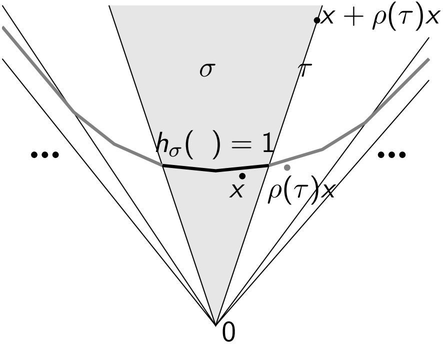
条件 III

III. $\exists h_\sigma : \sigma \to \mathbf{R}_{\geq 0}$ 区分的線形関数 s.t.

(i) $h_\sigma(x) = 0 \iff x = 0$

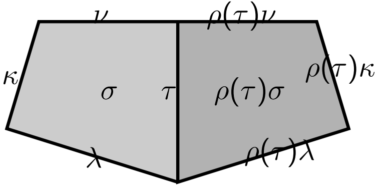
(ii) $h_\sigma(x+y) \geq h_\sigma(x) + h_\sigma(y)$ for $\forall x, y \in \sigma$

(iii) $h_\sigma(x + \rho(\tau)x) > 2h_\sigma(x)$ for $\forall \tau \in J_\sigma$ and for $\forall x \in \sigma \setminus \tau$



定理１の証明の概略

$\widehat{\sigma}^\tau := \sigma \cup \rho(\tau)\sigma$：有理強凸多面錐



$$I_{\widehat{\sigma}^\tau} = \{\nu \cup \rho(\tau)\nu, \kappa, \rho(\tau)\kappa, \lambda, \rho(\tau)\lambda\}$$

$$J_\sigma = \{\tau, \nu, \kappa\}$$

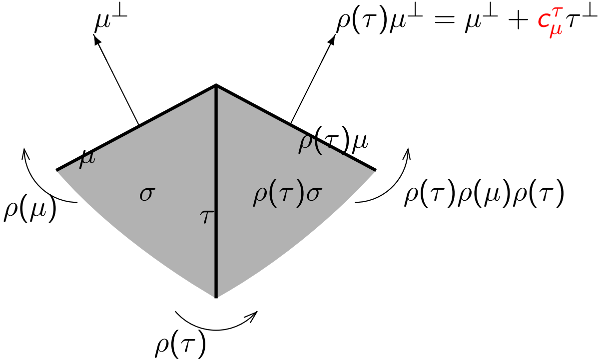
$$J_{\widehat{\sigma}^\tau} = \{\nu \cup \rho(\tau)\nu, \kappa, \rho(\tau)\kappa\}$$

$$\hat{\rho}^\tau(\mu) = \begin{cases} \rho(\nu) & \mu \supset \nu \in J_\sigma \\ \rho(\tau)\rho(\nu)\rho(\tau) & \mu \supset \rho(\tau)\nu \end{cases}$$

$c_\nu^\tau = 0$ のとき　$\rho(\nu) = \rho(\tau)\rho(\nu)\rho(\tau)$

$\widehat{\sigma}^\tau$, $\hat{\rho}^\tau$：条件 I, II, III をみたす

定理 1 の証明の概略 ($\widehat{\sigma}^\tau$ が条件 II をみたす)



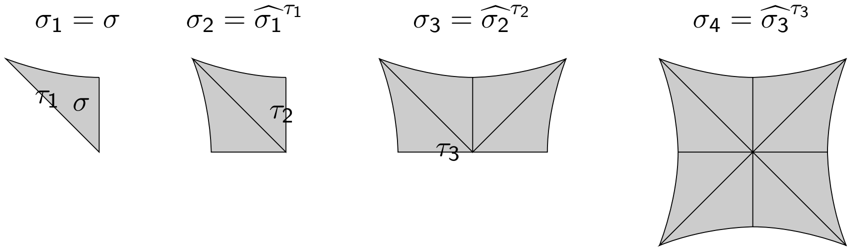
$$
\begin{aligned}
{}^t\rho(\mu)(\mu^\perp + c_\mu^\tau \tau^\perp) &= (\mu^\perp + c_\mu^\tau \tau^\perp) + (c_\mu^\tau c_\tau^\mu - 2)\mu^\perp \\
{}^t(\rho(\tau)\rho(\mu)\rho(\tau))\mu^\perp &= \mu^\perp + (c_\mu^\tau c_\tau^\mu - 2)(\mu^\perp + c_\mu^\tau \tau^\perp)
\end{aligned}
$$

$$\Longrightarrow \quad c_\mu^{\rho(\tau)\mu} = c_{\rho(\tau)\mu}^{\mu} = c_\mu^\tau c_\tau^\mu - 2$$

定理１の証明の概略の続き

$$h_{\widehat{\sigma}^{\tau}}(x) := \begin{cases} h_\sigma(x) & \text{if } x \in \sigma \\ h_\sigma(\rho(\tau)x) & \text{if } x \in \rho(\tau)\sigma \end{cases}$$

は条件 III をみたす



$\Sigma_1 = \{\text{faces of } \sigma\}$, $\Sigma_2 = \{\alpha, \rho(\tau_1)\alpha \mid \alpha \in \Sigma_1\}$, $\ldots$: 扇, $|\Sigma_j| = \sigma_j$

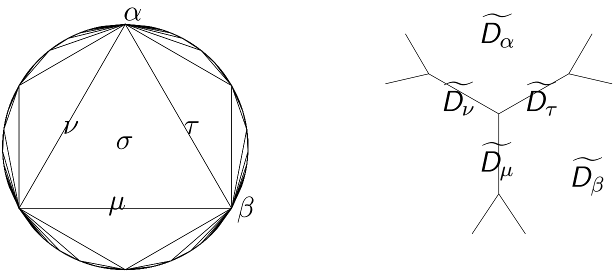
For $\forall \gamma \in \Gamma(\rho)$, $\exists \sigma_1 = \sigma$, $\sigma_2 = \widehat{\sigma_1}^{\tau_1}$, ..., $\sigma_k = \widehat{\sigma_{k-1}}^{\tau_{k-1}}$ s.t. $\gamma\sigma \in \Sigma_k$

扇から得られるトーリック多様体の構造

$C := |\Sigma(\sigma,\rho)|, \quad \widetilde{U} := \mathrm{Int}(\overline{\mathrm{ord}^{-1}(C)}) \quad (\mathrm{ord} : T_N\mathrm{emb}(\Sigma) \to \mathrm{Mc}(\Sigma) \supset N_{\mathbf{R}})$

$\widetilde{D^o_\alpha} := \mathrm{orb}(\alpha) \cap \widetilde{U}, \qquad \widetilde{D_\alpha} := \overline{\mathrm{orb}(\alpha)} \cap \widetilde{U}$ for $\alpha \in \Sigma(\sigma,\rho)$.

$\Longrightarrow \widetilde{U} = \bigsqcup_{\alpha\in\Sigma(\sigma,\rho)} \widetilde{D^o_\alpha}$



$\exists\, \Gamma \subset \Gamma(\rho)$ s.t. $\gamma\sigma \neq \sigma$ for $1 \neq {}^\forall\gamma \in \Gamma$ and ${}^\forall\sigma \not\subset \partial C, \quad [\Gamma(\rho) : \Gamma] < \infty$

$\Longrightarrow$ Γ acts on $\widetilde{U}$ without fixed points, $\quad U := \widetilde{U}/\Gamma, \quad q : \widetilde{U} \to U$

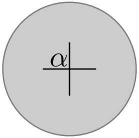
$D^o_{[\alpha]} := q(\widetilde{D^o_\alpha}), \quad D_{[\alpha]} := q(\widetilde{D_\alpha}) \Longrightarrow U = \bigsqcup_{[\alpha]\in\Sigma(\sigma,\rho)/\Gamma} D^o_{[\alpha]}$

Theorem 4.1

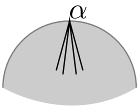
$\exists p_{[\alpha]} : D_{[\alpha]} \longrightarrow B_{[\alpha]}$: 固有正則写像 *for* $\forall\ \alpha \in \Sigma(\sigma, \rho)$
$B^o_{[\alpha]} := p_{[\alpha]}(D^o_{[\alpha]})$ 上の *fiber* $=$ *cpt toric var.* $\times$ *Abelian var.*

$\alpha \not\subset \partial C \iff D_{[\alpha]}$: cpt $\iff B_{[\alpha]}$: 1pt

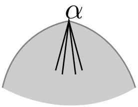
α の余次元が 2 の場合 $\alpha = \tau \cap \mu \ (\tau \in J_\sigma,\ \mu \in I_\sigma)$



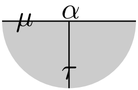
$c^\mu_\tau c^\tau_\mu = 4$



$c^\mu_\tau c^\tau_\mu > 4$



$\mu \notin J_\sigma$



$D_{[\alpha]}$: cpt
$B_{[\alpha]}$: 1 pt

$p_{[\alpha]}$ の fiber
は楕円曲線

$D^o_{[\alpha]} \simeq B^o_{[\alpha]}$

$p_{[\alpha]}$ の fiber
は有理曲線

Contraction

$$\Sigma^c := \{\alpha \in \Sigma(\sigma,\rho) \mid \alpha \not\subset \partial C\},\quad \widetilde{Z} := \bigcup_{\alpha\in\Sigma^c} \widetilde{D_\alpha},\ \ Z := \widetilde{Z}/\Gamma : \mathrm{cpt}$$

Theorem 5.1

$\exists W$: Z の近傍. $\exists\phi : W \longrightarrow V$: 固有正則写像

(V : a Stein analytic space)

$$\begin{array}{ccccc} W & \supset & D^o_{[\alpha]} \cap W & & \\ \phi\downarrow & & \downarrow p_{[\alpha]|D^o_{[\alpha]}\cap W} & & \\ V & \hookleftarrow & p_{[\alpha]}(D^o_{[\alpha]} \cap W) & \subset & B_{[\alpha]} \end{array}$$

例

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$: a basis of N, $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \ldots, \mathbf{e}_n^*\}$: dual

$\sigma := \mathbf{R}_{\geq 0}\mathbf{e}_1 + \mathbf{R}_{\geq 0}\mathbf{e}_2 + \cdots + \mathbf{R}_{\geq 0}\mathbf{e}_n, \quad I_\sigma = \{\tau_i \mid 1 \leq i \leq n\} \ (\tau_i \not\ni \mathbf{R}_{\geq 0}\mathbf{e}_i)$

$\gamma_i \mathbf{e}_j = \mathbf{e}_j \quad$ for $\forall i \neq j$,

$\gamma_i \mathbf{e}_i = \sum_{j=1}^n a_{ij}\mathbf{e}_j \qquad a_{ii} = -1, \ a_{ij} \in \mathbf{Z}_{\geq 0}$

$\Longrightarrow \ \gamma_i \in \mathrm{GL}(N), \quad \gamma_i^2 = 1, \quad \gamma_i\sigma \cap \sigma = \tau_i, \quad \gamma_i x = x \ $ for $\ \forall x \in \tau_i$

$\quad {}^t\gamma_i \mathbf{e}_i^* = -\mathbf{e}_i^*, \ \ {}^t\gamma_i \mathbf{e}_j^* = \mathbf{e}_j^* + a_{ij}\mathbf{e}_i^*$

$J_\sigma := I_\sigma, \ \ \rho(\tau_i) := \gamma_i \ \Longrightarrow \ c_{\tau_j}^{\tau_i} = a_{ij} \qquad (\because \ \tau_i^\perp = \mathbf{e}_i^*)$

$a_{ij} = a_{ji} = 0, \ \ a_{ij}a_{ji} = 2 \ $ or $\ a_{ij}a_{ji} \geq 4 \ $ for $\forall i \neq j \ \Longrightarrow$ 条件 I, II をみたす

$\sum_{j=1}^n a_{ij} > 1 \ \Longrightarrow$ 条件 III をみたす $(h_\sigma(x) = \langle \mathbf{e}_1^* + \cdots + \mathbf{e}_n^*, x \rangle)$

例

1. $a_{ij}a_{ji} > 4 \quad \text{for } \forall i \neq j, \quad \Gamma = \Gamma(\rho) \cap \mathrm{SL}(N)$
   $\Longrightarrow D^o_{[\alpha]} \simeq B^o_{[\alpha]} \quad \text{for } \forall \alpha \text{ with } \dim \alpha < n-1$
   $\Longrightarrow V$: 孤立特異点，例外集合は n 本の有理曲線

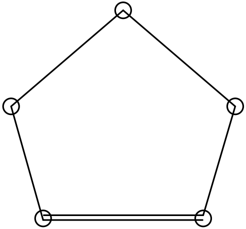
2. $a_{ij}a_{ji} = 4 \quad \text{for } \forall i \neq j, \quad \Gamma = \Gamma(\rho) \cap \mathrm{SL}(N)$
   $\Longrightarrow D^o_{[\alpha]} \simeq B^o_{[\alpha]} \quad \text{for } \forall \alpha \text{ with } \dim \alpha < n-2$
   $p_{[\alpha]}$ の fiber は楕円曲線 $\quad \text{for } \forall \alpha \text{ with } \dim \alpha = n-2$

3. $a_{ij}a_{ji} = 2 \quad \text{for } \forall i \neq j, \quad \Gamma = \Gamma(\rho) \cap \ker[\mathrm{SL}(N) \to \mathrm{SL}(N/3N)]$
   $\Longrightarrow D_{[\alpha]}$: cpt for $\forall \alpha$ with $\dim \alpha \geq n-2$
   $D^o_{[\alpha]} \simeq B^o_{[\alpha]} \quad \text{for } \forall \alpha \text{ with } \dim \alpha < n-3$
   $\Longrightarrow V$: 孤立特異点

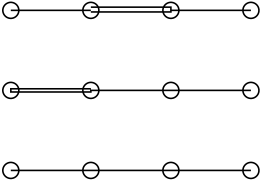
5次元の例

$$(a_{ij}) = \begin{pmatrix} -1 & 1 & 0 & 0 & 2 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix}$$ は条件 II をみたさないが

$\Sigma(\sigma, \rho)$ が扇ならば 5 次元 cusp 特異点が得られる。

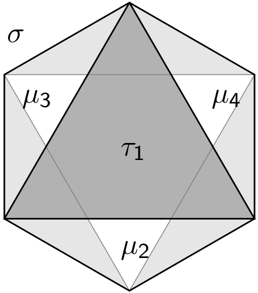


1 次元錐を固定する $\Gamma(\rho)$ の部分群の Dynkin 図形

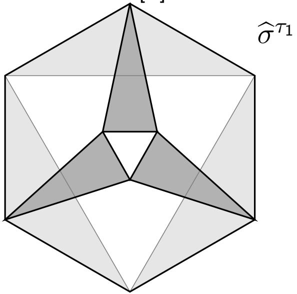


特異点の次元が余次元 2 の例

σ : $\mathbf{e}_i + \mathbf{e}_j$ $(1 \le i < j \le n)$ で張られる有理錐
τ_i : $\mathbf{e}_i + \mathbf{e}_j$ $(j \neq i)$ で張られる σ の面
μ_i : $\mathbf{e}_j + \mathbf{e}_k$ $(j \neq i \neq k \neq j)$ で張られる σ の面
$J_\sigma := \{\tau_i \mid 1 \le i \le n\}$
$\gamma_i \mathbf{e}_i = -\mathbf{e}_i, \quad \gamma_i \mathbf{e}_j = \mathbf{e}_j + 2\mathbf{e}_i \quad \text{if} \quad i \neq j$
$\Longrightarrow \quad \gamma_i \in \mathrm{GL}(N), \quad \gamma_i^2 = 1, \quad \gamma_i \sigma \cap \sigma = \tau_i, \quad \gamma_i x = x \quad \text{for} \quad \forall x \in \tau_i$
$\dim \tau_i \cap \mu_j = n - 1, \quad {}^t\gamma_i(\mu_j)^\perp = (\mu_j)^\perp, \quad \text{if } i \neq j$
$\alpha \in \Sigma(\sigma, \rho), \quad \dim \alpha = 1 \quad \Longrightarrow \quad \dim B_{[\alpha]} = n - 2, \quad p_{[\alpha]}$ の fiber は楕円曲線



$c_{\mu_j}^{\tau_i} = 0$



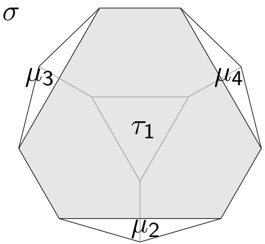
例

$$\sigma = \{y \in N_{\mathbf{R}} \mid \langle \mathbf{e}_i^*, y \rangle \geq 0, \langle \mathbf{f}_i^*, y \rangle \geq 0 \quad 1 \leq i \leq n\}$$

$$\mathbf{f}_i^* = \frac{1}{2}\mathbf{e}_1^* + \cdots + \frac{1}{2}\mathbf{e}_{i-1}^* - \mathbf{e}_i^* + \mathbf{e}_{i+1}^* + \cdots + \mathbf{e}_n^*$$

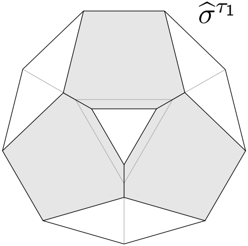
$$\mu_i = \{y \in \sigma \mid \langle \mathbf{e}_i^*, y \rangle = 0\},\ \tau_i = \{y \in \sigma \mid \langle \mathbf{f}_i^*, y \rangle = 0\},\ J_\sigma = \{\tau_i\}$$

$$\rho(\tau_i)\mathbf{e}_i = \mathbf{e}_i, \qquad \rho(\tau_i)\mathbf{e}_j = \begin{cases} \mathbf{e}_j + \mathbf{e}_i & j < i \\ \mathbf{e}_j + 2\mathbf{e}_i & i < j \end{cases}$$



$$c_{\mu_j}^{\tau_i} = 0$$

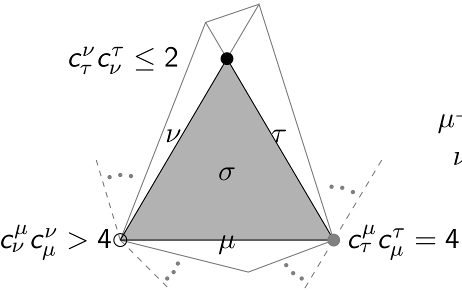
$$c_{\tau_j}^{\tau_i} c_{\tau_i}^{\tau_j} = 2$$

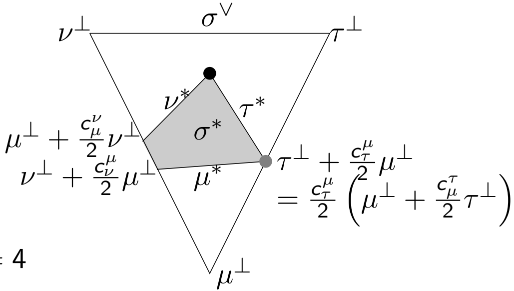


双対扇

$$
\begin{aligned}
\sigma^\vee &= \{x \in N_{\mathbf{R}} \mid \langle x, y\rangle \geq 0 \ \text{ for } \forall y \in \sigma\} \\
\sigma^* &= \{x \in \sigma^\vee \mid \langle x, v_\tau\rangle \geq 0 \text{ for } \forall \tau \in J_\sigma\} \quad \rho(\tau)v_\tau = -v_\tau \\
\tau^* &= \{x \in \sigma^* \mid \langle x, v_\tau\rangle = 0\} \quad \langle \tau^\perp, v_\tau\rangle < 0 \\
J_{\sigma^*} &= \{\tau^* \mid \tau \in J_\sigma\}, \quad \rho^*(\tau^*) = {}^t\rho(\tau)
\end{aligned}
$$

$$\Longrightarrow (\sigma^*)^* = \sigma, \qquad \overline{|\Sigma(\sigma^*, \rho^*)|} = |\Sigma(\sigma, \rho)|^\vee$$





補足

$$
\begin{aligned}
{}^t\rho(\mu)(\mu^{\perp} + c_{\mu}^{\tau}\tau^{\perp}) &= -\mu^{\perp} + c_{\mu}^{\tau}(\tau^{\perp} + c_{\mu}^{\tau}\mu^{\perp}) \\
&= (\mu^{\perp} + c_{\mu}^{\tau}\tau^{\perp}) + (c_{\mu}^{\tau}c_{\tau}^{\mu} - 2)\mu^{\perp}
\end{aligned}
$$

$$
\begin{aligned}
{}^t(\rho(\tau)\rho(\mu)\rho(\tau))\mu^{\perp} &= {}^t\rho(\tau){}^t\rho(\mu)(\mu^{\perp} + c_{\mu}^{\tau}\tau^{\perp}) \\
&= {}^t\rho(\tau)(c_{\mu}^{\tau}\tau^{\perp} + (c_{\mu}^{\tau}c_{\tau}^{\mu} - 1)\mu^{\perp}) \\
&= -c_{\mu}^{\tau}\tau^{\perp} + (c_{\mu}^{\tau}c_{\tau}^{\mu} - 1)(\mu^{\perp} + c_{\mu}^{\tau}\tau^{\perp}) \\
&= \mu^{\perp} + (c_{\mu}^{\tau}c_{\tau}^{\mu} - 2)(\mu^{\perp} + c_{\mu}^{\tau}\tau^{\perp})
\end{aligned}
$$