# Arithmetic Zariski pairs of line arrangements


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# Combinatorics and Topology

## Definition

*Combinatorics*: $\mathscr{C} := (\mathcal{L}, \mathcal{P})$, $\mathcal{L}$ finite set of *lines* and $\mathcal{P} \subset \{P \subset \mathcal{L} \mid \#P \geq 2\}$ finite set of *points* mimic arrangement of lines and multiple points.

## Definition (Realization of $\mathscr{C}$)

$\mathcal{A}$ line arrangement in $\mathbb{P}^2$: $(\mathcal{A}, \{\text{multiple points}\}) \leftrightarrow (\mathcal{L}, \mathcal{P})$

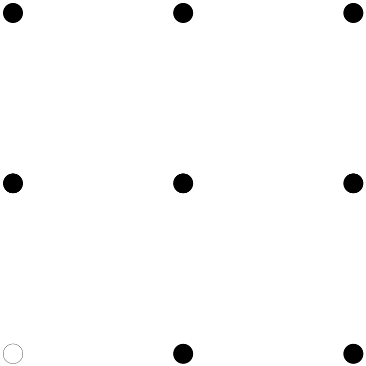
## Combinatorial objects

- $\mathbb{Z}^{\mathcal{L}} = \bigoplus_{L \in \mathcal{L}} \mathbb{Z} x_L$, $\dfrac{\mathbb{Z}^{\mathcal{L}}}{\mathbb{Z}\left(\sum_{L \in \mathcal{L}} x_L\right)} =: H_1^{\mathscr{C}} \cong H_1(\mathbb{P}^2 \setminus \mathcal{A}; \mathbb{Z})$
- $x_P = \sum_{P < L} x_L$

$$\{x_L \wedge x_P \in H_1^{\mathscr{C}} \wedge H_1^{\mathscr{C}} \mid P < L\} = H_2^{\mathscr{C}} \cong H_2(\mathbb{P}^2 \setminus \mathcal{A}; \mathbb{Z})$$

- $H^0_{\mathscr{C}} = \mathbb{Z}$, $H^j_{\mathscr{C}} \cong H^j(\mathbb{P}^2 \setminus \mathcal{A}; \mathbb{Z})$ dual of $H_j^{\mathscr{C}}$, $j = 1, 2$.

# McLane arrangements I



$\mathscr{ML}$

$\mathrm{Aut}\,\mathscr{ML} = \mathrm{GL}(2, \mathbb{F}_3)$

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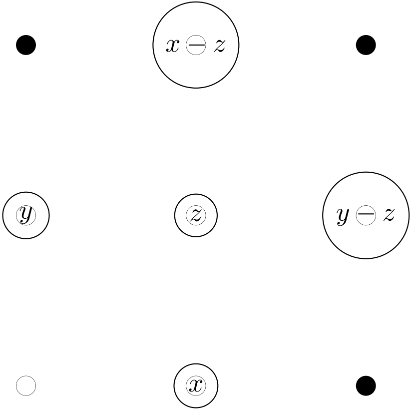


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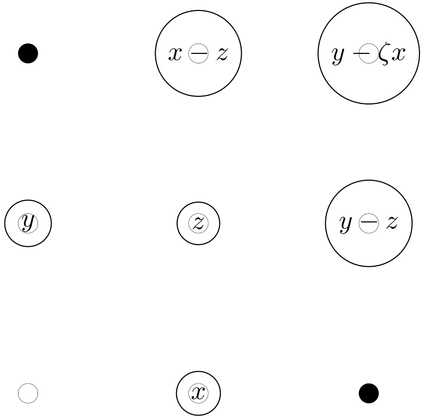


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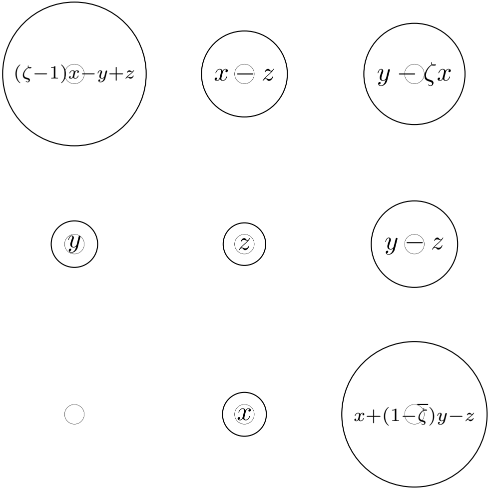
# McLane arrangements I




$\zeta^2 + \zeta + 1 = 0$

$\mathscr{ML}_\pm$

$\mathrm{Aut}\,\mathscr{ML} = \mathrm{GL}(2, \mathbb{F}_3)$

# McLane arrangements II

## Theorem (Rybnikov)

$\nexists \varphi : \pi_1(\mathbb{P}^2 \setminus \mathscr{ML}_+) \to \pi_1(\mathbb{P}^2 \setminus \mathscr{ML}_-)$ *group automorphism inducing the identity on homology.*

## Corollary

$\nexists \rho : (\mathbb{P}^2, \mathscr{ML}_+) \to (\mathbb{P}^2, \mathscr{ML}_-)$ *homeomorphism respecting orientations and ordering.*

## Orientation

$\exists \rho : (\mathbb{P}^2, \mathscr{ML}_+) \to (\mathbb{P}^2, \mathscr{ML}_-)$ homeomorphism respecting ordering and reversing orientation: *complex conjugation.*

## Order

$\exists \rho : (\mathbb{P}^2, \mathscr{ML}_+) \to (\mathbb{P}^2, \mathscr{ML}_-)$ homeomorphism respecting orientation: $\mathrm{GL}(2, \mathbb{F}_3) \setminus \mathrm{SL}(2, \mathbb{F}_3)$.

# Rybnikov

## Rybnikov's combinatorics

$\mathscr{RB} = \mathscr{ML}_1 \cup_{xz(x-z)=0} \mathscr{ML}_2$ (gluing *in general position*)

## Theorem

$G_{++} = \pi_1(\mathbb{P}^2 \setminus \mathscr{RB}_{++}) \not\cong \pi_1(\mathbb{P}^2 \setminus \mathscr{RB}_{+-}) = G_{--}$

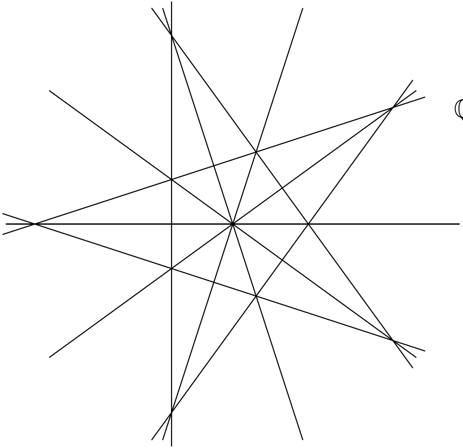
## Guidelines of the proof.

Assume they are isomorphic $\Longrightarrow G_{++}/\gamma_4(G_{++}) \cong G_{+-}/\gamma_4(G_{+-})$

1. The isomorphism induces the $\pm$*identity* on $H_1^{\mathscr{RB}}$ (purely combinatorial).
2. It does not happen using *truncated Alexander invariant.*

□

# Pentagon and Pentagram



$\mathscr{C}_0$

$\mathbb{Q}\left[\sqrt{5}\right]$

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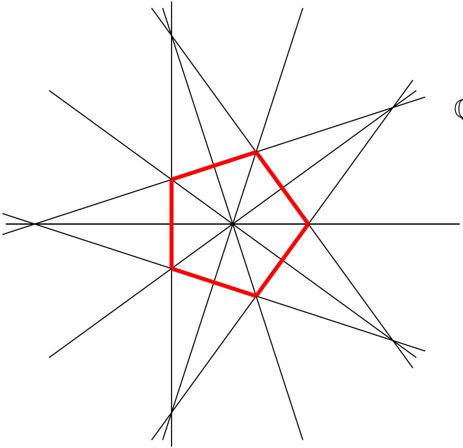

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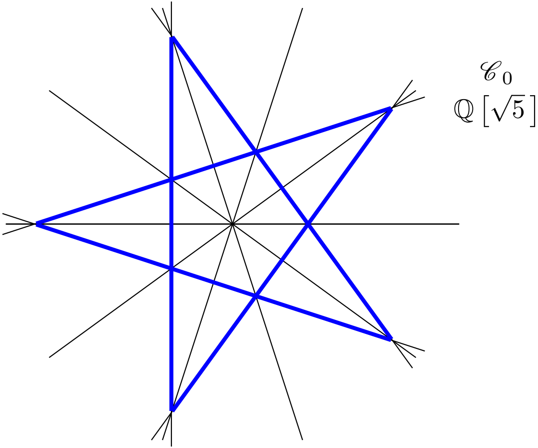

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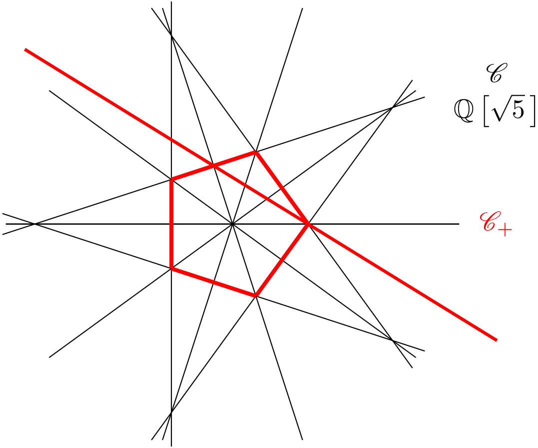

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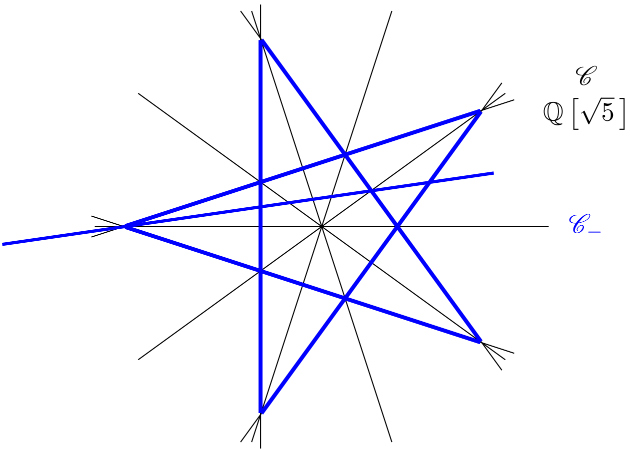

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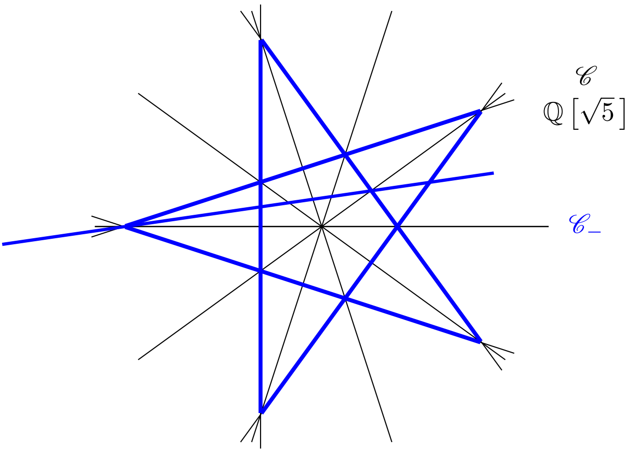
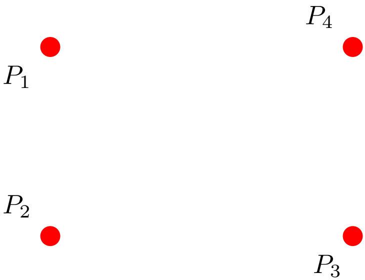



### Theorem

*There is no homeomorphism between* $(\mathbb{P}^2, \mathscr{C}_+)$ *and* $(\mathbb{P}^2, \mathscr{C}_-)$

# $\mathscr{G}_{91}$ combinatorics



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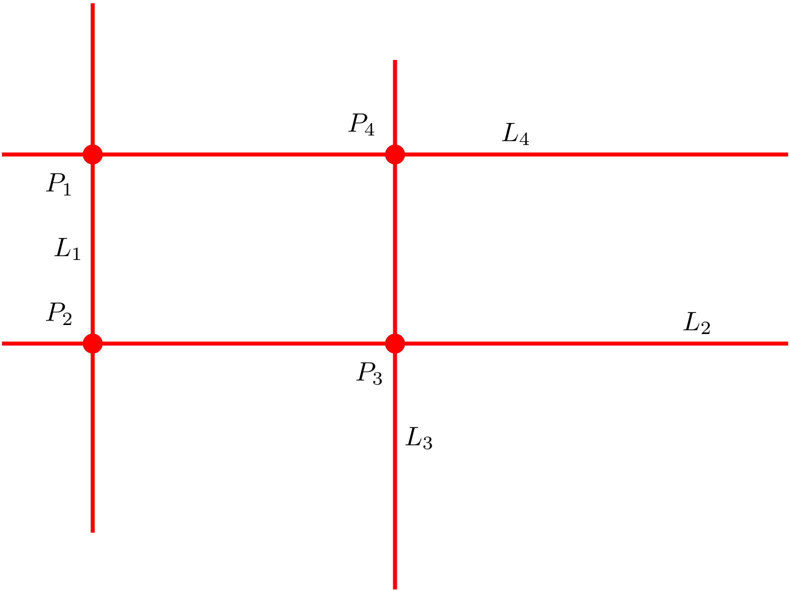

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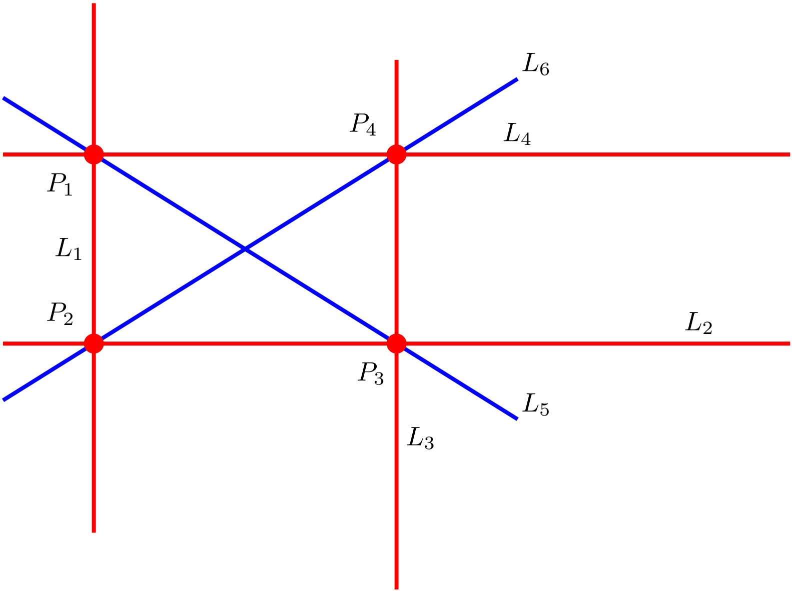

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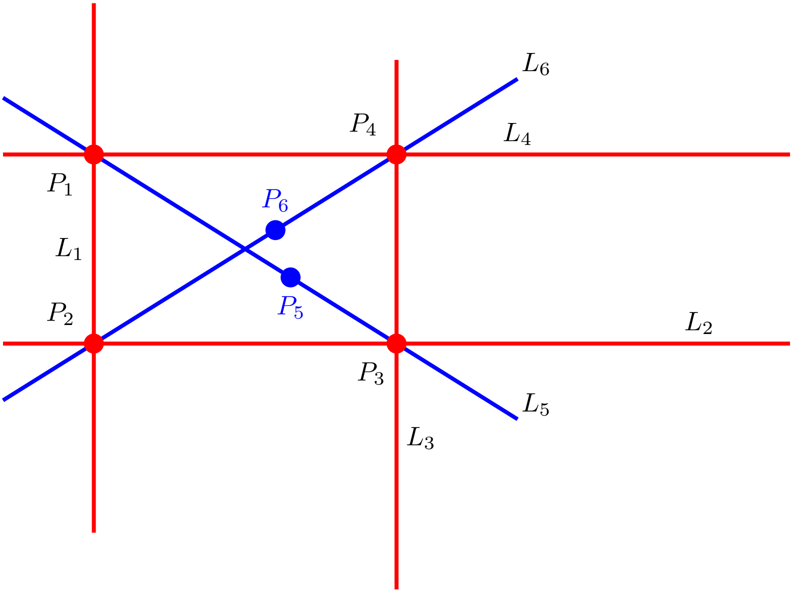

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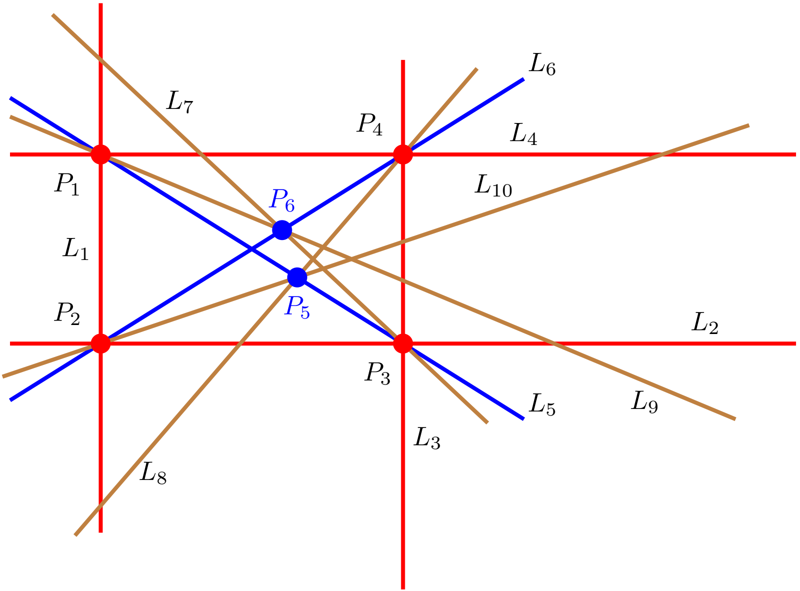

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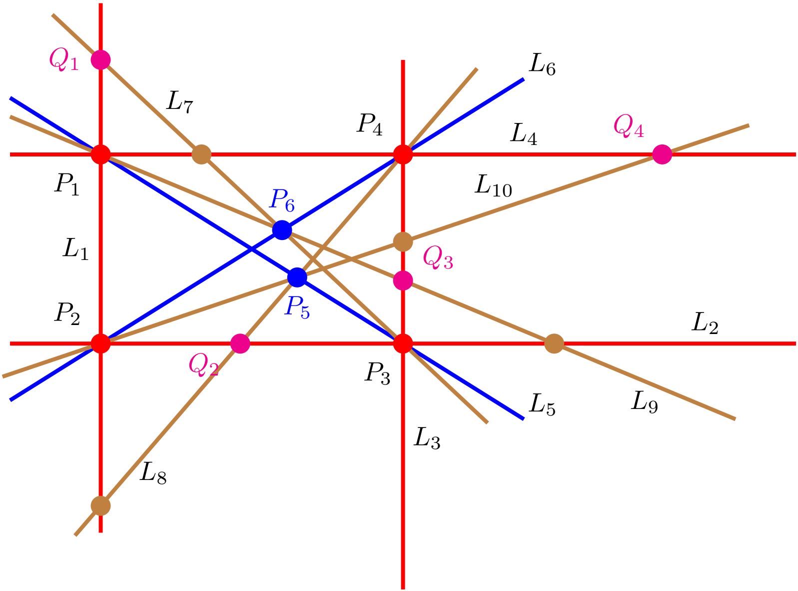

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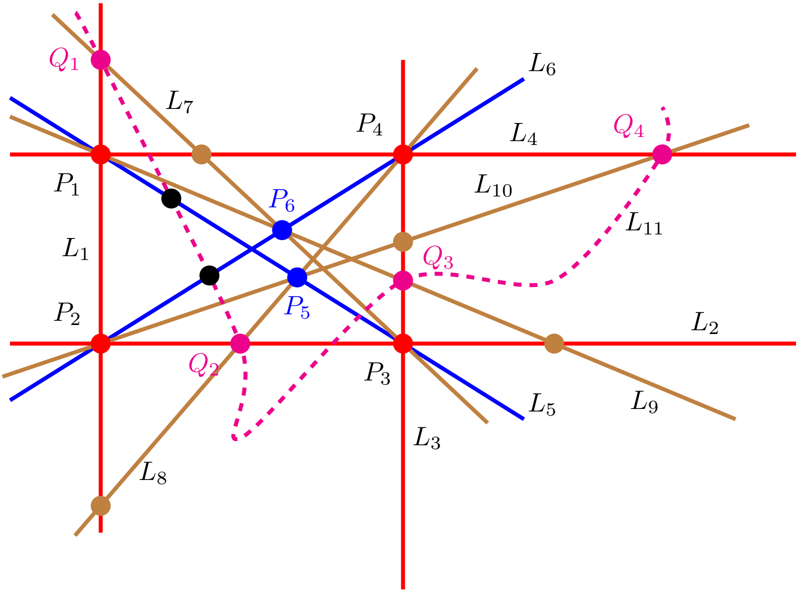

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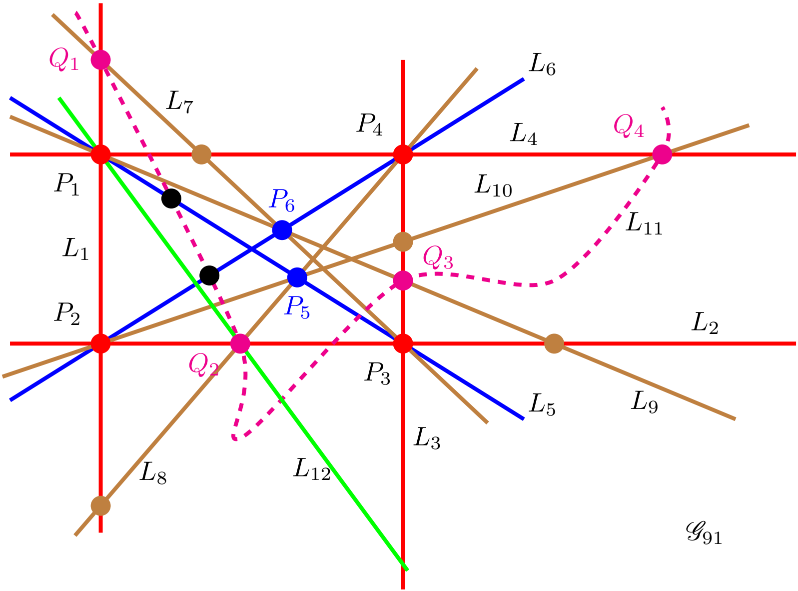

# Guerville's example

## Theorem

*$\mathscr{G}_{91}$ admits four (Galois-conjugate) realizations $\mathcal{A}_\zeta$ with equations in the cyclotomic field $\mathbb{K}_5$, for $\zeta$ a primitive fifth root of unity.*
*There is no oriented homeomorphism $(\mathbb{P}^2, \mathcal{A}_{\zeta_1}) \to (\mathbb{P}^2, \mathcal{A}_{\zeta_2})$ if $\zeta_1 \neq \zeta_2$.*

## Corollary

*There is no homeomorphism $(\mathbb{P}^2, \mathcal{A}_\zeta) \to (\mathbb{P}^2, \mathcal{A}_{\zeta^2})$.*

## Proof.

Use linking number over a character of order 5. □

# Main result I

## Theorem

*The groups* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ *and* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ *are not isomorphic (while their profinite completions are).*

## First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = \pm 1_{H_1^{\mathscr{G}_{91}}}$.

- Purely combinatorial statement.

# Homological rigidity

- $\rho : H_1^{\mathscr{C}} \to H_1^{\mathscr{C}}$ is an *admissible isomorphism* if $(\rho \wedge \rho)(H_2^{\mathscr{C}}) = H_2^{\mathscr{C}}$
- $\mathcal{A}_1, \mathcal{A}_2$ realizations of $\mathscr{C}$, $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_1) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2)$ isomorphism $\Longrightarrow \varphi_*$ admissible.
- $\mathscr{C}$ *homologically rigid* if

$$\pm \operatorname{Aut}(\mathscr{C}) = \{\text{admissible isomorphisms}\}$$

- $\rho$ admissible $\Longrightarrow \rho^* : H^1_{\mathscr{C}} \to H^1_{\mathscr{C}}$ respects the resonance varieties.
- $\{H_S$ irreducible components of resonance varieties in $H^1\} \leftrightarrow \{S$ combinatorial pencil$\}$ (type point, type Ceva, type Hesse,...)
- $\rho^*$ sends *triangles* to *triangles*

## Triangle

$S_1, S_2, S_3$ combinatorial pencils such that

$$\operatorname{codim} \bigcap_i H_{S_i} = \sum_i \operatorname{codim} H_{S_i} - 1.$$

# Homological rigidity

- $\rho : H_1^{\mathscr{C}} \to H_1^{\mathscr{C}}$ is an *admissible isomorphism* if $(\rho \wedge \rho)(H_2^{\mathscr{C}}) = H_2^{\mathscr{C}}$
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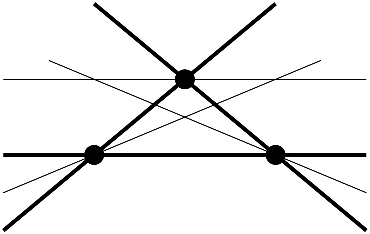


# Triangles in $\mathscr{G}_{91}$

| $i$ | $S_i$ | dim $H_S$ | $\Delta_S$ | $\Delta_{S,P_1}$ |
|---|---|---|---|---|
| 1 | 1, 7, 11 | 2 | 18 | 7 |
| 2 | 3, 9, 11 | 2 | 22 | 8 |
| 3 | 4, 10, 11 | 2 | 21 | 7 |
| 4 | 5, 8, 10 | 2 | 24 | 7 |
| 5 | 6, 9, 7 | 2 | 16 | 6 |
| 6 | 1, 2, 6, 10 | 3 | 53 | 12 |
| 7 | 2, 3, 5, 7 | 3 | 49 | 13 |
| 8 | 2, 8, 11, 12 | 3 | 57 | 15 |
| 9 | 4, 3, 6, 8 | 3 | 50 | 12 |
| 10 | 1, 4, 5, 9, 12 | 4 | 91 | 91 |
| 11 | 1, 2, 3, 4, 5, 6 | 2 | 24 | 8 |
| 12 | 1, 2, 4, 6, 8, 12 | 2 | 24 | 8 |
| 13 | 1, 2, 4, 10, 11, 12 | 2 | 20 | 7 |
| 14 | 1, 2, 5, 6, 7, 9 | 2 | 14 | 7 |
| 15 | 1, 2, 5, 7, 11, 12 | 2 | 14 | 7 |
| 16 | 1, 2, 5, 8, 10, 12 | 2 | 20 | 8 |
| 17 | 1, 3, 5, 7, 9, 11 | 2 | 14 | 7 |
| 18 | 1, 4, 5, 6, 8, 10 | 2 | 19 | 6 |
| 19 | 2, 3, 4, 5, 8, 12 | 2 | 20 | 8 |
| 20 | 2, 3, 5, 6, 8, 10 | 2 | 14 | 0 |
| 21 | 2, 3, 5, 9, 11, 12 | 2 | 18 | 9 |
| 22 | 2, 4, 6, 8, 10, 11 | 2 | 15 | 0 |
| 23 | 3, 4, 5, 6, 7, 9 | 2 | 12 | 6 |
| 24 | 3, 4, 8, 9, 11, 12 | 2 | 13 | 7 |
| 25 | 4, 5, 8, 10, 11, 12 | 2 | 15 | 7 |

# Behind homological rigidity

## Theorem (Marco)

- *$P := (L_1, \ldots, L_m)$ is a pencil (type point)*
- *It can be distinguished by triangles*
- *$\rho^* : H^1_{\mathscr{C}} \to H^1_{\mathscr{C}}$ admissible*

*Then $\rho^*(H_P) = H_P$, where*

$$H_P = \mathbb{Z}\langle x^*_{L_1} - x^*_{L_2}, \ldots, x^*_{L_1} - x^*_{L_m} \rangle$$

# Main result II

## Theorem

*The groups* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ *and* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ *are not isomorphic (while their profinite completions are).*

## First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = \pm 1_{H_1^{\mathscr{G}_{91}}}$.

## Second step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = 1_{H_1^{\mathscr{G}_{91}}}$

# Truncated Alexander Invariant

- $\mathscr{C}$ combinatorics, $\mathcal{A} = \{L_0, L_1, \ldots, L_\ell\}$ realization
- $G_{\mathcal{A}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{A})$, $\Lambda := \mathbb{Z}[H_1^{\mathscr{C}}] = \mathbb{Z}\left[t_1^{\pm 1}, \ldots, t_\ell^{\pm 1}\right]$, $t_0 = (t_1 \cdot \ldots \cdot t_\ell)^{-1}$
- $M_{\mathcal{A}} := G'_{\mathcal{A}}/G''_{\mathcal{A}}$ as $\Lambda$-module is the *Alexander invariant.*
- $\mathfrak{m} \subset \Lambda$ augmentation ideal of $\Lambda$:

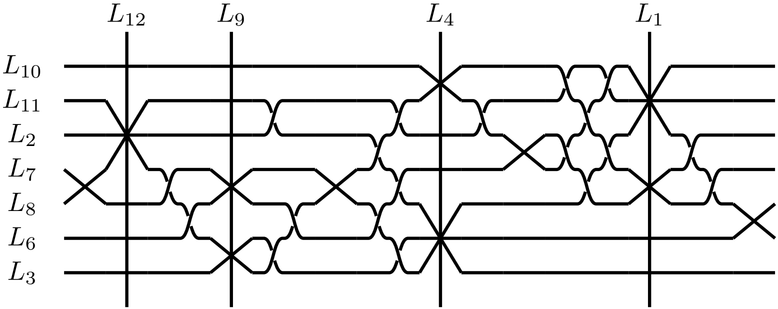
$$\ker\left(\Lambda \to \mathbb{Z}\right), \qquad \prod_{j=0}^{\ell} t_j^{n_j} \in H_1^{\mathscr{C}} \mapsto 1$$

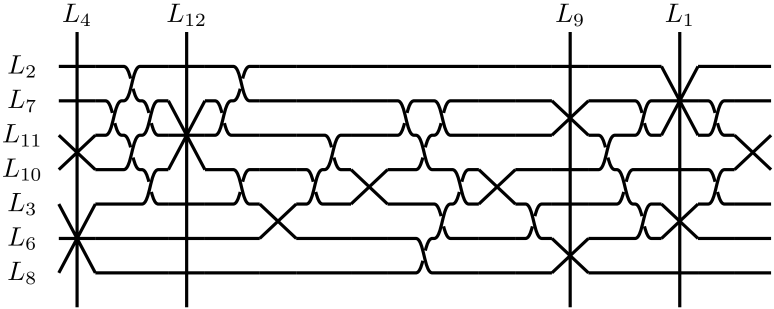
- $M_{\mathcal{A}}^k := M/\mathfrak{m}^k M = M \otimes_\Lambda \Lambda/\mathfrak{m}^k$ *truncated Alexander invariant.*
- Denote $s_i := t_i - 1 \in \mathfrak{m}$.
- $\theta_{k+2}(G_{\mathcal{A}}) := \ker(\varphi_k : G'_{\mathcal{A}} \to M_{\mathcal{A}}^k)$ Chen group

# Truncated Alexander Invariant

- $G_{\mathcal{A}} = \langle x_1, \ldots, x_\ell \mid R_1, \ldots, R_s \rangle$,
- $M_{\mathcal{A}}$ generated by $x_{i,j} = [x_i, x_j] \bmod G''$ and relators:
  - Rewriting $R_j$
    - $[x_i, x_j] = 1 \mapsto x_{i,j} = 0$
    - $[x_i, x_j x_k] = 1 \mapsto 0 = x_{i,j} + t_j \cdot x_{i,k} = x_{i,j} + x_{i,k} + s_j \cdot x_{i,k}$
    - $[x_i, x_k x_j x_k^{-1}] = 1 \mapsto s_i \cdot x_{k,j} + x_{i,j} = 0$
  - Jacobi relations: $s_i \cdot x_{j,k} + s_j \cdot x_{k,i} + s_k \cdot x_{i,j} = 0$
- $\operatorname{gr}^k M_{\mathcal{A}}$, $k = 0, 1$, is combinatorial.
  - $\operatorname{gr}^0 M_{\mathcal{A}} = \left( H_1^{\mathscr{C}} \wedge H_1^{\mathscr{C}} \right) / H_2^{\mathscr{C}}$ generated by $x_{i,j}$
  - $\operatorname{gr}^1 M_{\mathcal{A}} \subset M_{\mathcal{A}}^2$ generated by $s_i \cdot x_{j,k}$
  - $0 \to \operatorname{gr}^1 M_{\mathcal{A}} \to M_{\mathcal{A}}^2 \to \operatorname{gr}^0 M_{\mathcal{A}} \to 0$ splits non canonically.
- $g \in H_1$ and $p \in M_{\mathcal{A}}^k \Longrightarrow [g, p] \in M_{\mathcal{A}}^{k+1}$.

# Wiring diagrams



# Steps of the proof

- Assume an isomorphism $\varphi : G_{\mathcal{A}_\zeta} \to G_{\mathcal{A}_{\zeta^2}}$, $x'_i \mapsto x''_i \cdot g_i$, $g_i \in G'_{\mathcal{A}_{\zeta^2}}$
- $\operatorname{rk} M^1_{\mathcal{A}_\zeta} = \operatorname{rk} M^1_{\mathcal{A}_{\zeta^2}} = \operatorname{rk} \operatorname{gr}^0 M_2^{\mathscr{G}_{91}} = 23$, basis $\{x_{i,j} \mid (i,j) \in \mathcal{B}\}$.
- $\varphi_* : M^2_{\mathcal{A}_\zeta} \to M^2_{\mathcal{A}_{\zeta^2}}$. Need:

$$g_i \equiv \sum_{(j,k)\in\mathcal{B}} n_{i,j,k} x''_{j,k} \in M^1_{\mathcal{A}_{\zeta^2}}, \quad n_{i,j,k} \in \mathbb{Z}$$

$$x'_{i,j} \mapsto [x''_i \cdot g_i, x''_j \cdot g_j] = x''_{i,j} + s_i \cdot g_j - s_j \cdot g_i$$

- $R_i$, $i = 1, \ldots, 32$, relation of $G_{\mathcal{A}_\zeta}$ rewritten in $M^2_{\mathcal{A}_\zeta}$. For example if $L_i, L_j, L_k$ triple point:

$$x'_{i,j} + x'_{i,k} + \text{ terms in } \operatorname{gr}^1 M_{\mathcal{A}_\zeta}; \qquad x'_{i,k} + x'_{j,k} + \text{ terms in } \operatorname{gr}^1 M_{\mathcal{A}_\zeta}$$

- $\varphi_*(R_i) \in M^2_{\mathcal{A}_{\zeta^2}} \otimes \mathbb{Z}[n_{i,j,k}]$, more precisely

$$\varphi_*(R_i) \in \operatorname{gr}^1 M_{\mathcal{A}_{\zeta^2}} \otimes \mathbb{Z}[n_{i,j,k}], \quad \operatorname{rk} \operatorname{gr}^1 M_2^{\mathscr{G}_{91}} \cong \mathbb{Z}^{91}$$

# Linear equations

- We obtain a system $\mathcal{S}$ of $32 \times 91 = 2912$ linear equations in $11 \times 23 = 253$ unknowns
- Existence of $\varphi \implies$ existence of integer solutions in $\mathcal{S}$.
- Solve $\mathcal{S}$ with `Sagemath`.
- Solution over $\mathbb{Q}$: $\mathbb{Q}$-affine space of dim $= 12$
- Smallest ring where $\mathcal{S}$ admit solutions is $\mathbb{Z}\left[\frac{1}{5}\right]$.
- Whole process 314.02s CPU time (mostly for rewriting relations!).

# Main result III

## Theorem

*The groups* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ *and* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ *are not isomorphic (while their profinite completions are).*

## First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = \pm 1_{H_1^{\mathscr{G}_{91}}}$.

## Second step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = 1_{H_1^{\mathscr{G}_{91}}}$

## Third step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^3})$ isomorphism $\Longrightarrow \varphi_* = 1_{H_1^{\mathscr{G}_{91}}}$

Thank you

どうもありがとう