Michel Boileau

Profinite completion of groups and 3-manifolds I

Branched Coverings, Degenerations, and Related Topics

Hiroshima March 2016

Joint work with Stefan Friedl

17 mars 2016
Finite quotients

In this lecture $\pi$ will be a finitely generated and residually finite group.
Let $Q(\pi)$ be the set of finite quotients of $\pi$.
What properties of $\pi$ can be deduced from $Q(\pi)$?
For example if all finite quotient of $\pi$ are abelian, then $\pi$ is abelian.
Finite quotients of $\pi$ corresponds to finite index normal subgroups of $\pi$
So properties related to finite quotients of $\pi$ are encoded in the profinite completion of $\pi$. 
Profinite completion

Let $\mathcal{N}(\pi)$ be the collection of all finite index normal subgroups $\Gamma$ of $\pi$. $\mathcal{N}(\pi)$ is a directed set for the following pre-order: $\Gamma' \geq \Gamma$ if $\Gamma' \subset \Gamma$. If $\Gamma' \geq \Gamma$ there is an induced epimorphism $h_{\Gamma', \Gamma}: \pi/\Gamma' \to \pi/\Gamma$.

So to a group $\pi$ one can associate the inverse system:

$$\{\pi/\Gamma, h_{\Gamma', \Gamma}\}_{\Gamma} \text{ with } \Gamma \in \mathcal{N}(\pi)$$

The profinite completion of $\pi$ is defined as the inverse limit of this system:

$$\hat{\pi} = \lim_{\leftarrow} \pi/\Gamma$$
Profinite completion

Equip each finite quotient $\pi/\Gamma$, $\Gamma \in \mathcal{N}(\pi)$ with the discrete topology.

The set $\prod_{\Gamma \in \mathcal{N}(\pi)} \{\pi/\Gamma\}$ is compact.

Let $i_\pi : \pi \to \prod_{\Gamma \in \mathcal{N}(\pi)} \{\pi/\Gamma\}$ given by $\{g \in \pi \to \{g\Gamma\}_{\Gamma \in \mathcal{N}(\pi)}\}$.

Then $\hat{\pi}$ can be identified with the closure $\overline{i_\pi(\pi)}$ in $\prod_{\Gamma \in \mathcal{N}(\pi)} \{\pi/\Gamma\}$.

$i_\pi : \pi \to \hat{\pi}$ is injective since $\pi$ is residually finite.
Profinite completion

\( \hat{\pi} \) is a compact topological group.

A subgroup \( U < \hat{\pi} \) is open if and only if it is closed and of finite index.

A subgroup \( H < \hat{\pi} \) is closed if and only if it is the intersection of all open subgroups of \( \hat{\pi} \) containing it.

**Thm (N. Nikolov and D. Segal (2007))**

Let \( \pi \) be a finitely generated group. Then every finite index subgroup of \( \hat{\pi} \) is open. In particular \( \hat{\hat{\pi}} = \hat{\pi} \).

**Corollary**

Let \( \pi \) be a finitely generated and residually finite group, then :

(i) A finite index subgroup \( \Gamma \subset \pi \rightarrow \overline{\Gamma} \subset \hat{\pi}, \ [\pi : \Gamma] = [\hat{\pi} : \overline{\Gamma}] \) and \( \overline{\Gamma} \cong \hat{\Gamma} \).

(ii) Conversely an open subgroup \( H \subset \hat{\pi} \rightarrow H \cap \pi \in \pi \).

(iii) \( \Gamma \leq \pi \iff \overline{\Gamma} \leq \hat{\pi} \), and \( \pi / \Gamma \cong \hat{\pi} / \overline{\Gamma} \).
Homomorphisms

An important consequence is:

Lemma

For any finite group $G$ the map $i_{\pi} : \pi \rightarrow \hat{\pi}$ induces a bijection $i_{\pi}^* : \text{Hom}(\hat{\pi}, G) \rightarrow \text{Hom}(\pi, G)$.

A group homomorphism $\varphi : A \rightarrow B$ induces a continuous homomorphism $\hat{\varphi} : \hat{A} \rightarrow \hat{B}$.

If $A$ and $B$ are finitely generated, any homorphism $\hat{A} \rightarrow \hat{B}$ is continuous.

If $\varphi$ is an isomorphism, so is $\hat{\varphi}$.

On the other hand, an isomorphism $\phi : \hat{A} \rightarrow \hat{B}$ is not necessarily induced by a homomorphism $\varphi : A \rightarrow B$.

There are isomorphisms $\hat{\mathbb{Z}} \rightarrow \hat{\mathbb{Z}}$ that are not induced by an automorphism of $\mathbb{Z}$.
Isomorphisms

Let $A$ and $B$ be two finitely generated groups and $f : \hat{A} \to \hat{B}$ be an isomorphism.

For any finite group $G$ the isomorphism $f : \hat{A} \to \hat{B}$ induces a bijection:

$$i_A^* f^* i_B^{-1} : \text{Hom}(B, G) \xrightarrow{i_B^{-1}} \text{Hom}(\hat{B}, G) \xrightarrow{f^*} \text{Hom}(\hat{A}, G) \xrightarrow{i_A^*} \text{Hom}(A, G).$$

Given $\beta \in \text{Hom}(B, G)$ denote by $\beta \circ f$ the resulting homomorphism in $\text{Hom}(A, G)$.

Groups $A$ and $B$ with isomorphic profinite completions have the same set of finite quotients: $Q(A) = Q(B)$.

The converse also holds:

**Lemma**

*Two finitely generated groups $A$ and $B$ have isomorphic profinite completions if and only if they have the same set of finite quotients.*
Profinite rigidity

According to Grunwald and Zaleskii let define the genus of $\pi$ as :

**Definition**

$$G(\pi) = \{\text{finitely generated, residually finite groups } \Gamma \text{ such that } \hat{\Gamma} \cong \hat{\pi}\}, \text{ modulo isomorphisms.}$$

A residually finite and finitely generated group $\pi$ is profinitely rigid if $G(\pi) = \{\pi\}$.

**Question**

Which groups are profinitely rigid? Can $G(\pi)$ be infinite?

Surprisingingly, the following question is still open :

**Question**

Is a finitely generated free group profinitely rigid?
Profinite properties

One may ask a weaker question:

**Question**

What group theoretic properties are shared by groups in $G(\pi)$?

Such properties are called *profinite properties* of a group. For example, being abelian is a profinite property.

The next lemma says that the abelianizations are the same.

**Lemma**

$\hat{\Gamma} \cong \hat{\pi} \Rightarrow \Gamma^{ab} \cong \pi^{ab}$

**Corollary**

*If $\pi$ is abelian, then $G(\pi) = \{\pi\}$*
Examples

In general $G(\pi) \neq \{\pi\}$

**Thm (Baumslag 1974) ; Hirshon (1977))**

Let $\Gamma$ and $\pi$ two finitely generated groups. If $\Gamma \times \mathbb{Z} \cong \pi \times \mathbb{Z}$ then $\hat{\Gamma} \cong \hat{\pi}$.

Given a group $A$ and a class $\psi \in \text{Aut}(A)$, one can build the semidirect product $A_\psi := A \rtimes_\psi \mathbb{Z}$.

It corresponds to the split exact sequence

$$1 \to A \to A_\psi \to \mathbb{Z} \to 1,$$

where the action of $\mathbb{Z}$ on $A$ is given by $\psi$.

The isomorphism type of $A_\psi$ depends only on the class of $\psi$ in $\text{Out}(A)$.

As a consequence one gets examples of finitely generated and residually finite groups which are not profinitely rigid:
Examples

**Corollary**

Let $A$ be a finitely presented and residually finite group and $\psi \in \text{Aut}(A)$ such that $\psi^n$ is an inner automorphism for some $n \in \mathbb{Z}$. Then for any $k \in \mathbb{Z}$ relatively prime to $n$, $\hat{A}_{\psi^k} \cong \hat{A}_\psi$.

**Example**

Let $\pi_1 = \mathbb{Z}/25\mathbb{Z} \rtimes_\psi \mathbb{Z}$ and $\pi_2 = \mathbb{Z}/25\mathbb{Z} \rtimes_{\psi^2} \mathbb{Z}$, $\psi \in \text{Aut}(\mathbb{Z}/25\mathbb{Z})$ be given by $\psi(x) = x^6$ for a generator $x \in \mathbb{Z}/25\mathbb{Z}$. Then $\hat{\pi}_1 \cong \hat{\pi}_2$. In this example $\psi$ is of order 5 in $\text{Out}(\mathbb{Z}/25\mathbb{Z})$.

Since $A$ is residually finite and finitely generated, the profinite completion $\hat{A}_\psi$ can be computed from $\hat{A}$ and $\hat{\mathbb{Z}}$. 
Examples

The system of characteristic finite index subgroups $C(n) := \cap_{[A:\Gamma] \leq n} \Gamma$ is cofinal in $A$.

For each $n \in \mathbb{N}$ there exists some $m \in \mathbb{N}$ such that $\psi^m$ induces the identity on the characteristic quotient $A/C(n)$.

It follows that $C(n)_{\psi^m} := C(n) \rtimes_{\psi^m} \mathbb{Z}$ is a cofinal system of normal finite index subgroups of $A_{\psi}$, since $A \cap C(n)_{\psi^m} = C(n)$.

In particular $A_{\psi}$ is residually finite and its profinite topology induces that of $A$, so the closure $\overline{A} \subset \hat{A}_{\psi}$ can be identified with $\hat{A}$.

By using the automorphisms induced by the elements of $\text{Aut}(A)$ on the finite quotients $A/C(n)$ and the equality $\hat{A} = \lim_{\leftarrow} A/C(n)$, one can define an homomorphism $\text{Aut}(A) \to \text{Aut}(\hat{A})$.

Since $\text{Aut}(A)$ is itself residually finite, the above homomorphism extends to a homomorphism $\hat{\text{Aut}(A)} \to \text{Aut}(\hat{A})$. 

Examples

Therefore any homomorphism \( \psi : \mathbb{Z} \rightarrow \text{Aut}(A) \) extends to a homomorphism \( \hat{\psi} : \hat{\mathbb{Z}} \rightarrow \text{Aut}(\hat{A}) \rightarrow \text{Aut}(\hat{A}) \).

These are key observations for the proof of the following results:

**Proposition (Nikolov-Segal 2007)**

Let \( A \) be a finitely generated and residually finite group and \( \psi \in \text{Aut}(A) \), then:

1. \( \hat{A}_\psi = \hat{A} \times_{\psi} \mathbb{Z} = \hat{A} \times \hat{\psi} \mathbb{Z} \).
2. \( \hat{A}_\psi = \hat{A} \times \hat{\mathbb{Z}} \) if and only if \( \psi \) induces an inner automorphisms on the finite characteristic quotients of \( A \).

Nikolov and Segal have given an example of a finitely generated and residually finite group \( A \) with an automorphism \( \psi \in \text{Aut}(A) \) such that no positive power of \( \psi \) is an inner automorphism, but \( \hat{A}_\psi = \hat{A} \times \hat{\mathbb{Z}} \).
3-manifold groups

In these lectures $M$ will be a compact orientable aspherical 3-manifold with empty or toroidal boundary. For example the exterior $E(K)$ of a knot $k \subset S^3$.

By Perelman’s Geometrization Theorem $\pi_1(M)$ is residually finite.

**Definition**

An orientable compact 3-manifold $M$ is called profinitely rigid if $\hat{\pi}_1(M)$ distinguishes $\pi_1(M)$ from all other 3-manifold groups.

There are closed 3-manifolds which are not profinitely rigid.

The examples known at the moment are **Sol manifolds** (P. Stebe, L. Funar), or **Surface bundle with periodic monodromy**, i.e **Seifert fibered manifolds** (J. Hempel).
Examples : Seifert fibered

We describe now the Seifert fibered examples given by J. Hempel.

Let $F$ be a closed orientable surface, $h \in \text{Homeo}^+(F)$ and $M = F \times_h S^1$ be the surface bundle over $S^1$ with monodromy $h$.

Let $h_* \in \text{Aut}(\pi_1(F))$ be the automorphism induced by $h$, then
\[ \pi_1(F)_{h_*} = \pi_1(F) \times_{h_*} \mathbb{Z} \cong \pi_1(M). \]

Proposition (Hempel 2014)

If $M$ and $N$ are surface bundles with periodic monodromies $h$ and $h^k$, for $k$ coprime to the order of $h$, then $\hat{\pi_1(N)} \cong \hat{\pi_1(M)}$. 
Seifert fibered rigidity

**Thm (G. Wilkes (2015))**

Let $M$ be a closed orientable irreducible Seifert fibre space. Let $N$ be a compact orientable 3-manifold with $\pi_1(N) \cong \pi_1(M)$. Then either:

- $M$ is profinitely rigid, i.e. $\pi_1(N) \cong \pi(M)$, or
- $M$ and $N$ are Hempel examples.

**Corollary**

Let $F$ be a closed orientable surface. A homeomorphism $h$ of $F$ is homotopic to the identity if and only if it induces an inner automorphisms on every finite characteristic quotient of $\pi_1(F)$.

Does the action induced by $h$ on all the finite characteristic quotients of $\pi_1(F)$ determine $h_* \in Out(\pi_1(F))$ when $h$ is not periodic?

The next examples of torus bundles with Anosov monodromies show that it is not true.