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# Profinite completion of groups and 3-manifolds II

Branched Coverings, Degenerations, and Related Topics

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# 3-manifold groups

Throughout the lecture all 3-manifolds are understood to be connected, compact, orientable, aspherical and with empty or toroidal boundary.

By Perelman's Geometrization Theorem $\pi_1(M)$ is residually finite.

**Definition**

*An orientable compact 3-manifold $M$ is called profinitely rigid if for any 3-manifold $N$ $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)} \Rightarrow \pi_1(M) \cong \pi_1(N)$.*

There are closed 3-manifolds which are not profinitely rigid.

The examples known at the moment are **SOL manifolds** (P. Stebe, L. Funar), or **Surface bundle with periodic monodromy, i.e Seifert fibered manifolds** (J. Hempel).

The SOL examples are torus bundles with Anosov monodromies. They show that the action induced by the monodrmy $h$ on all the finite characteristic quotients of $\pi_1(F)$ does not determine $h_\star \in Out(\pi_1(F))$

# Examples : SOL

The examples of solvable fundamental groups do not follow from the Baumslag-Hirshon theorem :

**Proposition (P. Stebe(1972), L. Funar(2012))**

*There exist infinitely many pairs of torus bundles with Anosov monodromies whose fundamental groups have the same profinite completion, but are not isomorphic.*

In these examples $A \cong \mathbb{Z} \times \mathbb{Z}$ and $\psi, \varphi \in Out(A) = GL(2, \mathbb{Z})$ are represented by non conjugate Anosov matrices $\Psi$ and $\Phi$, whose images in $GL(2, \mathbb{Z}/n\mathbb{Z})$ are conjugate for every integer $n > 1$.

Stebe' s example : $\Psi = \begin{pmatrix} 188 & 275 \\ 121 & 177 \end{pmatrix}$ and $\Phi = \begin{pmatrix} 188 & 11 \\ 3025 & 177 \end{pmatrix}$

# Rigidity

There are no hyperbolic examples known, so the following question makes sense :

**Question (Rigidity)**

*Which compact, orientable, irreducible 3-manifolds are profinitely rigid ? In particular what about hyperbolic 3-manifolds ?*

The answer is positive for the figure-eight knot group by the work of M. Bridson and A. Reid :

**Thm (Bridson-Reid (2015))**

*The figure-eight knot group is detected by its profinite completion, among 3-manifold groups.*

This is true more generally for a puntured torus bundle (Bridson, Reid and Wilton 2015).

# Finiteness

The following finiteness problem is of interest :

**Question (Finiteness)**

*Given a compact orientable aspherical 3-manifold $M$ with empty or toroidal boundary, are there only finitely many 3-manifolds $N$ with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$ ?*

This is true for closed orientable Seifert fibered 3-manifolds :

**Thm (G. Wilkes (2015))**

*Let $M$ be a closed orientable irreducible Seifert fibre space. Let $N$ be a compact orientable 3-manifold with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$. Then either :*

- *$M$ is profinitely rigid, i.e. $\pi_1(N) \cong \pi(M)$, or*
- *$M$ and $N$ are surface bundles with periodic monodromies $h$ and $h^k$, for $k$ coprime to the order of $h$.*

# Profinite invariants

The main question addressed in the remaining of these lectures is :

## Question

*Which invariants or properties of $M$ are detected by $\widehat{\pi_1(M)}$ ?*

## Definition

*An invariant $\sigma$ (or a property $P$) is a profinite invariant (or a profinite property) if, given two compact, aspherical, orientable 3-manifold $M$ and $N$ with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$, $M$ and $N$ have the same invariant $\sigma$ (or $M$ has the property $P$ if and only if $N$ does).*

# Goodness

Following Serre a group $\pi$ is good if the following holds :

For any finite abelian group $A$ and any representation $\alpha : \pi \to Aut_{\mathbb{Z}}(A)$

the inclusion $\iota : \pi \to \widehat{\pi}$ induces an isomorphism $\iota^* : H^j_\alpha(\widehat{\pi}; A) \to H^j_\alpha(\pi; A)$, for any $j$.

The proof of the following theorem uses Agol's virtual fibration theorem :

**Thm (W. Cavendish (2012))**

*The fundamental group of any compact aspherical 3-manifold is good.*

**Corollary**

*For a compact aspherical 3-manifolds :*

*(i) the property of being closed is a profinite property.*

*(ii) the euler characteristic $\chi(M)$ is a profinite invariant.*

# Geometries

It is natural to ask whether the profinite completion detects Thurston' s geometric structures.

**Thm (H. Wilton-P.Zaleskii (2014))**

*Let $M$ be a closed aspherical orientable 3-manifold, then :*

1. *being hyperbolic is a profinite property.*
2. *being Seifert fibered is a profinite propery.*

The case (2) of this Theorem is used by Wilkes in the proof of his rigidity result for Seifert manifolds

The non-empty boundary case is still open in general.

With S. Friedl we settled the Seifert fibered case for knot exteriors.

# Geometries

### Corollary

*Let $M$ and $N$ two closed orientable aspherical 3-manifolds such that $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$. If $M$ admits a geometric structure then $N$ admits the same geometric structure.*

Profinite completion distinguish hyperbolic geometry among Thurston's geometries because hyperbolic manifold groups are residually non-abelian simple.

Coming back to the case of surface bundles over the circle, one gets the following corollary :

### Corollary

*Let $F$ be a closed orientable surface and $h$ a homeomorphism on $F$. Whether $h$ is pseudo-Anosov or periodic is detected by the actions induced by $h$ on all the finite characteristic quotients of $\pi_1(F)$.*

# Volume conjecture

For $M$ compact aspherical with empty or toroidal boundary, let define :

$Vol(M) =$ sum of volumes of hyperbolic pieces in the geometric decomposition of $M$.

Let $\mathcal{N}(\pi_1(M))$ be the set of all finite index normal subgroups $\Gamma$ of $\pi_1(M)$

### Conjecture (Asymptotique volume conjecture)

$$\limsup_{\Gamma \in \mathcal{N}(\pi_1(M))} \frac{\log(Tor(\Gamma^{ab}))}{[\pi_1(M):\Gamma]} = \frac{Vol(M)}{6\pi}$$

### Thm (T. Le (2014))

$$\limsup_{\Gamma \in \mathcal{N}(\pi_1(M))} \frac{\log(Tor(\Gamma^{ab}))}{[\pi_1(M):\Gamma]} \leq \frac{Vol(M)}{6\pi}$$

# volume conjecture

A much weaker question is :

**Question**

*Is $Vol(M)$ a profinite invariant ?*

A positive answer would settle the finiteness question for closed hyperbolic 3-manifolds.

It would also answer positively the following question :

**Question**

*Does profinite completion detects graph manifolds ?*

A *graph 3-manifold* is obtained by gluing along some boundary components finitely many elementary pieces homeomorphic to a *solid torus* $S^1 \times D^2$ or a *composite space* $S^1 \times \{\text{punctured disk}\}$.

It is obtained by gluing together geometric pieces which are not hyperbolic, hence its volume vanishes.

# Maps

A continuous map $f\colon M \to N$ induces an homomorphism $f_\star\colon \pi_1(M) \to \pi_1(N)$ and thus an homomorphism $\hat{f}_\star\colon \widehat{\pi_1(N)} \to \widehat{\pi_1(M)}$.

**Proposition (Dixon-Formanek-Poland-Ribes 1982)**

*Let $f\colon M \to N$ be a proper map between two compact aspherical 3-manifolds. If $f$ is a $\pi_1$-epimorphism and if there exists an isomorphism $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$(not necessarily induced by $f$), then $f$ is homotopic to a homeomorphism.*

Given a finitely generated group $\pi$, let $\pi(n) = \cap_{[\pi:K]\leq n} K$ the intersection of the subroups $K \subset \pi$ of index $\leq n$.

The system $\pi(n)$ of finite-index characteristic subgroups is cofinal and suffices to define the profinite completion : $\widehat{\pi} = \varprojlim \pi/\pi(n)$.

$\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)} \Rightarrow \pi_1(M)/\pi_1(M)(n) \cong \pi_1(N)/\pi_1(N)(n)$ for each $n \geq 1$.

# Maps

Therefore the induced epimorphisms

$f_*\colon \pi_1(M)/\pi_1(M)(n) \to \pi_1(N)/\pi_1(N)(n)$ are isomorphisms for all $n$.

Hence $\ker\{f_*\colon \pi_1(M) \to \pi_1(N)/\pi_1(N)(n)\} = \pi_1(M)(n)$ for all $n$.

$\ker f_* \subset \ker\{f_*\colon \pi_1(M) \to \pi_1(N)/\pi_1(N)(n)\} \Rightarrow$

$\ker f_* \subset \cap_{n\geq 1}\pi_1(M)(n) = \{1\}$, by residual finiteness of 3-manifold groups

So $f_*$ is injective and thus it is an isomorphism.

The result follows now from Waldhausen's theorem and from Mostow-Prasad rigidity theorem.

# Virtual rank

A subgroup $\Gamma \subset \pi$ is called *subnormal* if there exists a chain of subgroups $\pi = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \Gamma$, such that each $\Gamma_i$ is normal in $\Gamma_{i-1}$.

Given a finitely generated group $\pi$ we denote by $\mathrm{rk}(\pi)$ its rank, i.e. the smallest cardinality of a generating set of $\pi$.

If $f\colon M \to N$ is a $\pi_1$-epimorphism, then for any finite-index subgroup $\Gamma$ of $\pi_1(N)$ : $\mathrm{rk}(f_*^{-1}(\Gamma)) \geq \mathrm{rk}(\Gamma)$.

**Thm (Virtual rank, B-Friedl 2016)**

*Let $f\colon M \to N$ be a proper map between two aspherical 3-manifolds with empty or toroidal boundary. Assume that $N$ is not a closed graph manifold and that $f$ is a $\pi_1$-epimorphism such that for every finite-index subnormal subgroup $\Gamma$ of $\pi_1(N)$ :*

$$\mathrm{rk}\left((f_*)^{-1}(\Gamma)\right) = \mathrm{rk}(\Gamma).$$

*Then $f$ is homotopic to a homeomorphism.*

# Heegaard genus

Let $h(M)$ be the minimal number of one-handles in a handle-composition of $M$ with one zero-handle.

If $M$ is closed, $h(M)$ equals the *Heegaard genus*.

A covering $p\colon \widehat{N} \to N$ is *subregular* if $p$ can be written as a composition of regular coverings $p_i\colon N_i \to N_{i-1}$, $i = 1, \ldots, k$ with $N_k = \widehat{N}$ and $N_0 = N$.

Here is a variation on the previous theorem.

**Thm (virtual genus, B-Friedl 2016)**

*Let $f\colon M \to N$ be a proper map between two aspherical 3-manifolds with empty or toroidal boundary. Assume that $N$ is not a closed graph manifold and that $f$ is a $\pi_1$-epimorphism such that for every finite subregular cover $\widetilde{N}$ of $N$ and induced cover $\widetilde{M}$ :*

$$h(\widetilde{M}) = h(\widetilde{N}).$$

*Then $f$ is homotopic to a homeomorphism.*

# Heegaard genus

It is not known whether for every $\pi_1$-epimorphism $f : M \to N$ between two aspherical 3-manifolds the inequality $h(M) \geq h(N)$ holds.

The following result is a consequence of the proof of the previous Theorem and shows that the inequality holds virtually.

## Proposition

*Let $f : M \to N$ be a proper map between two aspherical 3-manifolds with empty or toroidal boundary. Assume that $N$ is not a closed graph manifold and that $f$ is a $\pi_1$-epimorphism. Then there exists a finite subregular cover $\widetilde{N}$ of $N$ such that the induced cover $\widetilde{M}$ satisfies the inequality $h(\widetilde{M}) \geq h(\widetilde{N})$.*

## Remark

*The proof is based on the Virtual Fibering Theorem of Agol, Przytycki-Wise and Wise : any aspherical 3-manifold with empty or toroidal boundary that is not a closed graph manifold is virtually a surface bundle.*