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Profinite completion of groups and 3-manifolds III

Branched Coverings, Degenerations, and Related Topics

Hiroshima March 2016

Joint work with Stefan Friedl

13 mars 2016

Thurston norm

We study now the relation between the profinite completion, and the Thurston norm of a 3-manifold.

M is still a compact, orientable, aspherical 3-manifold, with ∂M empty or an union of tori.

The complexity of a compact orientable surface F with connected components $F_1, \ldots, F_k$ is defined to be

$$\chi_-(F) := \sum_{i=1}^{d} \max\{-\chi(F_i), 0\}.$$

The Thurston norm of a cohomology class $\phi \in H^1(M;\mathbb{Z})$ is defined as

$$x_M(\phi) := \min\{\chi_-(F) \,|\, F \subset M \text{ properly embedded and dual to } \phi\}.$$

x_M extends to a seminorm on $H^1(M;\mathbb{R})$.

Regular isomorphism

Let M_1 and M_2 be two 3-manifolds such that there exists an isomorphism $f : \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$.

Such an isomorphism induces an isomorphism $\widehat{H_1(M_1;\mathbb{Z})} \to \widehat{H_1(M_2;\mathbb{Z})}$.

Thus $H_1(M_1;\mathbb{Z})$ and $H_1(M_2;\mathbb{Z})$ are **abstractly** isomorphic.

In general the isomorphism $\widehat{H_1(M_1;\mathbb{Z})} \to \widehat{H_1(M_2;\mathbb{Z})}$ is not induced by an isomorphism $H_1(M_1;\mathbb{Z}) \to H_1(M_2;\mathbb{Z})$.

To compare the Thurston norms of M_1 and M_2, let introduce the following :

Definition

An isomorphism $f : \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$ is regular if the induced isomorphism $\widehat{H_1(M_1;\mathbb{Z})} \to \widehat{H_1(M_2;\mathbb{Z})}$ is induced by an isomorphism $f_* : H_1(M_1;\mathbb{Z}) \to H_1(M_2;\mathbb{Z})$.

Fibered class

Definition

A class $\phi \in H^1(N;\mathbb{R})$ *is called fibered if there is a fibration* $p : M \to S^1$ *such that* $\phi = p_* : \pi_1(M) \to \mathbb{Z}$.

Thm (B-Friedl 2015)

Let M_1 *and* M_2 *be two aspherical 3-manifolds with empty or toroidal boundary. If* $f : \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$ *is a regular isomorphism, then :*

(1) for any class $\phi \in H^1(M_2;\mathbb{R})$, $x_{M_2}(\phi) = x_{M_1}(f^*\phi)$.

(2) $\phi \in H^1(M_2;\mathbb{R})$ *is fibered* $\iff$ $f^*\phi \in H^1(M_1;\mathbb{R})$ *is fibered.*

When $\partial M_1 \neq \emptyset$ and ϕ is a fibered class, this result has been obtained by A. Reid and M. Bridson, by a different method.

Twisted Alexander polynomials

Let X be a CW-complex, $\phi \in H^1(X;\mathbb{Z})$ and $\alpha\colon \pi_1(X) \to \mathrm{GL}(k,\mathbb{F})$, $\mathbb{F}$ being a field.

Set $\mathbb{F}[t^{\pm 1}]^k := \mathbb{F}^k \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$ and consider the tensor representation :

$\alpha \otimes \phi\colon \pi_1(X) \to \mathrm{Aut}_{\mathbb{F}[t^{\pm 1}]}(\mathbb{F}[t^{\pm 1}]^k)$, given by :

$g \mapsto \left(\sum_i v_i \otimes p_i(t) \mapsto \sum_i \alpha(g)(v_i) \otimes t^{\phi(g)} p_i(t)\right)$.

So one can view $\mathbb{F}[t^{\pm 1}]^k$ as a left $\mathbb{Z}[\pi_1(X)]$-module.

The twisted homology groups $H_i^{\alpha\otimes\phi}(X;\mathbb{F}[t^{\pm 1}]^k)$ are naturally $\mathbb{F}[t^{\pm 1}]$-modules.

Definition

The i-th twisted Alexander polynomial $\Delta^{\alpha}_{X,\phi,i} \in \mathbb{F}[t^{\pm 1}]$ is the order of the $\mathbb{F}[t^{\pm 1}]$-module $H_i^{\alpha\otimes\phi}(X;\mathbb{F}[t^{\pm 1}]^k)$.

The twisted Alexander polynomials are well-defined up to multiplication by some at^k where $a \in \mathbb{F} \setminus \{0\}$ and $k \in \mathbb{Z}$ (i.e. a unit in $\mathbb{F}[t^{\pm 1}]$).

Fiberedness and Thurston norm

For a polynomial $f(t) = \sum_{k=r}^{s} a_k t^k \in \mathbb{F}[t^{\pm 1}]$ with $a_r \neq 0$ and $a_s \neq 0$ define $\deg(f(t)) = s - r$. For the zero polynomial set $\deg(0) := +\infty$.

Thm (Friedl-Vidussi 2013 ; Friedl-Nagel 2015)

Let M be a compact, aspherical, orientable 3-manifold with empty or toroidal boundary and $\phi \neq 0 \in H^1(M;\mathbb{Z})$:

(1) The class ϕ is fibered $\Leftrightarrow \Delta^{\alpha}_{M,\phi,1} \neq 0$ for all primes p and all representations $\alpha : \pi_1(M) \to GL(k, \mathbb{F}_p)$.

(2) There exists a prime p and a representation $\alpha : \pi_1(M) \to GL(k, \mathbb{F}_p)$ such that

$$x_M(\phi) = \max\left\{0, \frac{1}{k}\left(-\deg\left(\Delta^{\alpha}_{M,\phi,0}\right) + \deg\left(\Delta^{\alpha}_{M,\phi,1}\right) - \deg\left(\Delta^{\alpha}_{M,\phi,2}\right)\right)\right\}.$$

The proof is building on the work of Agol, Przytycki-Wise and Wise.

Degrees of twisted Alexander polynomials

Given a group π, $\phi \in H^1(\pi;\mathbb{Z}) = \mathrm{Hom}(\pi,\mathbb{Z})$ and $n \in \mathbb{N}$, set $\phi_n : \pi \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}_n$

For a representation $\alpha : \pi \to GL(k,\mathbb{F})$ and $n \in \mathbb{N}$, let $\mathbb{F}[\mathbb{Z}_n]^k = \mathbb{F}^k \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}_n]$ and $\alpha \otimes \phi_n : \pi \to \mathrm{Aut}(\mathbb{F}[\mathbb{Z}_n]^k)$ the induced representation.

Proposition

Let X be a CW-complex, $\phi \in H^1(X;\mathbb{Z}) \setminus 0$, $\alpha : \pi_1(X) \to GL(k,\mathbb{F}_p)$, then :

(1) $\deg \Delta^{\alpha}_{X,\phi,0}(t) = \max\left\{ \dim_{\mathbb{F}_p}\left(H_0^{\alpha\otimes\phi_n}(X;\mathbb{F}_p[\mathbb{Z}_n]^k)\right) \,\middle|\, n \in \mathbb{N} \right\}$

(2) $\deg \Delta^{\alpha}_{X,\phi,1}(t) =$
$\max\left\{ \dim_{\mathbb{F}_p}\left(H_1^{\alpha\otimes\phi_n}(X;\mathbb{F}_p[\mathbb{Z}_n]^k)\right) - \dim_{\mathbb{F}_p}\left(H_0^{\alpha\otimes\phi_n}(X;\mathbb{F}_p[\mathbb{Z}_n]^k)\right) \,\middle|\, n \in \mathbb{N} \right\}$

twisted homology

Now the proof follows from the following two results :

Proposition

Let π_1 and π_2 be good groups and $f : \widehat{\pi_1} \xrightarrow{\cong} \widehat{\pi_2}$ an isomorphism. Let $\beta : \pi_2 \to \mathrm{GL}(k, \mathbb{F}_p)$ be a representation. Then for any i there is an isomorphism

$$H_i^{\beta \circ f}(\pi_1; \mathbb{F}_p^k) \cong H_i^{\beta}(\pi_2; \mathbb{F}_p^k).$$

Since 3-manifold groups are good, one gets :

Corollary

Let M_1 and M_2 be two 3-manifolds. Suppose $f : \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$ is a regular isomorphism. Then for any $\phi \neq 0 \in H^1(M_2, \mathbb{Z})$ and any representation $\alpha : \pi_1(M_2) \to GL(k, \mathbb{F}_p)$ one has :

$$\deg\left(\Delta^{\alpha \circ f}_{M_1, \phi \circ f, i}\right) = \deg\left(\Delta^{\alpha}_{M_2, \phi, i}\right), \quad i = 0, 1, 2.$$

$b_1 = 1$

When M_1 and M_2 have $b_1 = 1$, we do not need the regular assumption because of the following lemma :

Lemma

Let M be a 3-manifold with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ and $\beta\colon \pi_1(M) \to \mathrm{GL}(k, \mathbb{F}_p)$ a representation. Let $\phi_n\colon \pi_1(M) \to \mathbb{Z}_n$ and $\psi_n\colon \pi_1(M) \to \mathbb{Z}_n$ be two epimorphisms. Then given any i there exists an isomorphism $H_i^{\beta\otimes\phi_n}(M;\mathbb{F}_p[\mathbb{Z}_n]^k) \cong H_i^{\beta\otimes\psi_n}(M;\mathbb{F}_p[\mathbb{Z}_n]^k)$.

Knot exteriors in S^3 are typical examples of manifolds with first Betti number 1.

Knot exteriors

The exterior $E(K) = S^3 \setminus \mathcal{N}(K)$ of a knot $K \subset S^3$ is a compact orientable 3-manifold with $b_1 = 1$.

$\pi_1(E(K))$ is the group of the knot K.

There is a canonical epimorphism $\pi_1(E(K)) \to H_1(E(K);\mathbb{Z}) \cong \mathbb{Z}$.

For the corresponding class $\phi \in H^1(N;\mathbb{Z})$:

$x_{E(K)}(\phi) = 2g(K) + 1$, where $g(K)$ is the Seifert genus of K.

The knot K is said fibered if $\phi \in H^1(N;\mathbb{Z})$ is a fibered class.

The unknot is the only knot with abelian group. So :

Lemma

Let U be the unknot. If K is a knot with $\pi_1\widehat{(E(U))} \cong \pi_1\widehat{(E(K))}$, then $K = U$.

Knots

Thm (B-Friedl 2015)

Let K_1 and K_2 be two knots in S^3 with $\pi_1\widehat{(E(K_1))} \cong \pi_1\widehat{(E(K_2))}$. Then :

(1) They have the same Seifert genus : $g(K_1) = g(K_2)$

(2) K_1 is fibered iff K_2 is fibered

(3) If Δ_{K_1} has not a zero that is a root of unity, then $\Delta_{K_1} = \pm\Delta_{K_2}$

(4) If K_1 is a torus knot, $K_1 = K_2$.

(5) If K_1 is the figure-8 knot, $K_1 = K_2$.

(6) If $E(K_1)$ and $E(K_2)$ are hyperbolic and have a homeomorphic finite cyclic cover, either $K_1 = K_2$ or Δ_{K_1} and Δ_{K_2} are product of cyclotomic polynomials.

The statements (1) and (2) are direct consequence of the previous Theorem

cyclic coverings

The statement (3) follows from the next lemma.

Given a knot K denote by $E_n(K)$ the n-fold cyclic cover of $E(K)$.

$\pi_1(E_n(K)) = \ker(\pi_1(E(K)) \to H_1(E(K);\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z})$

Lemma

Let K_1 and K_2 be two knots such that $\widehat{\pi_1(E(K_1))} \cong \widehat{\pi_1(E(K_2))}$. Then :

(i) For any n $H_1(E_n(K_1;\mathbb{Z})) \cong H_1(E_n(K_2);\mathbb{Z})$.

(ii) Δ_{K_1} has a zero that is an n-th root of unity $\Leftrightarrow$ Δ_{K_2} has a zero that is an n-th root of unity.

Assertion (i) follows from the following fact :

$\widehat{\pi_1(E(K_1))} \cong \widehat{\pi_1(E(K_2))} \Rightarrow \widehat{\pi_1(E_n(K_1))} \cong \widehat{\pi_1(E_n(K_2))}$.

Cyclic coverings

Assertion (ii) follows from the Fox's formula :

$H_1(E_n(K);\mathbb{Z}) \cong \mathbb{Z} \oplus A$, with $|A| = \left|\prod_{k=1}^{n} \Delta_K\left(e^{2\pi ik/n}\right)\right|$

In particular $b_1(E_n(K)) = 1$ iff no n-th root of unity is a zero of Δ_K.

Statement (3) follows now from :

Thm (D. Fried 1988)

The Alexander polynomial of a knot can be recovered from the torsion parts of the first homology groups of the n-fold cyclic covers of its exterior, provided that no zero is a root of unity.

Since the trefoil and the figure-8 are the only fibered knots of genus 1 :

Corollary

Let J be the trefoil or the figure-8 knot. If K is a knot with $\widehat{\pi_1(E(J))} \cong \widehat{\pi_1(E(K))}$, then $J = K$.

Torus knots

Let $T_{p,q}$ be a torus knot of type (p, q) with $0 < p < q$. By the results above :

Corollary

$\pi_1(\widehat{E(T_{p,q})}) \cong \pi_1(\widehat{E(T_{r,s})}) \Leftrightarrow (p, q) = (r, s)$

Each torus knot is profinitely rigid because :

Proposition

Let J be a torus knot. If K is a knot with $\widehat{\pi_1(E(J))} \cong \widehat{\pi_1(E(K)})$, then K is a torus knot.

The proof of the last statement (6) uses the fact that the logarithmic Mahler measure of the Alexander polynomial is a profinite invariant and the study of knots with cyclically commensurable exteriors.

Rigidity of knots

Prime knots with isomorphic groups have homeomorphic complements. So the following question makes sense :

Question

Let K_1 and K_2 be two prime knots in S^3. If $\widehat{\pi_1(E(K_1))} \cong \widehat{\pi_1(E(K_2))}$, does it follow that $K_1 = K_2$?

The group of a prime knot K does not necessarily determine the knot exterior $E(k)$,among 3-manifolds, if it contains a properly embedded essential annulus.

However one may ask :

Question

Let M be a compact orientable aspherical 3-manifold and let $K \subset S^3$ be a knot. Does $\widehat{\pi_1(M)} \cong \widehat{\pi_1(E(K))}$ imply that $\pi_1(M)$ is isomorphic to a knot group ?

Virtual Thurston norms

We study to which degree does the virtual Thurston norms determine the type of the JSJ-decomposition of the 3-manifold.

Given a compact orientable aspherical 3-manifold M with empty or toroidal boundary, define :

- $b_1(M) = \dim_{\mathbb{R}}(H_1(M;\mathbb{R}))$,
- $k(M) = \dim_{\mathbb{R}}(\ker(x_M))$,
- $r(M) = \frac{k(M)}{b_1(M)}$ if $b_1(M) > 0$ and 0 otherwise.

A covering $f\colon \widetilde{M} \to M$ is *subregular* if f can be written as a composition of regular coverings $f_i\colon M_i \to M_{i-1}$, $i = 1, \ldots, k$ with $M_k = \widetilde{M}$ and $M_0 = M$.

Let $\mathcal{C}(M) =$ the class of all finite subregular covers $\widetilde{M}$ of M.

Volume

Definition

For a 3-manifold M let define :

- $\widehat{r}(M) = \sup_{\widetilde{M} \in \mathcal{C}(M)} r(\widetilde{M})$.
- $\rho(M) = \inf_{\widetilde{M} \in \mathcal{C}(M)} r(\widetilde{M})$.
- $\widehat{\rho}(M) = \sup_{\widetilde{M} \in \mathcal{C}(M)} \rho(\widetilde{M})$.

Thm (B-Friedl 2015)

Let M be a compact, connected, orientable, aspherical 3-manifold with empty or toroidal boundary.

- *M is a hyperbolic manifold* $\Leftrightarrow \widehat{r}(M) = 0$
- $vol(M) \neq 0 \Leftrightarrow \widehat{\rho}(M) = 0$.
- *M is a graph manifold* $\Leftrightarrow \widehat{\rho}(M) = 1$.

Virtual Thurston norms

Proposition (Graph manifolds)

Let M be an aspherical graph manifold. Then $\forall\ \epsilon > 0$ there exists a finite regular cover N of M such that for any finite cover $\overline{N}$ of N we have $r(\overline{N}) > 1 - \epsilon$.

The idea is to increase by finite coverings the Euler characteristic of the bases of the Seifert pieces of the JSJ-decomposition of M much more than the numbers of JSJ-tori in order that $r(N) = \frac{k(N)}{b_1(N)} \nearrow 1$.

Since the property of being aspherical and not being a graph manifold is preserved by finite cover, for $Vol(M) \neq 0$ it suffises to show :

Proposition ($Vol(M) \neq 0$)

If $Vol(M) \neq 0$, then given any $\epsilon > 0$, there exists a finite subregular cover N of M such that $r(N) < \epsilon$. In particular $\rho(M) = 0$.

Pro-virtual abelian completion

The pro-virtually abelian completion $\widehat{\pi}_{va}$ of a group π is defined in the same way as the profinite completion $\widehat{\pi}$ using virtually abelian quotients instead of finite quotients.

From the definition there exists a continuous homomorphism $\widehat{\pi}_{va} \to \widehat{\pi}$.

Co-virtually abelian normal subgroups of $\widehat{\pi}_{va}$ are open.

Any homomorphism between two finitely generated pro-virtually abelian groups is continuous.

Proposition

An isomorphism between the pro-virtual abelian completions of two groups induces regular isomorphisms between the profinite completions of their corresponding finite index subroups.

Corollary

The pro-virtually abelian completion $\widehat{\pi_1(M)}_{va}$ determines the Thurston norm of the finite coverings of M.