Non-Kähler complex structures on $\mathbb{R}^4$

Naohiko Kasuya
j.w.w. Antonio J. Di Scala and Daniele Zuddas

Aoyama Gakuin University

2016.3.8
1. **Introduction**
   - Problem and Motivation
   - Main Theorem

2. **Construction**
   - The Matsumoto-Fukaya fibration
   - Holomorphic models

3. **Further results**
Our problem

A complex mfd \((M, J)\) is said to be Kähler if there exists a symplectic form \(\omega\) compatible with \(J\), i.e.,

1. \(\omega(u, Ju) > 0\) for any \(u \neq 0 \in TM\),
2. \(\omega(u, v) = \omega(Ju, Jv)\) for any \(u, v \in TM\).

Problem

Is there any non-Kähler complex structure on \(\mathbb{R}^{2n}\)?
Our problem

A complex mfd \((M, J)\) is said to be Kähler if there exists a symplectic form \(\omega\) compatible with \(J\), i.e.,

1. \(\omega(u, Ju) > 0\) for any \(u \neq 0 \in TM\),
2. \(\omega(u, v) = \omega(Ju, Jv)\) for any \(u, v \in TM\).

Problem

Is there any non-Kähler complex structure on \(\mathbb{R}^{2n}\)?

- If \(n = 1\), the answer is “No”.  

Our problem

A complex mfd $(M, J)$ is said to be Kähler if there exists a symplectic form $\omega$ compatible with $J$, i.e.,

1. $\omega(u, Ju) > 0$ for any $u \neq 0 \in TM$,
2. $\omega(u, v) = \omega(Ju, Jv)$ for any $u, v \in TM$.

Problem

Is there any non-Kähler complex structure on $\mathbb{R}^{2n}$?

- If $n = 1$, the answer is “No”.
- If $n \geq 3$, “Yes” (Calabi-Eckmann).
Our problem

A complex mfd \((M, J)\) is said to be Kähler if there exists a symplectic form \(\omega\) compatible with \(J\), i.e.,

1. \(\omega(u, Ju) > 0\) for any \(u \neq 0 \in TM\),
2. \(\omega(u, v) = \omega(Ju, Jv)\) for any \(u, v \in TM\).

Problem

Is there any non-Kähler complex structure on \(\mathbb{R}^{2n}\)?

- If \(n = 1\), the answer is “No”.
- If \(n \geq 3\), “Yes” (Calabi-Eckmann).
- Then, what about if \(n = 2\)?
Calabi-Eckmann’s construction

\[ H_1 : S^{2p+1} \rightarrow \mathbb{C}P^p, \quad H_2 : S^{2q+1} \rightarrow \mathbb{C}P^q : \text{the Hopf fibrations.} \]

\[ H_1 \times H_2 : S^{2p+1} \times S^{2q+1} \rightarrow \mathbb{C}P^p \times \mathbb{C}P^q \text{ is a } T^2\text{-bundle.} \]

The Calabi-Eckmann manifold \( M_{p,q}(\tau) \) is a complex mfd diffeo to \( S^{2p+1} \times S^{2q+1} \) s.t. \( H_1 \times H_2 \) is a holomorphic torus bundle (\( \tau \) is the modulus of a fiber torus).

\( E_{p,q}(\tau) \): the top dim cell of the natural cell decomposition.
If \( p > 0 \) and \( q > 0 \), then it contains holomorphic tori.
So, it is diffeo to \( \mathbb{R}^{2p+2q+2} \) and non-Kähler.
Calabi-Eckmann’s construction

\[ H_1 : S^{2p+1} \to \mathbb{C}P^p, \quad H_2 : S^{2q+1} \to \mathbb{C}P^q : \text{the Hopf fibrations.} \]

\[ H_1 \times H_2 : S^{2p+1} \times S^{2q+1} \to \mathbb{C}P^p \times \mathbb{C}P^q \text{ is a } T^2\text{-bundle.} \]

The Calabi-Eckmann manifold \( M_{p,q}(\tau) \) is a complex mfd diffeo to \( S^{2p+1} \times S^{2q+1} \) s.t. \( H_1 \times H_2 \) is a holomorphic torus bundle (\( \tau \) is the modulus of a fiber torus).

\( E_{p,q}(\tau) \): the top dim cell of the natural cell decomposition.

If \( p > 0 \) and \( q > 0 \), then it contains holomorphic tori.

So, it is diffeo to \( \mathbb{R}^{2p+2q+2} \) and non-Kähler.

- This argument doesn’t work if \( p = 0 \) or \( q = 0 \).
Lemma (1)

If a complex manifold \((\mathbb{R}^{2n}, J)\) contains a compact holomorphic curve \(C\), then it is non-Kähler.

Proof.

Suppose it is Kähler. Then, there is a symplectic form \(\omega\) compatible with \(J\). Then, \(\int_C \omega > 0\). On the other hand, \(\omega\) is exact. By Stokes’ theorem, \(\int_C \omega = \int_C d\alpha = 0\). This is a contradiction.
Let $P = \{ 0 < \rho_1 < 1, 1 < \rho_2 < \rho_1^{-1} \} \subset \mathbb{R}^2$.

**Theorem**

*For any $(\rho_1, \rho_2) \in P$, there are a complex manifold $E(\rho_1, \rho_2)$ diffeomorphic to $\mathbb{R}^4$ and a surjective holomorphic map $f : E(\rho_1, \rho_2) \to \mathbb{C}P^1$ such that the only singular fiber $f^{-1}(0)$ is an immersed holomorphic sphere with one node, and the other fiber is either a holomorphic torus or annulus.*
The Matsumoto-Fukaya fibration

\[ f_{MF} : S^4 \rightarrow \mathbb{C}P^1 \] is a genus-1 achiral Lefschetz fibration with only two singularities of opposite signs.

- \( F_1 \): the fiber with the positive singularity \( ((z_1, z_2) \mapsto z_1z_2) \)
- \( F_2 \): the fiber with the negative singularity \( ((z_1, z_2) \mapsto z_1\bar{z}_2) \)
The Matsumoto-Fukaya fibration

\[ f_{MF} : S^4 \rightarrow \mathbb{C}P^1 \] is a genus-1 achiral Lefschetz fibration with only two singularities of opposite signs.

- \( F_1 \): the fiber with the positive singularity \( ((z_1, z_2) \mapsto z_1z_2) \)
- \( F_2 \): the fiber with the negative singularity \( ((z_1, z_2) \mapsto z_1\bar{z}_2) \)

\[ S^4 = N_1 \cup N_2, \text{ where } N_j \text{ is a tubular nbd of } F_j, \]
The Matsumoto-Fukaya fibration

\[ f_{MF} : S^4 \to \mathbb{C}P^1 \] is a genus-1 achiral Lefschetz fibration with only two singularities of opposite signs.

\( F_1 \): the fiber with the positive singularity \( ((z_1, z_2) \mapsto z_1 z_2) \)

\( F_2 \): the fiber with the negative singularity \( ((z_1, z_2) \mapsto z_1 \bar{z}_2) \)

- \( S^4 = N_1 \cup N_2 \), where \( N_j \) is a tubular nbd of \( F_j \),
- \( N_1 \cup (N_2 \setminus X) \cong \mathbb{R}^4 \) (\( X \) is a nbd of \(-\) sing),
The Matsumoto-Fukaya fibration

\[ f_{MF} : S^4 \to \mathbb{C}P^1 \] is a genus-1 achiral Lefschetz fibration with only two singularities of opposite signs.

- \( F_1 \): the fiber with the positive singularity \( ((z_1, z_2) \mapsto z_1z_2) \)
- \( F_2 \): the fiber with the negative singularity \( ((z_1, z_2) \mapsto z_1\bar{z}_2) \)

- \( S^4 = N_1 \cup N_2 \), where \( N_j \) is a tubular nbd of \( F_j \),
- \( N_1 \cup (N_2 \setminus X) \cong \mathbb{R}^4 \) (\( X \) is a nbd of \( -\) sing),
- In the smooth sense, \( f \) is a restriction of \( f_{MF} \).
Originally, it is constructed by taking the composition of the Hopf fibration \( H : S^3 \to \mathbb{C}P^1 \) and its suspension \( \Sigma H : S^4 \to S^3 \). \( f_{MF} = H \circ \Sigma H \).
The Matsumoto-Fukaya fibration 2

Originally, it is constructed by taking the composition of the Hopf fibration \( H : S^3 \to \mathbb{C}P^1 \) and its suspension \( \Sigma H : S^4 \to S^3 \). \( f_{MF} = H \circ \Sigma H \).


- **How to glue** \( \partial N_2 \) to \( \partial N_1 \) **is as the following pictures** (in the next page).
Gluing $N_1$ and $N_2$

Monodromy: left-handed Dehn twist

Monodromy: right-handed Dehn twist
Kirby diagrams

\[
\begin{align*}
\text{Figure:} & \quad \text{The Matsumoto-Fukaya fibration on } S^4. \\
\end{align*}
\]
Kirby diagrams 2

Figure: The map $f$ on $S^4 \setminus X \cong \mathbb{R}^4$. 

Naohiko Kasuya j.w.w. Antonio J. Di Scala and Daniele Zuddas

Non-Kähler complex structures on $\mathbb{R}^4$
Lemma (2)

Let us glue $A \times D^2$ to $N_1$ so that for each $t \in \partial D^2 = S^1$, the annulus $A \times \{t\}$ embeds in the fiber torus $f^{-1}(t)$ as a thickened meridian, and that it rotates in the longitude direction once as $t \in S^1$ rotates once. Then, the interior of the resultant manifold is diffeomorphic to $\mathbb{R}^4$. 

Naohiko Kasuya j.w.w. Antonio J. Di Scala and Daniele Zuddas

Non-Kähler complex structures on $\mathbb{R}^4$
Key Lemma

Lemma (2)

Let us glue $A \times D^2$ to $N_1$ so that for each $t \in \partial D^2 = S^1$, the annulus $A \times \{t\}$ embeds in the fiber torus $f^{-1}(t)$ as a thickened meridian, and that it rotates in the longitude direction once as $t \in S^1$ rotates once. Then, the interior of the resultant manifold is diffeomorphic to $\mathbb{R}^4$.

- We will realize this gluing by complex manifolds!
Kodaira’s holomorphic model

\[ \Delta(r) = \{ z \in \mathbb{C} \mid |z| < r \}, \]
\[ \Delta(r_1, r_2) = \{ z \in \mathbb{C} \mid r_1 < |z| < r_2 \}. \]

Consider an elliptic fibration

\[ \pi : \mathbb{C}^* \times \Delta(0, \rho_1)/\mathbb{Z} \to \Delta(0, \rho_1), \]

where the action is \( n \cdot (z, w) = (zw^n, w) \).

It naturally extends to \( f_1 : W \to \Delta(\rho_1) \). 

\( W \) is a tubular neighborhood of a singular elliptic fiber of type \( I_1 \). It is a holomorphic model of \( N_1 \).
The model for $N_2 \setminus X$ is $\Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$.

The gluing domain is

$$V_2 := \Delta(1, \rho_2) \times \Delta(\rho_1^{-1}, \rho_0^{-1}) \subset \Delta(1, \rho_2) \times \Delta(\rho_0^{-1}).$$

$$Y := \{(z\varphi(w), w) \in \mathbb{C}^* \times \Delta(\rho_0, \rho_1) \mid z \in \Delta(1, \rho_2)\},$$

where $\varphi(w) = \exp \left( \frac{1}{4\pi i} (\log w)^2 - \frac{1}{2} \log w \right)$.

Define the gluing domain $V_1 \subset W$ by $V_1 = Y/\mathbb{Z}$. 
The model for $N_2 \setminus X$ is $\Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$.

The gluing domain is $V_2 := \Delta(1, \rho_2) \times \Delta(\rho_1^{-1}, \rho_0^{-1}) \subset \Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$.

$$Y := \{(z \varphi(w), w) \in \mathbb{C}^* \times \Delta(\rho_0, \rho_1) \mid z \in \Delta(1, \rho_2)\},$$

where $\varphi(w) = \exp \left( \frac{1}{4\pi i} (\log w)^2 - \frac{1}{2} \log w \right)$.

$\varphi(re^{i(\theta+2\pi)}) = re^{i\theta} \varphi(re^{i\theta}) = w \varphi(w)$.

Define the gluing domain $V_1 \subset W$ by $V_1 = Y/\mathbb{Z}$. 

---
Gluing domains in the two pieces 2

\[ V_2 = \Delta(1, \rho_2) \times \Delta(\rho_i^{-1}, \rho_0^{-1}) \cap \Delta(1, \rho_2) \times \Delta(\rho_0^{-1}) \]

the section of \( \pi \) defined by \( \varphi \)
Gluing the two pieces

By the biholomorphism between the gluing domains

$$\Phi : V_2 \rightarrow V_1; \ (z, w^{-1}) \mapsto [(z\phi(w), w)],$$

we obtain a complex manifold

$$E(\rho_1, \rho_2) = (\Delta(1, \rho_2) \times \Delta(\rho^{-1}_0)) \cup_{\Phi} W.$$
Gluing the two pieces

By the biholomorphism between the gluing domains

$$\Phi : V_2 \rightarrow V_1; \ (z, w^{-1}) \mapsto [(z\phi(w), w)],$$

we obtain a complex manifold

$$E(\rho_1, \rho_2) = (\Delta(1, \rho_2) \times \Delta(\rho_0^{-1})) \cup_{\Phi} W.$$  

$\Delta(\rho_1)$ and $\Delta(\rho_0^{-1})$ are glued to become $\mathbb{C}P^1$. 
Gluing the two pieces

By the biholomorphism between the gluing domains

$$\Phi : V_2 \rightarrow V_1; \ (z, w^{-1}) \mapsto [(z\phi(w), w)],$$

we obtain a complex manifold

$$E(\rho_1, \rho_2) = (\Delta(1, \rho_2) \times \Delta(\rho_0^{-1})) \cup \Phi W.$$

- $\Delta(\rho_1)$ and $\Delta(\rho_0^{-1})$ are glued to become $\mathbb{C}P^1$.
- $f$ is defined to be $f_1 : W \rightarrow \Delta(\rho_1)$ on $W$, and the second projection on $\Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$. 
Classification of holomorphic curves

Lemma (3)

Any compact holomorphic curve in $E(\rho_1, \rho_2)$ is a compact fiber of the map $f : E(\rho_1, \rho_2) \to \mathbb{C}P^1$.

Proof.

Let $i : C \to E(\rho_1, \rho_2)$ be a compact holomorphic curve. The composition $f \circ i : C \to \mathbb{C}P^1$ is a holomorphic map between compact Riemann surfaces. It is either a branched covering or a constant map. Since it is homotopic to a constant map, it is a constant map.
Properties of $E(\rho_1, \rho_2)$

Thanks to the existence of the fibration $f$ and the previous lemma, we can show the following properties.
Properties of $E(\rho_1, \rho_2)$

Thanks to the existence of the fibration $f$ and the previous lemma, we can show the following properties.

- $E(\rho_1, \rho_2) \ncong E(\rho'_1, \rho'_2)$ if $(\rho_1, \rho_2) \neq (\rho'_1, \rho'_2)$.
Thanks to the existence of the fibration $f$ and the previous lemma, we can show the following properties.

- $E(\rho_1, \rho_2) \not\cong E(\rho'_1, \rho'_2)$ if $(\rho_1, \rho_2) \neq (\rho'_1, \rho'_2)$.
- $E(\rho_1, \rho_2) \times \mathbb{C}^{n-2}$ give uncountably many non-Kähler complex structures on $\mathbb{R}^{2n}$ ($n \geq 3$).
Thanks to the existence of the fibration $f$ and the previous lemma, we can show the following properties.

- $E(\rho_1, \rho_2) \nmid E(\rho'_1, \rho'_2)$ if $(\rho_1, \rho_2) \neq (\rho'_1, \rho'_2)$.
- $E(\rho_1, \rho_2) \times \mathbb{C}^{n-2}$ give uncountably many non-Kähler complex structures on $\mathbb{R}^{2n}$ ($n \geq 3$).
- Any holomorphic function is constant.
Properties of $E(\rho_1, \rho_2)$

Thanks to the existence of the fibration $f$ and the previous lemma, we can show the following properties.

- $E(\rho_1, \rho_2) \not\cong E(\rho'_1, \rho'_2)$ if $(\rho_1, \rho_2) \neq (\rho'_1, \rho'_2)$.
- $E(\rho_1, \rho_2) \times \mathbb{C}^{n-2}$ give uncountably many non-Kähler complex structures on $\mathbb{R}^{2n}$ ($n \geq 3$).
- Any holomorphic function is constant.
- Any meromorphic function is the pullback of that on $\mathbb{C}P^1$ by $f$. 
Properties of $E(\rho_1, \rho_2)$ 2
Properties of $E(\rho_1, \rho_2)$

- $f^* : \text{Pic}(\mathbb{C}P^1) \to \text{Pic}(E(\rho_1, \rho_2))$ is injective.
Properties of $E(\rho_1, \rho_2)$ 2

- $f^* : \text{Pic}(\mathbb{C}P^1) \rightarrow \text{Pic}(E(\rho_1, \rho_2))$ is injective.
- It cannot be holomorphically embedded in any compact complex surface.
Thank you for your attention!