RAAGs in knot groups

Takuya Katayama

Hiroshima University

March 8, 2016
In this talk, we consider the following question.

**Question**

For a given non-trivial knot in the 3-sphere, which right-angled Artin group admits an embedding into the knot group?

**The goal of this talk**

To give a complete classification of right-angled Artin groups which admit embeddings into the knot group, for each non-trivial knot in the 3-sphere by means of Jaco-Shalen-Johnnson decompositions.
Definition of RAAGs

\( \Gamma \): a finite simple graph (\( \Gamma \) has no loops and multiple-edges)
\( V(\Gamma) = \{v_1, v_2, \ldots, v_n\} \): the vertex set of \( \Gamma \)
\( E(\Gamma) \): the edge set of \( \Gamma \)

**Definition**

The right-angled Artin group (RAAG), or the graph group on \( \Gamma \) is a group given by the following presentation:

\[
A(\Gamma) = \langle v_1, v_2, \ldots, v_n \mid [v_i, v_j] = 1 \text{ if } \{v_i, v_j\} \in E(\Gamma) \rangle.
\]

**Example**

\[
A(\bullet \bullet \bullet \cdots \bullet) \cong F_n.
\]
\[
A(\text{the complete graph on } n \text{ vertices}) \cong \mathbb{Z}^n.
\]
\[
A(\bullet \bullet \bullet \cdots \bullet) \cong \mathbb{Z} \times F_n.
\]
Theorem (Crisp-Wiest, 2004)

\( S: \) a connected surface

If \( S \not\cong \#\mathbb{RP}^2 \) (\( n = 1, 2, 3 \)), then

\[ \exists \text{ a RAAG } A \text{ s.t. } \pi_1(S) \hookrightarrow A. \]

Theorem (Agol, Liu, Przytycki, Wise...et al.)

\( M: \) a compact aspherical 3-manifold

The interior of \( M \) admits a complete Riemannian metric with non-positive curvature

\[ \Leftrightarrow \pi_1(M) \text{ admits a virtual embedding into a RAAG}. \]

i.e.,

\[ \pi_1(M) \text{ finite index} \]

\[ \exists H \hookrightarrow \exists A: \text{ a RAAG} \]
Theorem (Jaco-Shalen, Johannson, Thurston’s hyperbolization thm)

If \( K \) is a knot in \( S^3 \), then the knot exterior \( E(K) \) of \( K \) has a canonical decomposition by tori into hyperbolic pieces and Seifert pieces. Moreover, each Seifert piece is homeomorphic to one of the following spaces: a composing space, a cable space and a torus knot exterior.

Each cable space has a finite covering homeomorphic to a composing space, and \( \pi_1 \) of a composing space is isomorphic to \( A(\ ) \). Hence \( \pi_1 \) of the cable space is virtually a RAAG.
\( K := (\text{figure eight knot}) \# (\text{cable on trefoil knot}) \)

\[
E(K) =
\]

We now cut \( E(K) \) along tori...
$E(K) =$

- figure 8
- composing
- cable
- trefoil

Seifert-Seifert gluing
Question (recall)

For a given non-trivial knot in the 3-sphere, which RAAG admits an embedding into the knot group?
An answer to the question

Main Theorem (K.)

$K$: a non-trivial knot,  $G(K) := \pi_1(E(K))$,  $\Gamma$: a finite simple graph

Case 1. If $E(K)$ has only hyperbolic pieces,

then $A(\Gamma) \hookrightarrow G(K)$ iff $\Gamma$ is a disjoint union of $\bullet \cdots \bullet$ and $\bullet \bullet \bullet \cdots \bullet$.

Case 2. If $E(K)$ is Seifert fibered (i.e., $E(K)$ is a torus knot exterior),

then $A(\Gamma) \hookrightarrow G(K)$ iff $\Gamma$ is a star graph or $\bullet \bullet \bullet \cdots \bullet$.

Case 3. If $E(K)$ has both a Seifert piece and a hyperbolic piece, and has no Seifert-Seifert gluing,

then $A(\Gamma) \hookrightarrow G(K)$ iff $\Gamma$ is a disjoint union of star graphs.

Case 4. If $E(K)$ has a Seifert-Seifert gluing,

then $A(\Gamma) \hookrightarrow G(K)$ iff $\Gamma$ is a forest.

Here a simplicial graph $\Gamma$ is said to be a forest if each connected component of $\Gamma$ is a tree.
**Definition**

\[ \Gamma: \text{ a simple graph.} \]

A subgraph \( \Lambda \subset \Gamma \): full

\[ \begin{align*}
\text{def} & \iff \\
\forall e \in E(\Gamma), \ e^{(0)} \subset \Lambda & \implies e \in E(\Lambda). 
\end{align*} \]

**Lemma**

\( \Gamma: \text{ a finite simple graph.} \)

If \( \Lambda \) is a full subgraph of \( \Gamma \), then \( \langle V(\Lambda) \rangle \cong A(\Lambda) \).

**Lemma**

\( A(\Gamma): \text{ the RAAG on a finite simple graph } \Gamma \)

If \( A(\Gamma) \) admits an embedding into a knot group, then \( \Gamma \) is a forest.
**Theorem (Papakyriakopoulos-Conner, 1956)**

$G(K)$: the knot group of a non-trivial knot $K$

Then there is an embedding $\mathbb{Z}^2 \hookrightarrow G(K)$ and is no embedding $\mathbb{Z}^3 \hookrightarrow G(K)$.

**Theorem (Droms, 1985)**

$A(\Gamma)$: the RAAG on a finite simple graph $\Gamma$

Then $A(\Gamma)$ is a 3-manifold group iff each connected component of $\Gamma$ is a **triangle** or a **tree**.

Hence, in the proof of Main Theorem, we may assume $\Gamma$ is a finite **forest**, and so every connected subgraph $\Lambda$ of $\Gamma$ is a full subgraph ($A(\Lambda) \hookrightarrow A(\Gamma)$).
Proof of Main Theorem(2)

Main Theorem(2)

M: a Seifert piece in a knot exterior, \( \Gamma \): a finite simple graph

Then \( A(\Gamma) \hookrightarrow \pi_1(M) \) iff \( \Gamma \) is a star graph \( \bullet \ldots \bullet \) or \( \bullet \bullet \bullet \ldots \).

We treat only the case \( M \) is a non-trivial torus knot exterior (because the other case can be treated similarly). Let \( G(p, q) \) be the \((p, q)\)-torus knot group.

**Proof of the if part.** It is enough to show that

\[
A(\bullet \ldots) \cong \mathbb{Z} \times F_n \hookrightarrow G(p, q)
\]

for some \( n \geq 2 \).

Note that \([G(p, q), G(p, q)] \cong F_n\) for some \( n \geq 2 \).

Then \( Z(G(p, q)) \times [G(p, q), G(p, q)] \) is a subgroup of \( G(p, q) \) isomorphic to \( \mathbb{Z} \times F_n \), as required.
The only if part of Main Theorem(2)

M: a Seifert piece in a knot exterior, Γ: a finite simple graph

Suppose $A(\Gamma) \hookrightarrow \pi_1(M)$.

Then $\Gamma$ is a star graph $\bullet\bullet\bullet\bullet\bullet$ or $\bullet\bullet\bullet\bullet\bullet\bullet$.

Note that, in general, the following three facts hold.

1. If $\Gamma$ is disconnected, then $A(\Gamma)$ is centerless.
2. $A(\bullet\cdots\bullet\bullet)$ is centerless.
3. If $\Gamma$ has $\bullet\cdots\bullet\bullet\bullet\bullet$ as a (full) subgraph, then $A(\bullet\cdots\bullet\bullet\bullet\bullet) \hookrightarrow A(\Gamma)$.

Now suppose that $A(\Gamma) \hookrightarrow G(p, q)$ and $E(\Gamma) \neq \emptyset$.

Then $\Gamma$ is a forest.

On the other hand, our assumptions imply that $A(\Gamma)$ has a non-trivial center.

Hence (1) implies that $\Gamma$ is a tree.

Moreover, (2) together with (3) implies that $\Gamma$ does not contain $\bullet\cdots\bullet\bullet\bullet\bullet$ as a subgraph.

Thus $\Gamma$ is a star graph.
Main Theorem(4)

\( \Gamma \): a finite simple graph, \( \{C_1, C_2\} \): a Seifert-Seifert gluing in a knot exterior, \( T \): the JSJ torus \( C_1 \cap C_2 \)

If \( \Gamma \) is a forest, then \( A(\Gamma) \hookrightarrow \pi_1(C_1) \ast_{\pi_1(T)} \pi_1(C_2) \).

It is enough to show the following two lemmas.

(A) If \( \Gamma \) is a forest, then \( A(\Gamma) \hookrightarrow A(\bullet\bullet\bullet\bullet\bullet) \).

(B) \( A(\bullet\bullet\bullet\bullet\bullet) \hookrightarrow \pi_1(C_1) \ast_{\pi_1(T)} \pi_1(C_2) \).
Main Theorem (4)

\( \Gamma: \) a finite simple graph, \( \{C_1, C_2\}: \) a Seifert-Seifert gluing in a knot exterior, \( T: \) the JSJ torus \( C_1 \cap C_2 \)

If \( \Gamma \) is a forest, then \( A(\Gamma) \hookrightarrow \pi_1(C_1) \star_{\pi_1(T)} \pi_1(C_2). \)

It is enough to show the following two lemmas.

(A) If \( \Gamma \) is a forest, then \( A(\Gamma) \hookrightarrow A(\bullet \cdots \bullet). \) (Kim-Koberda)

(B) \( A(\bullet \cdots \bullet) \hookrightarrow \pi_1(C_1) \star_{\pi_1(T)} \pi_1(C_2). \) (Niblo-Wise)
Let $\Gamma$ be a finite simple graph and $v$ a vertex of $\Gamma$.

$\text{St}(v)$: the full subgraph induced by $v$ and the vertices adjacent to $v$.

$D_v(\Gamma)$: the *double* of $\Gamma$ along the full subgraph $\text{St}(v)$, namely, $D_v(\Gamma)$ is obtained by taking two copies of $\Gamma$ and gluing them along copies of $\text{St}(v)$.

The Seifert-van Kampen theorem implies the following.

**Lemma**

$A(D_v(\Gamma)) \leftrightarrow A(\Gamma)$. 

Lemma (A)

If $\Gamma$ is a finite forest, then $A(\Gamma) \hookrightarrow A(\bullet\bullet\bullet\bullet)$. 

Proof.
Since every finite forest is a full subgraph of a finite tree $T$, we may assume that $\Gamma = T$.
We shall prove this theorem by induction on the ordered pair $(\text{diam}(T), \# \text{ of geodesic edge-paths of length diam}(T))$ and by using doubled graphs.
If $\text{diam}(T) \leq 2$, then $T$ is a star graph, and so we have $A(\bullet\cdots\bullet) \hookrightarrow A(\bullet\bullet\bullet) \hookrightarrow A(\bullet\bullet\bullet\bullet)$. We now consider the case where the diameter of $T$ is at least 3.
\( v \): a pendant vertex on a geodesic edge-path of length \( \text{diam}(T) \)

\( v' \): the (unique) vertex adjacent to \( v \)

\( T' := T \setminus (v \cup \{v, v'\}) \)

Case 1. The degree of \( v' \) is at least 3.
Hence, we have $A(T) \leftrightarrow A(D_{v_1}(T')) \leftrightarrow A(T')$.

Removing away $v$ and $\{v, v'\}$ from $T$ implies that either the diam decreases or # of geodesic edge-paths of length diam decreases.
Case 2. The degree of $v'$ is equal to 2. We can assume $\text{diam}(T) \geq 4$.

Thus we have $A(T) \leftrightarrow A(D_{v'}(T')) \leftrightarrow A(T')$. 
A proof of [Niblo-Wise, 2000]

Lemma (B)

$\Gamma$: a finite simple graph, $\{C_1, C_2\}$: a Seifert-Seifert gluing in a knot exterior, $T$: the JSJ torus $C_1 \cap C_2$

Then $A(\bullet-\bullet-\bullet-\bullet) \hookrightarrow \pi_1(C_1) \ast_{\pi_1(T)} \pi_1(C_2)$. 
Proof. We shall construct an embedding
\[ A(\bullet\rightarrow) \hookrightarrow \pi_1(C_1) *_{\pi_1(T)} \pi_1(C_2) \]
as follows.

For each \( i = 1,2 \), we take a finite index subgroup of \( \pi_1(C_i) \), which is isomorphic to \( A(\text{St}_{m_i}) \) for some \( m_i \geq 2 \).

Here, \( \text{St}_{m_i} \) = \includegraphics{tikz.png}.

(i) \( \psi(\bullet) \in A(\text{St}_{m_1}) \cap A(\text{St}_{m_2}) \cap \pi_1(T) \cap Z(\pi_1(C_1)) \).
(ii) \( \psi(\bullet) \in A(\text{St}_{m_1}) \cap A(\text{St}_{m_2}) \cap \pi_1(T) \cap Z(\pi_1(C_2)) \).
(iii) \( \psi(\bullet) \in A(\text{St}_{m_1}) \).
(iv) \( \psi(\bullet) \in A(\text{St}_{m_2}) \).

Then the normal form theorem says that \( \psi \) is injective, as desired.
Main Theorem (K.)

\( K: \) a non-trivial knot, \( \Gamma: \) a finite simplicial graph

Case 1. If \( E(K) \) has only hyperbolic pieces,

\[ A(\Gamma) \rightarrow G(K) \iff \Gamma \text{ is a disjoint union of } \bullet \bullet \ldots \bullet \text{ and } \bullet \bullet \ldots \bullet. \]

Case 2. If \( E(K) \) is Seifert (i.e. \( M \) is a torus knot exterior),

\[ A(\Gamma) \rightarrow G(K) \iff \Gamma \text{ is a star graph } \bullet \bullet \ldots \bullet \text{ or } \bullet \bullet \bullet \ldots \bullet. \]

Case 3. If \( E(K) \) has both a Seifert piece and a hyperbolic piece and has no Seifert-Seifert gluing,

\[ A(\Gamma) \rightarrow G(K) \iff \Gamma \text{ is a disjoint union of star graphs.} \]

Case 4. If \( E(K) \) has a Seifert-Seifert gluing,

\[ A(\Gamma) \rightarrow G(K) \iff \Gamma \text{ is a forest.} \]
Future work

Question
Which knot group admits an embedding into a RAAG?

Every knot group admits a \textit{virtual} embedding into a RAAG. This question seems to be connected with the following question.

Question
Which knot group is bi-orderable?

Since every RAAG is bi-orderable (Duchamp-Thibon), every knot group which admits an embedding into a RAAG must be bi-orderable.
Thank you.