RAAGs in knot groups

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In this talk, we consider the following question.

Question

For a given non-trivial knot in the 3-sphere, which right-angled Artin group admits an embedding into the knot group?

The goal of this talk

To give a complete classification of right-angled Artin groups which admit embeddings into the knot group, for each non-trivial knot in the 3-sphere by means of Jaco-Shalen-Johnnson decompositions.

Definition of RAAGs

Γ: a finite simple graph (Γ has no loops and multiple-edges)
$V(\Gamma) = \{v_1, v_2, \cdots, v_n\}$: the vertex set of Γ
$E(\Gamma)$: the edge set of Γ

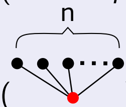
Definition

The right-angled Artin group (RAAG), or the graph group on Γ is a group given by the following presentation:

$$A(\Gamma) = \langle\ v_1, v_2, \ldots, v_n \ \mid\ [v_i, v_j] = 1 \ \mathit{if}\ \{v_i, v_j\} \in E(\Gamma)\ \rangle.$$

Example

$A(\bullet \bullet \bullet \cdots \bullet) \cong F_n$.
$A(\mathit{the\ complete\ graph\ on\ n\ vertices}) \cong \mathbb{Z}^n$.

$A($ 

$) \cong \mathbb{Z} \times F_n$.

Embeddings of low dim manifold groups into RAAGs

Theorem (Crisp-Wiest, 2004)

S: a connected surface
If $S \not\cong \overset{n}{\#}\mathbb{RP}^2$ $(n = 1,\ 2,\ 3)$, *then*
$\exists$ *a RAAG A s.t.* $\pi_1(S) \hookrightarrow A$.

Theorem (Agol, Liu, Przytycki, Wise...et al.)

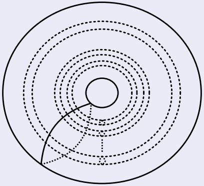
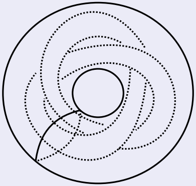
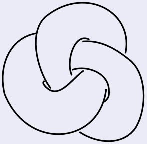
M : a compact aspherical 3-manifold
The interior of M admits a complete Riemannian metric with non-positive curvature
$\Leftrightarrow \pi_1(M)$ *admits a virtual embedding into a RAAG.*

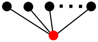
i.e., $\pi_1(M)$
finite index $\vee\!|$
$\exists\ H \hookrightarrow \exists\ A$*: a RAAG*

Jaco-Shalen-Johannson decompositions of knot exteriors

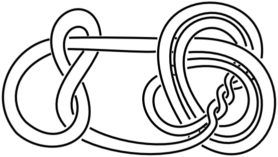
Theorem (Jaco-Shalen, Johannson, Thurston's hyperbolization thm)

If K is a knot in S^3, then the knot exterior $E(K)$ of K has a canonical decomposition by tori into hyperbolic pieces and Seifert pieces. Moreover, each Seifert piece is homeomorphic to one of the following spaces: a composing space, a cable space and a torus knot exterior.

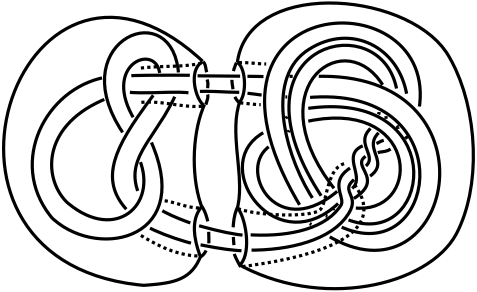




Each cable space has a finite covering homeomorphic to a composing space, and π_1 of a composing space is isomorphic to A(). Hence π_1 of the cable space is virtually a RAAG.

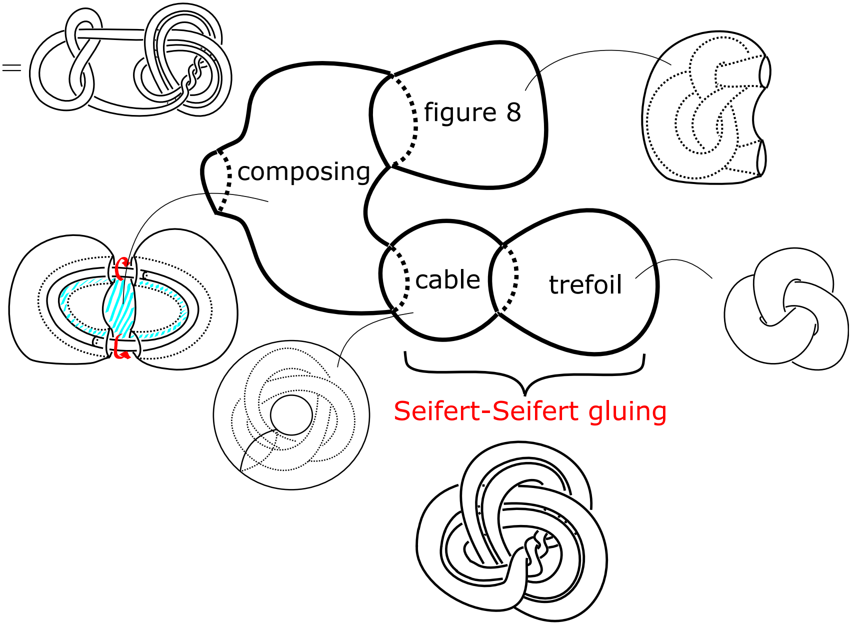
$K :=$ (figure eight knot)#(cable on trefoil knot)

$E(K) =$ 

We now cut $E(K)$ along tori...



$E(K) =$



Question (recall)

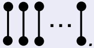
For a given non-trivial knot in the 3-sphere, which RAAG admits an embedding into the knot group?

An answer to the question

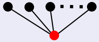
Main Theorem (K.)

K: a non-trivial knot, $G(K) := \pi_1(E(K))$, Γ: a finite simple graph

Case 1. *If $E(K)$ has only hyperbolic pieces,*

then $A(\Gamma) \hookrightarrow G(K)$ iff Γ is a disjoint union of ● ● ● · · · ● *and* .

Case 2. *If $E(K)$ is Seifert fibered (i.e., $E(K)$ is a torus knot exterior),*

then $A(\Gamma) \hookrightarrow G(K)$ iff Γ is a star graph  *or* ● ● ● · · · ●.

Case 3. *If $E(K)$ has both a Seifert piece and a hyperbolic piece, and has no Seifert-Seifert gluing,*

then $A(\Gamma) \hookrightarrow G(K)$ iff Γ is a disjoint union of star graphs.

Case 4. *If $E(K)$ has a Seifert-Seifert gluing,*

then $A(\Gamma) \hookrightarrow G(K)$ iff Γ is a forest.

Here a simplicial graph Γ is said to be a forest if each connected component of Γ is a tree.

Definition

Γ: a simple graph.

A subgraph $\Lambda \subset \Gamma$: full

$\overset{def}{\Leftrightarrow}$ $\forall e \in E(\Gamma)$, $e^{(0)} \subset \Lambda \Rightarrow e \in E(\Lambda)$.

Lemma

Γ: a finite simple graph.

If Λ is a full subgraph of Γ, then $\langle V(\Lambda)\rangle \cong A(\Lambda)$.

Lemma

$A(\Gamma)$: the RAAG on a finite simple graph Γ

If $A(\Gamma)$ admits an embedding into a knot group, then Γ is a forest.

Theorem (Papakyriakopoulos-Conner, 1956)

$G(K)$: the knot group of a non-trivial knot K
Then there is an embedding $\mathbb{Z}^2 \hookrightarrow G(K)$ and is no embedding $\mathbb{Z}^3 \hookrightarrow G(K)$.

Theorem (Droms, 1985)

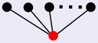
$A(\Gamma)$: the RAAG on a finite simple graph Γ
Then $A(\Gamma)$ is a 3-manifold group iff each connected component of Γ is a triangle or a tree.

Hence, in the proof of Main Theorem, we may assume Γ is a finite forest, and so every connected subgraph Λ of Γ is a full subgraph ($A(\Lambda) \hookrightarrow A(\Gamma)$).

Proof of Main Theorem(2)

Main Theorem(2)

M: a Seifert piece in a knot exterior, Γ: a finite simple graph

Then $A(\Gamma) \hookrightarrow \pi_1(M)$ iff Γ is a star graph  or ●●●···●.

We treat only the case M is a non-trivial torus knot exterior (because the other case can be treated similarly). Let $G(p,q)$ be the (p,q)-torus knot group.

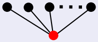
Proof of the if part. It is enough to show that

$A($$) \cong \mathbb{Z} \times F_n \hookrightarrow G(p,q)$ for some $n \geq 2$.

Note that $[G(p,q), G(p,q)] \cong F_n$ for some $n \geq 2$.

Then $Z(G(p,q)) \times [G(p,q), G(p,q)]$ is a subgroup of $G(p,q)$ isomorphic to $\mathbb{Z} \times F_n$, as required.

The only if part of Main Theorem(2)

M: a Seifert piece in a knot exterior, Γ: a finite simple graph
Suppose $A(\Gamma) \hookrightarrow \pi_1(M)$.
Then Γ is a star graph  or ●●●· · ·●.

Note that, in general, the following three facts hold.
(1) If Γ is disconnected, then $A(\Gamma)$ is centerless.
(2) $A($●—●—●—●$)$ is centerless.
(3) If Γ has ●—●—●—● as a (full) subgraph, then $A($●—●—●—●$) \hookrightarrow A(\Gamma)$.
Now suppose that $A(\Gamma) \hookrightarrow G(p, q)$ and $E(\Gamma) \neq \emptyset$.
Then Γ is a forest.
On the other hand, our assumptions imply that $A(\Gamma)$ has a non-trivial center.
Hence (1) implies that Γ is a tree.
Moreover, (2) together with (3) implies that Γ does not contain ●—●—●—● as a subgraph.
Thus Γ is a star graph.

Main Theorem(4)

Γ: a finite simple graph, $\{C_1, C_2\}$: a Seifert-Seifert gluing in a knot exterior, T: the JSJ torus $C_1 \cap C_2$
If Γ is a forest, then $A(\Gamma) \hookrightarrow \pi_1(C_1) \underset{\pi_1(T)}{*} \pi_1(C_2)$.

It is enough to show the following two lemmas.
(A) If Γ is a forest, then $A(\Gamma) \hookrightarrow A(\bullet - \bullet - \bullet - \bullet)$.
(B) $A(\bullet - \bullet - \bullet - \bullet) \hookrightarrow \pi_1(C_1) \underset{\pi_1(T)}{*} \pi_1(C_2)$.

Main Theorem(4)

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It is enough to show the following two lemmas.

(A) If Γ is a forest, then $A(\Gamma) \hookrightarrow A(\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet)$. (Kim-Koberda)

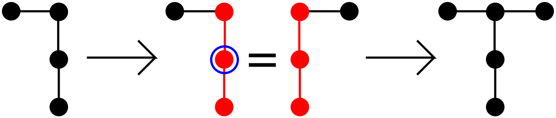
(B) $A(\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet) \hookrightarrow \pi_1(C_1) \underset{\pi_1(T)}{*} \pi_1(C_2)$. (Niblo-Wise)

Double of a graph along a star

Let Γ be a finite simple graph and v a vertex of Γ.

$\mathrm{St}(v)$: the full subgraph induced by v and the vertices adjacent to v.

$D_v(\Gamma)$: the *double* of Γ along the full subgraph $\mathrm{St}(v)$, namely, $D_v(\Gamma)$ is obtained by taking two copies of Γ and gluing them along copies of $\mathrm{St}(v)$.



The Seifert-van Kampen theorem implies the following.

Lemma

$A(D_v(\Gamma)) \hookrightarrow A(\Gamma)$.

An elementary proof of [Kim-Koberda, 2013]

Lemma(A)

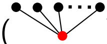
If Γ is a finite forest, then $A(\Gamma) \hookrightarrow A(\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet)$.

Proof.

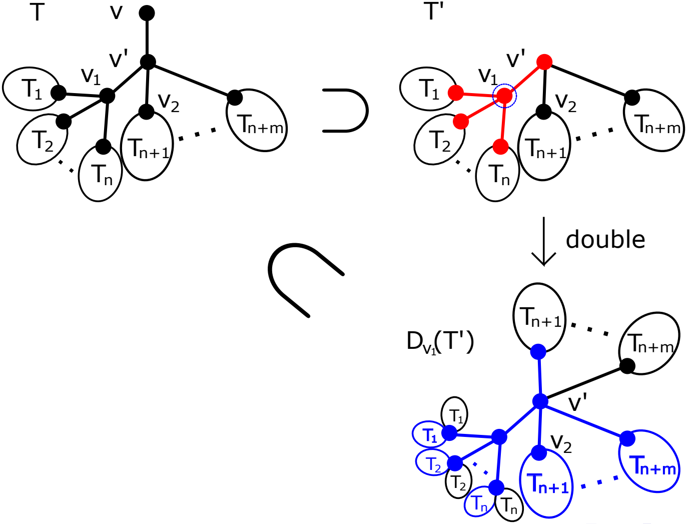
Since every finite forest is a full subgraph of a finite tree T, we may assume that $\Gamma = T$.

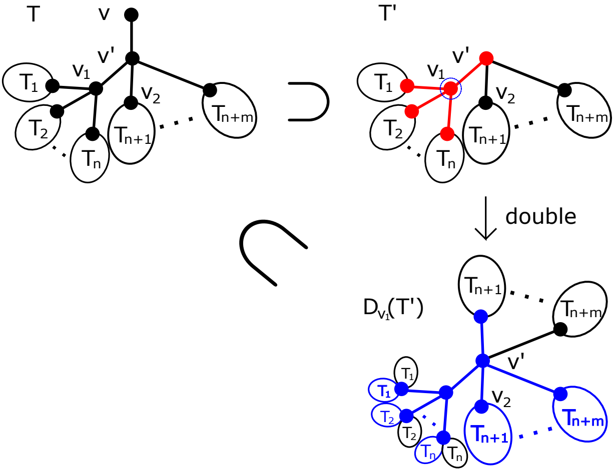
We shall prove this theorem by induction on the ordered pair
(diam(T), # of geodesic edge-paths of length diam(T))
and by using doubled graphs.

If diam(T)≤ 2, then T is a star graph, and so we have

$A($  $) \hookrightarrow A(\bullet\!-\!\bullet\!-\!\bullet) \hookrightarrow A(\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet)$. We now consider the case where the diameter of T is at least 3.

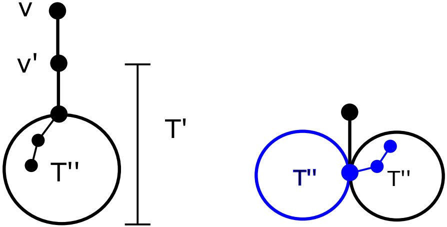
v: a pendant vertex on a geodesic edge-path of length diam(T)
v': the (unique) vertex adjacent to v
$T' := T \setminus (v \cup \{v, v'\})$
Case 1. The degree of v' is at least 3.





Hence, we have $A(T) \hookrightarrow A(D_{v_1}(T')) \hookrightarrow A(T')$.
Removing away v and $\{v, v'\}$ from T implies that either the diam decreases or $\#$ of geodesic edge-paths of length diam decreases.

Case 2. The degree of v' is equal to 2.
We can assume diam$(T) \geq 4$.



Thus we have $A(T) \hookrightarrow A(D_{v'}(T')) \hookrightarrow A(T')$.

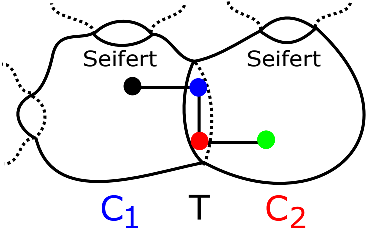
A proof of [Niblo-Wise, 2000]

Lemma(B)

Γ: a finite simple graph, $\{C_1, C_2\}$: a Seifert-Seifert gluing in a knot exterior, T: the JSJ torus $C_1 \cap C_2$
Then $A(\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet) \hookrightarrow \pi_1(C_1) \underset{\pi_1(T)}{*} \pi_1(C_2)$.

Proof. We shall construct an embedding

$A($$) \hookrightarrow \pi_1(C_1) \underset{\pi_1(T)}{*} \pi_1(C_2)$ as follows.



ψ(●): an element of Z(π_1(C$_1$))

ψ(●): an element of Z(π_1(C$_2$))

ψ(●) ψ(●) generate F$_2$

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For each $i = 1, 2$, we take a finite index subgroup of $\pi_1(C_i)$, which is isomorphic to $A(\mathrm{St}_{m_i})$ for some $m_i \geq 2$.

Here, $\mathrm{St}_{m_i} =$  .

(i) ψ(●) $\in A(\mathrm{St}_{m_1}) \cap A(\mathrm{St}_{m_2}) \cap \pi_1(T) \cap Z(\pi_1(C_1))$.

(ii) ψ(●) $\in A(\mathrm{St}_{m_1}) \cap A(\mathrm{St}_{m_2}) \cap \pi_1(T) \cap Z(\pi_1(C_2))$.

(iii) ψ(●) $\in A(\mathrm{St}_{m_1})$.

(iv) ψ(●) $\in A(\mathrm{St}_{m_2})$.

Then the normal form theorem says that ψ is injective, as desired.

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Main Theorem (K.)

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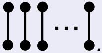
then $A(\Gamma) \hookrightarrow G(K)$ iff Γ is a disjoint union of  *and* .

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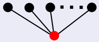
then $A(\Gamma) \hookrightarrow G(K)$ iff Γ is a star graph  *or* .

Case 3. *If $E(K)$ has both a Seifert piece and a hyperbolic piece and has no Seifert-Seifert gluing,*

then $A(\Gamma) \hookrightarrow G(K)$ iff Γ is a disjoint union of star graphs.

Case 4. *If $E(K)$ has a Seifert-Seifert gluing,*

then $A(\Gamma) \hookrightarrow G(K)$ iff Γ is a forest.

Future work

Question

Which knot group admits an embedding into a RAAG?

Every knot group admits a virtual embedding into a RAAG. This question seems to be connected with the following question.

Question

Which knot group is bi-orderable?

Since every RAAG is bi-orderable (Duchamp-Thibon), every knot group which admits an embedding into a RAAG must be bi-orderable.

Thank you.