On algebraic description of the Goldman-Turaev Lie bialgebra

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(joint work with Nariya Kawazumi (University of Tokyo))
The Goldman-Turaev Lie bialgebra

$\Sigma$: a compact oriented surface

$\hat{\pi} = \hat{\pi}(\Sigma) := \pi_1(\Sigma)/\text{conjugacy} \cong \text{Map}(S^1, \Sigma)/\text{homotopy}$

Two operations to loops on $\Sigma$

1. Goldman bracket (‘86)

$[\ , \ ]: (\mathbb{Q}\hat{\pi}/\mathbb{Q}1) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}1) \rightarrow \mathbb{Q}\hat{\pi}/\mathbb{Q}1, \ \ \alpha \otimes \beta \mapsto [\alpha, \beta]$ 

$1 \in \hat{\pi}$: the class of a constant loop

2. Turaev cobracket (‘91)

$\delta: \mathbb{Q}\hat{\pi}/\mathbb{Q}1 \rightarrow (\mathbb{Q}\hat{\pi}/\mathbb{Q}1) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}1)$

Theorem (Goldman (bracket) + Turaev (cobracket, Lie bialgebra) + Chas (involutivity))

The triple $(\mathbb{Q}\hat{\pi}/\mathbb{Q}1, [\ , \ ], \delta)$ is an involutive Lie bialgebra.
Introduction

Lie bialgebra

The operation $[\ , \ ]$ is defined by using the intersection of two loops, while the operation $\delta$ by using the self-intersection of a loop.

Theorem (bis)

The triple $(\mathbb{Q}\hat{\pi}/\mathbb{Q}\mathbf{1}, [\ , \ ], \delta)$ is an involutive Lie bialgebra.

Definition

A triple $(\mathfrak{g}, [\ , \ ], \delta)$ is a Lie bialgebra if

1. the pair $(\mathfrak{g}, [\ , \ ])$ is a Lie algebra,
2. the pair $(\mathfrak{g}, \delta)$ is a Lie coalgebra, and
3. the maps $[\ , \ ]$ and $\delta$ satisfy a compatibility condition:

$$\forall \alpha, \beta \in \mathfrak{g}, \quad \delta[\alpha, \beta] = \alpha \cdot \delta(\beta) - \beta \cdot \delta(\alpha).$$

Moreover, if $[\ , \ ] \circ \delta = 0$ then $(\mathfrak{g}, [\ , \ ], \delta)$ is called involutive.
Fundamental group and tensor algebra

We have a binary operation \([ \ , \ ]\) and a unary operation \(\delta\) on \(\mathbb{Q} \hat{\pi} / \mathbb{Q} 1\). The goal is to express them algebraically, i.e., by using tensors.

Assume \(\partial \Sigma \neq \emptyset\) (e.g., \(\Sigma = \Sigma_{g,1}, \Sigma = \Sigma_{0,n+1}\)). Then any “group-like” Magnus expansion \(\theta\) gives an isomorphism (of complete Hopf algebras)

\[
\theta : \mathbb{Q} \pi_1(\Sigma) \xrightarrow{\cong} \hat{T}(H)
\]

onto the complete tensor algebra generated by \(H := H_1(\Sigma; \mathbb{Q})\).
Moreover, we have an isomorphism (of \(\mathbb{Q}\)-vector spaces)

\[
\theta : \mathbb{Q} \hat{\pi} \xrightarrow{\cong} \hat{T}(H)^{cyc}.
\]

Here,

1. the source \(\mathbb{Q} \hat{\pi}\) is a certain completion of \(\mathbb{Q} \hat{\pi}\),
2. \(cyc\) means taking the space of cyclic invariant tensors.
Algebraic description of the Goldman bracket

We can define \([ , ]^\theta\) by the commutativity of the following diagram.

\[
\begin{array}{ccc}
\mathbb{Q}\hat{\pi} \otimes \mathbb{Q}\hat{\pi} & \xrightarrow{[ , ]} & \mathbb{Q}\hat{\pi} \\
\theta \otimes \theta \downarrow & & \downarrow \theta \\
\hat{T}(H)^{\text{cyc}} \otimes \hat{T}(H)^{\text{cyc}} & \xrightarrow{[ , ]^\theta} & \hat{T}(H)^{\text{cyc}}
\end{array}
\]

Theorem (Kawazumi-K., Massuyeau-Turaev), stated roughly

For some choice of \(\theta\), \([ , ]^\theta\) has a simple, \(\theta\)-independent expression.

1. For \(\Sigma = \Sigma_{g,1}\), it equals the associative version of the Lie algebra of symplectic derivations introduced by Kontsevich.
2. For \(\Sigma = \Sigma_{0,n+1}\), it equals the Lie algebra of special derivations in the sense of Alekseev-Torossian (c.f. the work of Ihara).
Algebraic description of the Turaev cobracket

Similarly, we can define $\delta^\theta$ by the commutativity of the following diagram.

\[
\begin{CD}
\mathbb{Q}\hat{\pi} / \mathbb{Q}1 @>\delta>> (\mathbb{Q}\hat{\pi} / \mathbb{Q}1) \otimes (\mathbb{Q}\hat{\pi} / \mathbb{Q}1) \\
\theta @VVV @VV\theta \otimes \theta V \\
\widehat{T}(H)^{cy} @>\delta^\theta>> \widehat{T}(H)^{cy} \otimes \widehat{T}(H)^{cy}
\end{CD}
\]

Question

Can we have a simple expression for $\delta^\theta$?

Our motivation: the Johnson homomorphism

$\mathcal{I}(\Sigma)$: the Torelli group of $\Sigma$

$\mathfrak{h}(\Sigma)$: Morita’s Lie algebra (Kontsevich’s “lie”)

\[
\mathcal{I}(\Sigma) \xleftarrow{\tau} \mathfrak{h}(\Sigma) \xrightarrow{\text{Kawazumi-K}} \mathbb{Q}\hat{\pi} \xrightarrow{\delta} \mathbb{Q}\hat{\pi} \otimes \mathbb{Q}\hat{\pi}.
\]

Then $\text{Im}(\tau) \subset \text{Ker}(\delta)$. For instance, the Morita trace factors through $\delta$. 
1 Introduction

2 Goldman bracket

3 Turaev cobracket
Definition of the Goldman bracket

Recall: \( \hat{\pi} = \hat{\pi}(\Sigma) = \text{Map}(S^1, \Sigma)/\text{homotopy}. \)

**Definition (Goldman)**

\( \alpha, \beta \in \hat{\pi} \): represented by free loops in general position

\[
[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta) \alpha_p \beta_p \in \mathbb{Q}\hat{\pi}.
\]

Here, \( \varepsilon_p(\alpha, \beta) = \pm 1 \) is the local intersection number of \( \alpha \) and \( \beta \) at \( p \), and \( \alpha_p \) is the loop \( \alpha \) based at \( p \).

This formula induces a Lie bracket on \( \mathbb{Q}\hat{\pi} \), and \( 1 \in \hat{\pi} \) is central.

**Background**

Study of the Poisson structures on \( \text{Hom}(\pi_1(\Sigma), G)/G \).
The action $\sigma$

For $*_{0}, *_{1} \in \partial \Sigma$, $\Pi \Sigma(*_{0}, *_{1}) := \text{Map}(([0, 1], 0, 1), (\Sigma, *_{0}, *_{1}))/\text{homotopy}$.

**Definition (Kawazumi-K.)**

For $\alpha \in \hat{\Pi}$ and $\beta \in \Pi \Sigma(*_{0}, *_{1})$,

$$\sigma(\alpha)\beta := \sum_{p \in \alpha \cap \beta} \varepsilon_{p}(\alpha, \beta)\beta_{*_{0}}p\alpha_{p}\beta_{p*_{1}} \in \mathbb{Q}\Pi \Sigma(*_{0}, *_{1}).$$

This formula induces a $\mathbb{Q}$-linear map

$$\sigma = \sigma_{*_{0}, *_{1}} : \mathbb{Q}\hat{\Pi} \rightarrow \text{End}(\mathbb{Q}\Pi \Sigma(*_{0}, *_{1})).$$

The Leibniz rule holds: for $\beta_{1} \in \Pi \Sigma(*_{0}, *_{1})$ and $\beta_{2} \in \Pi \Sigma(*_{1}, *_{2})$,

$$\sigma(\alpha)(\beta_{1}\beta_{2}) = (\sigma(\alpha)\beta_{1})\beta_{2} + \beta_{1}(\sigma(\alpha)\beta_{2}).$$
The action $\sigma$ (continued)

Write $\partial \Sigma = \bigsqcup_i \partial_i \Sigma$ with $\partial_i \Sigma \cong S^1$. For each $i$, choose $*_i \in \partial_i \Sigma$.

The small category $Q \Pi \Sigma$
- **Objects:** $\{*_i\}_i$
- **Morphisms:** $Q \Pi \Sigma(*_i,*_j)$

Consider the Lie algebra $\text{Der}(Q \Pi \Sigma)$

$$\text{Der}(Q \Pi \Sigma) := \{(D_{i,j})_{i,j} \mid D_{i,j} \in \text{End}(Q \Pi \Sigma(*_i,*_j)), D_{i,j} \text{ satisfy the Leibniz rule.}\}$$

Then the collection $(\sigma_{*_i,*_j})_{i,j}$ defines a Lie algebra homomorphism

$$\sigma : \hat{Q} \to \text{Der}_{\partial}(Q \Pi \Sigma).$$

**Example**

If $\partial \Sigma = S^1$, we have $\sigma : \hat{Q} \to \text{Der}_{\partial}(Q \pi_1(\Sigma))$. 
Completions

We have a Lie algebra homomorphism

\[ \sigma : \hat{Q} \rightarrow \text{Der}_\partial (\hat{Q} \Pi \Sigma). \]

The augmentation ideal \( I \subset \hat{Q}_{\pi_1}(\Sigma) \) defines a filtration \( \{I^m\} \) of \( \hat{Q}_{\pi_1}(\Sigma) \). We set

\[ \hat{Q}_{\pi_1}(\Sigma) := \varprojlim \frac{\hat{Q}_{\pi_1}(\Sigma)}{I^m}. \]

Likewise, we can consider the completions of \( \hat{Q} \) and \( \hat{Q} \Pi \Sigma \).

For example,

1. the Goldman bracket induces a complete Lie bracket
   \[ [ , ] : \hat{Q} \hat{\otimes} \hat{Q} \rightarrow \hat{Q}. \]

2. we get a Lie algebra homomorphism
   \[ \sigma : \hat{Q} \rightarrow \text{Der}_\partial (\hat{Q} \Pi \Sigma). \]
Magnus expansion

Let $\pi$ be a free group of finite rank. Set $H := \pi^{\text{abel}} \otimes \mathbb{Q} \cong H_1(\pi; \mathbb{Q})$ and $\hat{T}(H) := \prod_{m=0}^{\infty} H^\otimes m$.

Definition (Kawazumi)

A map $\theta: \pi \to \hat{T}(H)$ is called a (generalized) Magnus expansion if
1. $\theta(x) = 1 + [x] + (\text{terms with } \text{deg } \geq 2)$,
2. $\theta(xy) = \theta(x)\theta(y)$.

Definition (Massuyeau)

A Magnus expansion $\theta$ is called group-like if $\theta(\pi) \subset \text{Gr}(\hat{T}(H))$.

If $\theta$ is a group-like Magnus expansion, then we have an isomorphism

$$\theta: \mathbb{Q}[\pi] \cong \hat{T}(H)$$

of complete Hopf algebras.
The case of $\Sigma = \Sigma_{g,1}$

Definition (Massuyeau)

A group-like expansion $\theta : \pi_1(\Sigma) \to \hat{T}(H)$ is called symplectic if

$$\theta(\partial \Sigma) = \exp(\omega),$$

where $\omega \in H \otimes^2$ corresponds to $1_H \in \text{Hom}(H, H) = H^* \otimes H \cong H \otimes H.$

Fact: symplectic expansions do exist.

The Lie algebra of symplectic derivations (Kontsevich):

$$\text{Der}_\omega(\hat{T}(H)) := \{ D \in \text{End}(\hat{T}(H)) \mid D \text{ is a derivation and } D(\omega) = 0 \}.$$

The restriction map

$$\text{Der}_\omega(\hat{T}(H)) \to \text{Hom}(H, \hat{T}(H)) \cong H \otimes \hat{T}(H) \subset \hat{T}(H), \quad D \mapsto D|_H$$

induces a $\mathbb{Q}$-linear isomorphism $\text{Der}_\omega(\hat{T}(H)) \cong \hat{T}(H)^\text{cyc}.$
The case of $\Sigma = \Sigma_{g,1}$: the Goldman bracket

Consider the diagram

$$
\begin{array}{ccc}
\mathbb{Q}\hat{\pi} \otimes \mathbb{Q}\hat{\pi} & \xrightarrow{[\ , \ ]} & \mathbb{Q}\hat{\pi} \\
\theta \otimes \theta & \downarrow & \theta \\
\hat{T}(H)^{cyc} \otimes \hat{T}(H)^{cyc} & \xrightarrow{[\ , \ ]^\theta} & \hat{T}(H)^{cyc}
\end{array}
$$

where the vertical map $\theta$ is induced by $\pi \ni x \mapsto -(\theta(x) - 1) \in \hat{T}(H)$.

**Theorem (Kawazumi-K., Massuyeau-Turaev)**

If $\theta$ is symplectic, $[\ , \ ]^\theta$ equals the Lie bracket in $\hat{T}(H)^{cyc} = \text{Der}_\omega(\hat{T}(H))$.

Explicit formula: for $X_1, \ldots, X_m, Y_1, \ldots, Y_n \in H$,

$$
[X_1 \cdots X_m, Y_1 \cdots Y_n]^\theta = \sum_{i,j} (X_i \cdot Y_j)X_{i+1} \cdots X_mX_1 \cdots X_{i-1}Y_{j+1} \cdots Y_nY_1 \cdots Y_{j-1}.
$$
The case of $\Sigma = \Sigma_{g,1}$: the action $\sigma$

Consider the diagram

$$
\begin{array}{ccc}
\mathbb{Q} \hat{T} \otimes \mathbb{Q} \pi_1(\Sigma) & \xrightarrow{\sigma} & \mathbb{Q} \pi_1(\Sigma) \\
\downarrow \theta \otimes \theta & & \downarrow \theta \\
\hat{T}(H)^{cy} \otimes \hat{T}(H) & \longrightarrow & \hat{T}(H)
\end{array}
$$

Here, the bottom horizontal arrow is the action of $\hat{T}(H)^{cy} = \text{Der}_\omega(\hat{T}(H))$ by derivations.

Theorem (Kawazumi-K., Massuyeau-Turaev)

*If $\theta$ is symplectic, this diagram is commutative.*

- Kawazumi-K.: use (co)homology theory of Hopf algebras
- Massuyeau-Turaev: use the notion of Fox paring (see the next page)
The case of $\Sigma = \Sigma_{g,1}$: a refinement

Homotopy intersection form (Turaev, Papakyriakopoulos)

For $\alpha, \beta \in \pi_1(\Sigma)$, set $\eta(\alpha, \beta) := \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta) \alpha_* p \beta_* p \in \mathbb{Q}\pi_1(\Sigma)$.

Theorem (Massuyeau-Turaev)

*If $\theta$ is symplectic, then the following diagram is commutative.*

\[
\begin{array}{ccc}
\mathbb{Q}\pi_1(\Sigma) \times \mathbb{Q}\pi_1(\Sigma) & \xrightarrow{\eta} & \mathbb{Q}\pi_1(\Sigma) \\
\theta \otimes \theta & \downarrow & \downarrow \theta \\
\widehat{T}(H) \otimes \widehat{T}(H) & \xrightarrow{(\sim) + \rho_s} & \widehat{T}(H).
\end{array}
\]

Here, $X_1 \cdots X_m \sim Y_1 \cdots Y_n = (X_m \cdot Y_1)X_1 \cdots X_{m-1} Y_2 \cdots Y_n$ and

\[
\rho_s(a, b) = (a - \varepsilon(a))s(\omega)(b - \varepsilon(b)), \text{ where}
\]

\[
s(\omega) = \frac{1}{\omega} + \frac{1}{(e^{-\omega} - 1)} = -\frac{1}{2} - \frac{\omega}{12} + \frac{\omega^3}{720} - \frac{\omega^5}{30240} + \cdots. \quad (\text{Bernoulli numbers appear!})
\]
The case of $\Sigma = \Sigma_{0,n+1}$

We regard $\Sigma_{0,n+1} = D^2 \setminus \bigcup_{i=1}^{n} \text{Int}(D_i)$. Then $H \cong \bigoplus_{i=1}^{n} \mathbb{Q}[\partial D_i]$.

Definition (Massuyeau (implicit in the work of Alekseev-Enriquez-Torossian))

A Magnus expansion $\theta$ is called special if

1. $\exists g_i \in \text{Gr}(\hat{T}(H))$ such that $\theta(\partial D_i) = g_i \exp([\partial D_i])g_i^{-1}$,
2. $\theta(\partial D^2) = \exp([\partial D^2])$.

The Lie algebra of special derivations (in the sense of Alekeev-Torossian):

$$\text{sder}(\hat{T}(H)) = \{ D \in \text{Der}(\hat{T}(H)) \mid D([\partial D_i]) = [[\partial D_i], \exists u_i], D([\partial D^2]) = 0 \}.$$  

We can naturally identify $\text{sder}(\hat{T}(H))$ with $\hat{T}(H)^{\text{cycl}}$.

Theorem (Kawazumi-K., Massuyeau-Turaev)

If $\theta$ is special, then $[\ , \ ]^\theta$ equals the Lie bracket in $\text{sder}(\hat{T}(H))$.
General case ($\partial \Sigma \neq \emptyset$)

Write $\Sigma = \Sigma_{g,n+1}$ and $\partial \Sigma = \bigsqcup_{i=0}^{n} \partial_i \Sigma$.
Put $\overline{\Sigma} := \Sigma \cup (\bigsqcup_{i=0}^{n} D^2) \cong \Sigma_g$.
Choose a section $s$ of $i_* : H_1(\Sigma) \to H_1(\overline{\Sigma})$.
We need

1. a notion of Magnus expansion for the small category $\mathcal{Q}\mathcal{P}\Sigma$,
2. a ($s$-dependent) boundary condition for such an expansion $\theta$.

Then, we have a simple ($s$-dependent) expression for $[ , ]^\theta$ and $\sigma^\theta$.

An application:

**Theorem (Kawazumi-K., the infinitesimal Dehn-Nielsen theorem)**

*For any $\Sigma$ with $\partial \Sigma \neq \emptyset$, the map $\sigma : \widehat{\mathcal{Q}\hat{\pi}} \to \text{Der}_\partial(\widehat{\mathcal{Q}\mathcal{P}\Sigma})$ is a Lie algebra isomorphism.*
1. Introduction

2. Goldman bracket

3. Turaev cobracket
Definition of the Turaev cobracket

Definition (Turaev)

\( \alpha \in \hat{\pi} \): represented by a generic immersion

\[
\delta(\alpha) := \sum_{p \in \Gamma_{\alpha}} \alpha_p^1 \otimes \alpha_p^2 - \alpha_p^2 \otimes \alpha_p^1 \in (\mathbb{Q}\hat{\pi}/\mathbb{Q}1) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}1).
\]

Here:

- \( \Gamma_{\alpha} \) is the set of double points of \( \alpha \),
- \( \alpha_p^1, \alpha_p^2 \) are two branches of \( \alpha \) created by \( p \). They are arranged so that \( (\alpha_p^1, \alpha_p^2) \) gives a positive frame of \( T_p(\Sigma) \).

This formula induces a Lie cobracket on \( \mathbb{Q}\hat{\pi}/\mathbb{Q}1 \).

Background

A skein quantization of Poisson algebras on surfaces.
Self-intersection $\mu$

Definition (essentially introduced by Turaev)

$\alpha \in \pi_1(\Sigma)$: represented by a generic immersion

$$\mu(\alpha) := - \sum_{p \in \Gamma_\alpha} \varepsilon_p(\alpha) \alpha_* p \alpha_* \otimes \alpha_p \in \mathbb{Q}\pi_1(\Sigma) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}1).$$

This formula induces a $\mathbb{Q}$-linear map

$$\mu: \mathbb{Q}\pi_1(\Sigma) \to \mathbb{Q}\pi_1(\Sigma) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}1).$$

1. $\mu$ is a refinement of $\delta$; we have

$$\delta(|\alpha|) = \text{Alt}(| | \otimes \text{id}) \mu(\alpha),$$

where $| |: \mathbb{Q}\pi_1(\Sigma) \to \mathbb{Q}\hat{\pi}/\mathbb{Q}1$ is the natural projection.

2. The operations $\mu$ and $\delta$ extends naturally to completions.

3. There is a framed version of $\delta$, related to the Enomoto-Satoh trace and Alekseev-Torossian’s divergence cocycle (Kawazumi).
Algebraic description of $\mu$ at the graded level

Define $\mu^\theta$ by the commutativity of the following diagram.

\[
\begin{array}{ccc}
\mathbb{Q}\pi_1(\Sigma) & \xrightarrow{\mu} & \mathbb{Q}\pi_1(\Sigma) \otimes (\mathbb{Q}\hat{\pi}/\mathbb{Q}1) \\
\theta & \downarrow & \theta \otimes \theta \\
\hat{T}(H) & \xrightarrow{\mu^\theta} & \hat{T}(H) \otimes \hat{T}(H)_{\text{cyc}}
\end{array}
\]

Theorem (Kawazumi-K., Massuyeau-Turaev)

For $\Sigma = \Sigma_{g,1}$ and for any symplectic expansion $\theta$,

\[
\mu^\theta = \mu^{\text{alg}} + \mu^{\theta}_{(0)} + \mu^{\theta}_{(1)} + \cdots,
\]

where $\mu^{\theta}_{(i)}$ is a map of degree $i$ and $\mu^{\text{alg}}$ a map of degree $-2$. For $i \geq 1$, $\mu^{\theta}_{(i)}$ depend on the choice of $\theta$, but $\mu^{\text{alg}}$ does not.
Algebraic description of $\delta$ at the graded level

Corollary

For $\Sigma = \Sigma_{g,1}$ and for any symplectic expansion $\theta$,

$$\delta^\theta = \delta^{\text{alg}} + \delta^\theta_{(1)} + \cdots.$$ 

Explicit formula: for $X_1, \ldots, X_m \in H$,

$$\delta^{\text{alg}}(X_1 \cdots X_m)$$

$$= - \sum_{i<j} (X_i \cdot X_j) \text{Alt}(X_{i+1} \cdots X_{j-1} \otimes X_{j+1} \cdots X_m X_1 \cdots X_{i-1}).$$

Open question

Is there a symplectic expansion $\theta$ such that $\delta^\theta = \delta^{\text{alg}}$?

Note: for $g = 1$, there is a $\theta$ such that $\delta^\theta \neq \delta^{\text{alg}}$. Namely,

$$\{\text{symplectic expansions}\} \supset \{\theta \mid \delta^\theta = \delta^{\text{alg}}\}.$$
Algebraic description of $\delta$: the case of $\Sigma_{0,n+1}$

For $\Sigma = \Sigma_{0,n+1}$ and for any special expansion $\theta$,

$$\delta^\theta = \delta^\text{alg} + \delta^\theta_{(1)} + \cdots,$$

where $\delta^\text{alg}$ is a map of degree $-1$.

Explicit formula: for $X_1, \ldots, X_m \in H$,

$$\delta^\text{alg}(X_1 \cdots X_m) = \sum_{i<j} \delta_{X_i, X_j} \text{Alt} \left( X_i \cdots X_{j-1} \otimes X_{j+1} \cdots X_m X_1 \cdots X_{i-1} \right) + X_j \cdots X_m X_1 \cdots X_{i-1} \otimes X_{i+1} \cdots X_{j-1}$$

The proof uses a capping argument: consider the embedding

$$\Sigma_{0,n+1} \hookrightarrow \Sigma_{0,n+1} \cup \left( \bigsqcup_{i=1}^n \Sigma_{1,1} \right) = \Sigma_{n,1}.$$
Recent development

Why is $\delta^\theta$ more difficult than $[,]^\theta$? The main reason is that

$$\text{Self}(\alpha\beta) = \text{Self}(\alpha) \sqcup \text{Self}(\beta) \sqcup (\alpha \cap \beta).$$

Partial results

1. For $\Sigma = \Sigma_{0,n+1}$, Kawazumi obtained a description of $\delta^\theta$ with respect to the exponential Magnus expansion ($\theta(x_i) = \exp([x_i])$).

2. For $\Sigma = \Sigma_{1,1}$, there is a symplectic expansion $\theta$ such that $\delta^\theta = \delta^\text{alg}$ modulo terms of degree $\geq 9$. (K., using computer)

Theorem (Massuyeau ‘15)

Let $\Sigma = \Sigma_{0,n+1}$. For a special expansion $\theta$ arising from the Kontsevich integral, $\delta^\theta$ equals $\delta^\text{alg}$. (Actually a description for $\mu^\theta$ is obtained.)
Summary

Two operations to loops on $\Sigma$

\[
[~,~]: \hat{Q}\hat{\pi} \hat{\otimes} \hat{Q}\hat{\pi} \rightarrow \hat{Q}\hat{\pi},
\]
\[
\delta: \hat{Q}\hat{\pi} \rightarrow \hat{Q}\hat{\pi} \hat{\otimes} \hat{Q}\hat{\pi},
\]

refinement $\rightsquigarrow$

\[
\eta: \hat{Q}\pi_1(\Sigma) \hat{\otimes} \hat{Q}\pi_1(\Sigma) \rightarrow \hat{Q}\pi_1(\Sigma)
\]
\[
\mu: \hat{Q}\pi_1(\Sigma) \rightarrow \hat{Q}\pi_1(\Sigma) \hat{\otimes} \hat{Q}\hat{\pi}
\]

Current status of finding a simple expression for $[~,~]^\theta$ and $\delta^\theta$:

<table>
<thead>
<tr>
<th></th>
<th>Magnus expansion</th>
<th>$[<del>,</del>]^\theta$</th>
<th>$\delta^\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_{g,1}$</td>
<td>symplectic</td>
<td>OK</td>
<td>?</td>
</tr>
<tr>
<td>$\Sigma_{0,n+1}$</td>
<td>special</td>
<td>OK</td>
<td>OK (Massuyeau)</td>
</tr>
<tr>
<td>general case</td>
<td>a $\partial$-condition</td>
<td>OK</td>
<td>?</td>
</tr>
</tbody>
</table>

1. We know that $\text{gr}(\delta^\theta) = \delta^{\text{alg}}$.
2. To get “?” we need a refinement of symplectic/special condition.