Reidemeister torsion and exceptional surgeries along the figure eight knot

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Reidemeister torsion

Definition (R-trosion $\mathrm{Tor}(W;\rho)$)

W	:	a finite CW-complex,
$\rho\colon \pi_1(W) \to \mathrm{SL}_n(\mathbb{C})$	:	$\mathrm{SL}_n(\mathbb{C})$-representation of π_1
$C_*(W;\mathbb{C}^n_\rho)$	:	local system given by ρ
$= \mathbb{C}^n \otimes_\rho C_*(\widetilde{W};\mathbb{Z}[\pi_1])$		($\widetilde{W}$:universal cover)
		$v \otimes \gamma\sigma = \rho(\gamma)^{-1} v \otimes \sigma$

Under $H_*(W;\mathbb{C}^n_\rho) = 0$,

$$\mathrm{Tor}(W;\rho) := \prod_{i \geq 0} \det(\partial \boldsymbol{b}_{i+1} \cup \boldsymbol{b}_i / \boldsymbol{c}_i)^{(-1)^{i+1}}$$

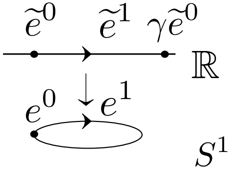
via the decomposition

$$C_i(W;\mathbb{C}^n_\rho) = \mathrm{Ker}\,\partial_i \oplus (\text{a lift of } \mathrm{Im}\,\partial_i) = \mathrm{Im}\,\partial_{i+1} \oplus (\text{a lift of } \mathrm{Im}\,\partial_i).$$

Example of Reidemeister torsion

Local system for S^1

$$\rho : \pi_1 S^1 = \langle \gamma \rangle \to \mathrm{SL}_2(\mathbb{C}),$$
$$\gamma \mapsto A$$



$$\mathbb{C}^2 \simeq C_1(S^1; \mathbb{C}^2_\rho) \xrightarrow{\partial_1} C_0(S^1; \mathbb{C}^2_\rho) \simeq \mathbb{C}^2$$
$$v \otimes \widetilde{e}^1 \mapsto v \otimes \gamma\widetilde{e}^0 - v \otimes \widetilde{e}^0 = (A^{-1} - I)v \otimes \widetilde{e}^0$$

C_0 has the new basis $(A^{-1} - I)\begin{pmatrix}1\\0\end{pmatrix} \otimes \widetilde{e}^0$, $(A^{-1} - I)\begin{pmatrix}0\\1\end{pmatrix} \otimes \widetilde{e}^0$.

R-torsion for (S^1, ρ)

$$\mathrm{Tor}(S^1; \rho) = \frac{1}{\det(A^{-1} - I)} \left(= (\det \partial_1)^{-1} \right)$$

Sequence of R-torsion

We can make a sequence by the following procedure:

1. Choose an $\mathrm{SL}_2(\mathbb{C})$-rep. $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$.

2. Take composition with $\sigma_n : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$,

   $\rho = \rho_2, \quad \rho_3, \quad \ldots, \quad \rho_n = \sigma_n \circ \rho, \quad \ldots$ (a seq. of reps.)

3. Consider $\mathrm{Tor}(W; \rho_n)$ if $H_*(W; \mathbb{C}^n_{\rho_n}) = 0 \quad (\forall \rho_n)$, i.e.,

   $\mathrm{Tor}(M; \rho_2), \mathrm{Tor}(M; \rho_3), \ldots, \mathrm{Tor}(M; \rho_n), \ldots$ (a seq. of invs.)

Remark

The behavior of $\{|\mathrm{Tor}(W; \rho_n)| \mid n = 1, 2, \ldots\}$ ($n \to \infty$)
is related to the geometric feature of W.

Example of $\mathrm{Tor}(W;\rho_n)$

$\sigma_n(A)$: the action of A on $\{p(x,y)\mid \text{homog.}, \deg p = n-1\}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x,y) = p(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}) = p(dx-by, -cx+ay)$$

For example, if $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, then

$$\sigma_3(A) = \begin{pmatrix} a^{-2} & & \\ & 1 & \\ & & a^2 \end{pmatrix}, \quad \sigma_4(A) = \begin{pmatrix} a^{-3} & & & \\ & a^{-1} & & \\ & & a & \\ & & & a^3 \end{pmatrix}.$$

$$\begin{aligned} \mathrm{Tor}(S^1;\rho_{2N}) &= \{\det(\sigma_{2N}(A) - I)\}^{-1} \\ &= \left\{ \prod_{k=1}^{N} (a^{2k-1}-1)(a^{-2k+1}-1) \right\}^{-1} \end{aligned}$$

Previous work I for the asymptotics of R-torsion

The asymptotics for Hyperbolic manifolds
by W. Müller, P. Menal–Ferrer & J. Porti

M: a hyperbolic 3-manifold of finite volume

$$\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C}) \quad \text{(holonomy rep.)}$$

$$\Rightarrow \rho_{2N} = \sigma_{2N} \circ \rho \text{ satisfies } H_*(M; \rho_{2N}) = 0 \ (\forall N)$$

Moreover

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M; \rho_{2N})|}{(2N)^2} = \frac{\mathrm{Vol}(M)}{-4\pi} \left(= \frac{v_3\|M\|}{-4\pi} \right)$$

$\mathrm{Vol}(M)$: hyperbolic vol. of M, $\|M\|$: simplicial vol. of M

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Previous work II for the asymptotics of R-torsion

Asymptotic behavior for a Seifert fibered space (Y)

M: a Seifert fibered space with m exceptional fibers

$$\lim_{N\to\infty} \frac{\log |\mathrm{Tor}(M;\rho_{2N})|}{(2N)^2} = 0$$

$$\lim_{N\to\infty} \frac{\log |\mathrm{Tor}(M;\rho_{2N})|}{2N} = \log |\mathrm{Tor}(\text{regular fiber};\rho)|^{-\chi'}$$

$$\rho_{2N} = \sigma_{2N} \circ \rho : \pi_1(M) \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\sigma_{2N}} \mathrm{SL}_{2N}(\mathbb{C})$$

$$s.t. \quad \text{regular fiber} \mapsto -I \mapsto -I_{2N},$$

g : the genus of the base orbifold,

$2\lambda_j$: the order of the 2×2-matrix corresponding to j-th exceptional fiber

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$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\rho_{2N})|}{2N} = -\left(2 - 2g - \sum_{j=1}^{m} \frac{\lambda_j - 1}{\lambda_j}\right)\log 2$$

$$\rho_{2N} = \sigma_{2N} \circ \rho : \pi_1(M) \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\sigma_{2N}} \mathrm{SL}_{2N}(\mathbb{C})$$

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Motivations of this research

1. $M = M_1 \cup_{T^2} \cdots \cup_{T^2} M_k$: a graph manifold

   What is the limit of the following?

$$\lim_{N \to \infty} \frac{\log |\mathrm{Tor}(M; \rho_{2N})|}{2N}$$

2. What happens about R-torsion

   when a hyperbolic structure $\rho : \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C})$ moves to a degenerate one?

Exceptional surgeries along 4_1 (Figure eight knot)

Fact (Exceptional surgeries along 4_1)

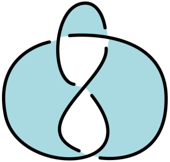
$1/0$	0	± 1	± 2	± 3
S^3	T^2–b'dle	$S^2(2,3,7)$	$S^2(2,4,5)$	$S^2(3,3,4)$

± 4 *: Graph manifold M*

$M =$ *Exterior of* $3_1 \cup$ *twisted I–b'dle over the Klein's bottle*

$($ M_1 $\cup_{T^2}$ M_2 $)$

± 4-slope $=$ the boundary of
punctured Klein' bottle
by checkerboard coloring



Main result

Theorem (the limit of leading coefficient)

- *M = Exterior of $3_1 \cup$ twisted I–b'dle over the Klein's bottle*
- *$\bar{\rho}$: $\mathrm{SL}_2(\mathbb{C})$-representation induced from*

$$\rho\colon \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C}),$$

Then

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\bar{\rho}_{2N})|}{2N} = \frac{2}{5}\log 3 - \frac{1}{5}\log 2$$

Our approach

Fact

- *In the case of $K = 4_1$,*
  the set of $\rho\colon \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C})$ is well–known.

- $\mathrm{Tor}(M) = \mathrm{Tor}(E_K) \cdot \mathrm{Tor}(D^2 \times S^1) = \mathrm{Tor}(M_1) \cdot \mathrm{Tor}(M_2)$

  This is useful to observe the local systems for M_1 and M_2.

- $\mathrm{Tor}(M_1)$ *and* $\mathrm{Tor}(M_2)$ *are also well–known.*

We will see the behavior of $\mathrm{Tor}(M_1)$ and $\mathrm{Tor}(M_2)$ by

$$\rho\colon \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C}),\ \mathrm{Tor}(E_K) \text{ and } \mathrm{Tor}(D^2 \times S^1)$$

Toroidal surgery and Representation

Induced representation $\bar{\rho}\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$

M: resulting manifold by ± 4-surgery along $K = 4_1$

$$\begin{array}{ccc} \pi_1(E_K) & \xrightarrow{\quad\rho\quad} & \mathrm{SL}_2(\mathbb{C}) \\ \downarrow & \nearrow_{\bar{\rho}} & \\ \pi_1(M) = \pi_1(E_K) \,/\, \langle\!\langle m^{\pm 4}\ell \rangle\!\rangle & & \end{array}$$

Therefore

$$\rho(m)^{\pm 4}\rho(\ell) = I \Leftrightarrow \bar{\rho} \text{ is induced}$$

Problem

Which $\rho\colon \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C})$ induces $\bar{\rho}\colon : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$?

Equivalence condition for reps. related to ± 4-surgery

Necessary condition for $\rho(m)^{\pm 4}\rho(\ell) = I$

For $K = 4_1$,

$$\begin{aligned}\rho(m)^{\pm 4}\rho(\ell) = I &\Rightarrow \operatorname{tr}\rho(m)^4 = \operatorname{tr}\rho(\ell) \\ &\Leftrightarrow \operatorname{tr}\rho(m) = 0.\end{aligned}$$

$$(\because) \quad \operatorname{tr}\rho(\ell) = \big(\operatorname{tr}\rho(m)\big)^4 - 5\big(\operatorname{tr}\rho(m)\big)^2 + 2.$$

Sufficient condition for $\rho(m)^{\pm 4}\rho(\ell) = I$

For K = any two–bridge knot,

$$\operatorname{tr}\rho(m) = 0 \Leftrightarrow \rho(m) \overset{\text{conj.}}{\sim} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(\ell) = I$$

$$\Rightarrow \rho(m)^{\pm 4}\rho(\ell) = I$$

Representation for our graph manifold

$M = E_K \cup_{\pm 4} D^2 \times S^1$

ρ induces $\bar{\rho} \Leftrightarrow \mathrm{tr}\,\rho(m) = 0$,

$$\begin{array}{ccc} \pi_1(E_K) & \xrightarrow{\rho} & \mathrm{SL}_2(\mathbb{C}) \\ \downarrow & \nearrow_{\bar{\rho}} & \\ \pi_1(E_K)\,/\,\langle\!\langle m^{\pm 4}\ell \rangle\!\rangle & & \end{array}$$

Proposition

$M =$ Exterior of $3_1 \cup$ twisted I–b'dle over the Klein's bottle

$(\quad M_1 \quad \cup \quad M_2 \quad)$

- $\bar{\rho}|_{\pi_1(M_1)} : \pi_1(M_1) \to \mathrm{SL}_2(\mathbb{C})$*: reducible (abelian),*
- $\bar{\rho}|_{\pi_1(M_2)} : \pi_1(M_2) \to \mathrm{SL}_2(\mathbb{C})$*: non–abelian.*

($\because$) M. Teragaito's presentation:
$\pi_1(M) = \langle a, b, x, y \mid a^2 = b^3, x^{-1}yx = y^{-1}, \mu = y^{-1}, h = y^{-1}x^2 \rangle$
and R-torsions.

R-torsion for a graph manifold

Result by surgery formula

$$\mathrm{Tor}(M;\bar\rho) = \mathrm{Tor}(E_K)\mathrm{Tor}(D^2\times S^1) = \frac{-2(\mathrm{tr}\,\rho(m)-1)}{2-\mathrm{tr}\,\rho(m)} = 1$$

$(\because)\quad \mathrm{tr}\,\rho(m) = 0.$

Result by JSJ-decomposition

$$\begin{aligned}\mathrm{Tor}(M;\bar\rho) &= \mathrm{Tor}(M_1;\bar\rho)\cdot\mathrm{Tor}(M_2;\bar\rho)\\ &= \frac{\Delta_{3_1}(\zeta)\Delta_{3_1}(\zeta^{-1})}{(\zeta-1)(\zeta^{-1}-1)}\cdot 1\end{aligned}$$

where $\Delta_{3_1}(t) = t - 1 + t^{-1}$.

$$\mathrm{Tor}(M;\bar\rho) = 1 \Rightarrow \zeta = \exp(\pi\sqrt{-1}/5) : \text{10-th root of unity.}$$

R-torsion for $2N$-dim representations

$M = M_1 \cup M_2$: JSJ-decomposition,

$\bar{\rho}|_{\pi_1(M_1)}$:abelian, $\bar{\rho}|_{\pi_1(M_2)}$:non–abelian

Result by JSJ-decomposition

In $\mathrm{Tor}(M;\bar{\rho}_{2N}) = \mathrm{Tor}(M_1;\bar{\rho}_{2N}) \cdot \mathrm{Tor}(M_2;\bar{\rho}_{2N})$,

$$\mathrm{Tor}(M_1;\bar{\rho}_{2N}) = \frac{\prod_{k=1}^{N} \Delta_{3_1}(\zeta^{2k-1})\Delta_{3_1}(\zeta^{-2k+1})}{\prod_{k=1}^{N}(\zeta^{2k-1}-1)(\zeta^{-2k+1}-1)}$$

$$\mathrm{Tor}(M_2;\bar{\rho}_{2N}) = 1$$

where

$\Delta_{3_1}(t) = t - 1 + t^{-1}$, $\zeta = \exp(\pi\sqrt{-1}/5)$: 10-th root of unity.

Remark

$$\Delta_{3_1}(\zeta^{\pm 1}) = \frac{-1+\sqrt{5}}{2},\ \Delta_{3_1}(\zeta^{\pm 3}) = \frac{-1-\sqrt{5}}{2},\ \Delta_{3_1}(\zeta^{\pm 5}) = -3$$

The asymptotic behavior for a graph manifold

Theorem (the limit of leading coefficient)

$$\lim_{N\to\infty} \frac{\log|\mathrm{Tor}(M;\bar{\rho}_{2N})|}{2N}$$

$$= \frac{2}{5}\log\left|\left(\frac{-1+\sqrt{5}}{2}\right)^2\left(\frac{-1-\sqrt{5}}{2}\right)^2(-3)\right| - \frac{1}{5}\log 2$$

$$= \frac{2}{5}\log 3 - \frac{1}{5}\log 2$$

Remark

- M: a Seifert fibered space
  $\Rightarrow$ the limit of leading coeff. $= (\dots)\log 2$
- $\dfrac{\pm 1+\sqrt{5}}{2}$: the square roots of $\dfrac{3\pm\sqrt{5}}{2}$
  which is the root of $\Delta_{4_1}(t) = t^2 - 3t + 1$.