Reidemeister torsion and exceptional surgeries along the figure eight knot

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Reidemeister torsion

Definition (R-torsion $\text{Tor}(W; \rho)$)

\[ W \]: a finite CW-complex,
\[ \rho: \pi_1(W) \to \text{SL}_n(\mathbb{C}) \]: $\text{SL}_n(\mathbb{C})$-representation of $\pi_1$
\[ C_*(W; \mathbb{C}_\rho^n) \]: local system given by $\rho$

\[ C_*(W; \mathbb{C}_\rho^n) = \mathbb{C}_n \otimes_{\rho} C_*(\widetilde{W}; \mathbb{Z}[\pi_1]) \] (\( \widetilde{W} \): universal cover)

Under $H_*(W; \mathbb{C}_\rho^n) = 0$,

\[ \text{Tor}(W; \rho) := \prod_{i \geq 0} \det(\partial b_{i+1} \cup b_i/c_i)^{-1}(i+1) \]

via the decomposition

\[ C_i(W; \mathbb{C}_\rho^n) = \text{Ker} \partial_i \oplus (\text{a lift of Im} \partial_i) = \text{Im} \partial_{i+1} \oplus (\text{a lift of Im} \partial_i). \]
Example of Reidemeister torsion

Local system for $S^1$

$\rho : \pi_1 S^1 = \langle \gamma \rangle \to \text{SL}_2(\mathbb{C})$, 
\[ \gamma \mapsto A \]

$\mathbb{C}^2 \simeq C_1(S^1; \mathbb{C}_\rho^2) \xrightarrow{\partial_1} C_0(S^1; \mathbb{C}_\rho^2) \simeq \mathbb{C}^2$

$v \otimes \tilde{e}^1 \mapsto v \otimes \gamma \tilde{e}^0 - v \otimes \tilde{e}^0 = (A^{-1} - I)v \otimes \tilde{e}^0$

$C_0$ has the new basis $(A^{-1} - I) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \tilde{e}^0, (A^{-1} - I) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \tilde{e}^0$.

$\mathbb{R}$-torsion for $(S^1, \rho)$

$\text{Tor}(S^1; \rho) = \frac{1}{\det(A^{-1} - I)} \left( = (\det \partial_1)^{-1} \right)$
Sequence of R-torsion

We can make a sequence by the following procedure:

1. Choose an $\text{SL}_2(\mathbb{C})$-rep. $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$.

2. Take composition with $\sigma_n : \text{SL}_2(\mathbb{C}) \to \text{SL}_n(\mathbb{C})$, 
   
   $\rho = \rho_2, \rho_3, \ldots, \rho_n = \sigma_n \circ \rho, \ldots$ (a seq. of reps.)

3. Consider $\text{Tor}(W; \rho_n)$ if $H_*(W; \mathbb{C}_{\rho_n}) = 0$ (\forall \rho_n), i.e., 
   
   $\text{Tor}(M; \rho_2), \text{Tor}(M; \rho_3), \ldots, \text{Tor}(M; \rho_n), \ldots$ (a seq. of invs.)

Remark

The behavior of $\{|\text{Tor}(W; \rho_n)| \mid n = 1, 2, \ldots\}$ ($n \to \infty$) 

is related to the geometric feature of $W$. 
Example of $\text{Tor}(W; \rho_n)$

$\sigma_n(A)$: the action of $A$ on $\{p(x, y) \mid \text{homog.}, \deg p = n - 1\}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x, y) = p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\right) = p(dx - by, -cx + ay)$$

For example, if $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, then

$$\sigma_3(A) = \begin{pmatrix} a^{-2} & 1 \\ a^2 & \end{pmatrix}, \quad \sigma_4(A) = \begin{pmatrix} a^{-3} & & \\ & a^{-1} & \\ & & a \end{pmatrix}. \quad \text{Tor}(S^1; \rho_{2N}) = \left\{ \det(\sigma_{2N}(A) - I) \right\}^{-1}$$

$$= \left\{ \prod_{k=1}^{N} (a^{2k-1} - 1)(a^{-2k+1} - 1) \right\}^{-1}$$
Previous work I for the asymptotics of R-torsion

The asymptotics for Hyperbolic manifolds
by W. Müller, P. Menal–Ferrer & J. Porti

$M$: a hyperbolic 3-manifold of finite volume

$\rho : \pi_1(M) \to SL_2(\mathbb{C})$ (holonomy rep.)

$\Rightarrow \rho_{2N} = \sigma_{2N} \circ \rho$ satisfies $H_\ast(M; \rho_{2N}) = 0$ ($\forall N$)

Moreover

$$\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{(2N)^2} = \frac{\text{Vol}(M)}{-4\pi} \left( = \frac{v_3 \|M\|}{-4\pi} \right)$$

$\text{Vol}(M)$: hyperbolic vol. of $M$, $\|M\|$: simplicial vol. of $M$
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Moreover

\[\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{(2N)^2} = \frac{\text{Vol}(M)}{-4\pi} \left(= \frac{v_3 \|M\|}{-4\pi}\right)\]

\(\text{Vol}(M)\): hyperbolic vol. of \(M\), \(\|M\|\): simplicial vol. of \(M\)
Previous work II for the asymptotics of R-torsion

Asymptotic behavior for a Seifert fibered space (Y)

\( M \): a Seifert fibered space with \( m \) exceptional fibers

\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{(2N)^2} = 0
\]

\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{2N} = \log |\text{Tor}(\text{regular fiber}; \rho)|^{-\chi'}
\]

\( \rho_{2N} = \sigma_{2N} \circ \rho : \pi_1(M) \xrightarrow{\rho} \text{SL}_2(\mathbb{C}) \xrightarrow{\sigma_{2N}} \text{SL}_{2N}(\mathbb{C}) \)

s.t. \( \text{regular fiber} \leftrightarrow -I \leftrightarrow -I_{2N} \),

\( g \): the genus of the base orbifold,

\( 2\lambda_j \): the order of the \( 2 \times 2 \)-matrix corresponding to \( j \)-th exceptional fiber
Previous work II for the asymptotics of R-torsion

Asymptotic behavior for a Seifert fibered space $(Y)$

$M$: a Seifert fibered space with $m$ exceptional fibers

\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{(2N)^2} = 0
\]

\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{2N} = - \left( 2 - 2g - \sum_{j=1}^{m} \frac{\lambda_j - 1}{\lambda_j} \right) \log 2
\]

\[
\rho_{2N} = \sigma_{2N} \circ \rho : \pi_1(M) \xrightarrow{\rho} \text{SL}_2(\mathbb{C}) \xrightarrow{\sigma_{2N}} \text{SL}_{2N}(\mathbb{C})
\]

s.t. regular fiber $\mapsto -I \mapsto -l_{2N}$,

$g$: the genus of the base orbifold,

$2\lambda_j$: the order of the $2 \times 2$-matrix corresponding to $j$-th exceptional fiber
Motivations of this research

1. $M = M_1 \cup T_2 \cdots \cup T_2 M_k$: a graph manifold

   What is the limit of the following?

   $$\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{2N}$$

2. What happens about R-torsion when a hyperbolic structure $\rho : \pi_1(E_K) \to \text{SL}_2(\mathbb{C})$ moves to a degenerate one?
Exceptional surgeries along $4_1$ (Figure eight knot)

Fact (Exceptional surgeries along $4_1$)

\[
\begin{array}{cccccc}
1/0 & 0 & \pm 1 & \pm 2 & \pm 3 \\
S^3 & T^2-b’dle & S^2(2,3,7) & S^2(2,4,5) & S^2(3,3,4) \\
\end{array}
\]

$\pm 4: \text{Graph manifold } M$

\[M = \text{Exterior of } 3_1 \cup \text{twisted } I-b’dle \text{ over the Klein’s bottle}
\]

\[
\begin{array}{c}
M_1 \\
\cup_{T^2}
\end{array}
\begin{array}{c}
M_2
\end{array}
\]

$\pm 4$-slope = the boundary of punctured Klein’ bottle by checkerboard coloring
Main result

Theorem (the limit of leading coefficient)

- $M = \text{Exterior of } 3_1 \cup \text{twisted I–b’dle over the Klein’s bottle}$

- $\bar{\rho} : \text{SL}_2(\mathbb{C})\text{-representation induced from } \rho : \pi_1(E_K) \to \text{SL}_2(\mathbb{C}),$

Then

$$\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \bar{\rho}2N)|}{2N} = \frac{2}{5} \log 3 - \frac{1}{5} \log 2$$
Our approach

Fact

- In the case of $K = 4_1$, the set of $\rho: \pi_1(E_K) \to \text{SL}_2(\mathbb{C})$ is well–known.

- $\text{Tor}(M) = \text{Tor}(E_K) \cdot \text{Tor}(D^2 \times S^1) = \text{Tor}(M_1) \cdot \text{Tor}(M_2)$

  This is useful to observe the local systems for $M_1$ and $M_2$.

- $\text{Tor}(M_1)$ and $\text{Tor}(M_2)$ are also well–known.

We will see the behavior of $\text{Tor}(M_1)$ and $\text{Tor}(M_2)$ by

\[ \rho: \pi_1(E_K) \to \text{SL}_2(\mathbb{C}), \text{Tor}(E_K) \text{ and } \text{Tor}(D^2 \times S^1) \]
Induced representation $\bar{\rho}: \pi_1(M) \to \text{SL}_2(\mathbb{C})$

$M$: resulting manifold by $\pm 4$-surgery along $K = 4_1$

$$\pi_1(E_K) \xrightarrow{\rho} \text{SL}_2(\mathbb{C})$$

$$\pi_1(M) = \pi_1(E_K) / \langle \langle m^{\pm 4} \ell \rangle \rangle$$

Therefore

$$\rho(m)^{\pm 4} \rho(\ell) = I \iff \bar{\rho} \text{ is induced}$$

Problem

Which $\rho: \pi_1(E_K) \to \text{SL}_2(\mathbb{C})$ induces $\bar{\rho}: \pi_1(M) \to \text{SL}_2(\mathbb{C})$?
Equivalence condition for reps. related to $\pm 4$-surgery

**Necessary condition for** $\rho(m)^{\pm 4} \rho(\ell) = I$

For $K = 4_1$,

$$\rho(m)^{\pm 4} \rho(\ell) = I \Rightarrow \text{tr} \rho(m)^4 = \text{tr} \rho(\ell) \iff \text{tr} \rho(m) = 0.$$ 

($\therefore$) $\text{tr} \rho(\ell) = (\text{tr} \rho(m))^4 - 5(\text{tr} \rho(m))^2 + 2.$

**Sufficient condition for** $\rho(m)^{\pm 4} \rho(\ell) = I$

For $K = \text{any two–bridge knot}$,

$$\text{tr} \rho(m) = 0 \iff \rho(m) \cong \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(\ell) = I \Rightarrow \rho(m)^{\pm 4} \rho(\ell) = I.$$
Representation for our graph manifold

\[ M = E_K \cup_{\pm 4} D^2 \times S^1 \]

\[ \rho \text{ induces } \bar{\rho} \iff \text{tr } \rho(m) = 0, \]

Proposition

\[ \pi_1(E_K) \xrightarrow{\rho} \text{SL}_2(\mathbb{C}) \]

\[ \pi_1(E_K) / \langle \langle m^{\pm 4} \ell \rangle \rangle \]

\[ M = \text{Exterior of } 3_1 \cup \text{twisted l–b’dle over the Klein’s bottle} \]

(\ M_1 \cup M_2 \ )

\[ \bar{\rho}|_{\pi_1(M_1)} : \pi_1(M_1) \to \text{SL}_2(\mathbb{C}) : \text{reducible (abelian)} , \]

\[ \bar{\rho}|_{\pi_1(M_2)} : \pi_1(M_2) \to \text{SL}_2(\mathbb{C}) : \text{non–abelian}. \]

(\ :: \ ) M. Teragaito’s presentation:

\[ \pi_1(M) = \langle a, b, x, y \mid a^2 = b^3, x^{-1}yx = y^{-1}, \mu = y^{-1}, h = y^{-1}x^2 \rangle \]

and R-torsions.
R-torsion for a graph manifold

Result by surgery formula

\[
\text{Tor}(M; \tilde{\rho}) = \text{Tor}(E_K) \text{Tor}(D^2 \times S^1) = \frac{-2(\text{tr} \rho(m) - 1)}{2 - \text{tr} \rho(m)} = 1
\]

\((\because)\) \quad \text{tr} \rho(m) = 0.

Result by JSJ-decomposition

\[
\text{Tor}(M; \tilde{\rho}) = \text{Tor}(M_1; \tilde{\rho}) \cdot \text{Tor}(M_2; \tilde{\rho}) = \frac{\Delta_3(\zeta) \Delta_3(\zeta^{-1})}{(\zeta - 1)(\zeta^{-1} - 1)} \cdot 1
\]

where \(\Delta_3(t) = t - 1 + t^{-1}\).

\[
\text{Tor}(M; \tilde{\rho}) = 1 \Rightarrow \zeta = \exp(\pi \sqrt{-1}/5) : 10\text{-th root of unity.}
\]
R-torsion for 2N-dim representations

\[ M = M_1 \cup M_2 : \text{JSJ-decomposition,} \]
\[ \bar{\rho} \mid_{\pi_1(M_1)} : \text{abelian,} \quad \bar{\rho} \mid_{\pi_1(M_2)} : \text{non-abelian} \]

Result by JSJ-decomposition

\[ \text{In } \text{Tor}(M; \bar{\rho}_{2N}) = \text{Tor}(M_1; \bar{\rho}_{2N}) \cdot \text{Tor}(M_2; \bar{\rho}_{2N}), \]
\[ \text{Tor}(M_1; \bar{\rho}_{2N}) = \frac{\prod_{k=1}^{N} \Delta_{3_1}(\zeta^{2k-1}) \Delta_{3_1}(\zeta^{-2k+1})}{\prod_{k=1}^{N} (\zeta^{2k-1} - 1)(\zeta^{-2k+1} - 1)} \]
\[ \text{Tor}(M_2; \bar{\rho}_{2N}) = 1 \]

where
\[ \Delta_{3_1}(t) = t - 1 + t^{-1}, \quad \zeta = \exp(\pi \sqrt{-1}/5): 10\text{-th root of unity.} \]

Remark

\[ \Delta_{3_1}(\zeta^{\pm 1}) = \frac{-1 + \sqrt{5}}{2}, \quad \Delta_{3_1}(\zeta^{\pm 3}) = \frac{-1 - \sqrt{5}}{2}, \quad \Delta_{3_1}(\zeta^{\pm 5}) = -3 \]
The asymptotic behavior for a graph manifold

**Theorem (the limit of leading coefficient)**

\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \tilde{\rho}_{2N})|}{2N} = \frac{2}{5} \log \left| \left( \frac{-1 + \sqrt{5}}{2} \right)^2 \left( \frac{-1 - \sqrt{5}}{2} \right)^2 (-3) \right| - \frac{1}{5} \log 2
\]

\[
= \frac{2}{5} \log 3 - \frac{1}{5} \log 2
\]

**Remark**

- \(M\): a Seifert fibered space
  \[\Rightarrow\] the limit of leading coeff. \(= (\ldots) \log 2\)

- \[\pm \frac{1 + \sqrt{5}}{2}\] : the square roots of \[\frac{3 \pm \sqrt{5}}{2}\]
  which is the root of \(\Delta_{41}(t) = t^2 - 3t + 1\).