Corks, exotic 4-manifolds and knot concordance

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I. Background and Main results
   Exotic 4-manifolds represented by framed knots
   Application to knot concordance

II. Brief review of corks

III. Proof of the main results
1.A. Exotic framed knots

**Problem**
Does every smooth 4-manifold admit an exotic (i.e. homeo but non-diffeo) smooth structure?

We consider a special class of 4-manifolds:

A framed knot (i.e. knot + integer) in $S^3$ gives a 4-mfd by attaching $2\text{-handle } D^2 \times D^2$ to $D^4$ along the framed knot.

A pair of framed knots in $S^3$ is said to be *exotic* if they represent homeo but non-diffeo 4-mfds.

**Problem**
Find exotic pairs of framed knots!

**Remark.** $\exists$ framed knot admitting NO exotic framed knot
1.A. Exotic framed knots

**Problem**
Find exotic pairs of framed knots!

**Theorem** (Akbulut ’91)
∃ an exotic pair of $-1$-framed knots.

**Theorem** (Kalmár-Stipsicz ’13)
∃ an infinite family of exotic pairs of $-1$-framed knots.

**Remark.** Framings of these examples are all $-1$. For each pair, one 4-mfd is Stein, but the other is non-Stein.
1.A. Exotic framed knots

**Theorem (Y)**

\[ \forall n \in \mathbb{Z}, \exists \text{ infinitely many exotic pairs of } n\text{-framed knots.} \]

Furthermore, both knots in each pair gives Stein 4-mfds.

Moreover, we give machines which produce vast examples.

Recall:

A knot \( P \) in \( S^1 \times D^2 \) induces a **satellite map**

\[ P : \{ \text{knot in } S^3 \} \to \{ \text{knot in } S^3 \} \]

by identifying reg. nbd of a knot with \( S^1 \times D^2 \) via 0-framing.
1.A. Exotic framed knots

Machines producing vast examples:

**Main Theorem** $\text{(Y)}$

$\forall n \in \mathbb{Z}, \exists$ satellite maps $P_n, Q_n$ s.t.

for any knot $K$ in $S^3$ with

$2g_4(K) - 2 = \overline{ad}(K)$ and $n \leq \hat{tb}(K),$

$n$-framed $P_n(K)$ and $Q_n(K)$ are an exotic pair.

**Remark.**

For each $n$, there are many $K$ satisfying the assumption.

If $K$ satisfies the assumption, then $P_n(K)$ and $Q_n(K)$ satisfy.
1.A. Different viewpoint: exotic satellite maps

For a satellite map $P : \{\text{knot}\} \rightarrow \{\text{knot}\}$ and $n \in \mathbb{Z}$, we define a **4-dimensional $n$-framed satellite map**

$$P^{(n)} : \{\text{knot in } S^3\} \rightarrow \{\text{smooth 4-mfd}\}$$

by $P^{(n)}(K) =$ 4-manifold represented by $n$-framed $P(K)$.

$P^{(n)}$ and $Q^{(n)}$ are called smoothly the same, if $P^{(n)}(K)$ and $Q^{(n)}(K)$ are diffeo for any knot $K$.

New difference between smooth and topological categories:

**Theorem $(Y)$**

$$\forall n \in \mathbb{Z}, \exists \text{ 4-dim } n\text{-framed satellite maps}$$

which are topologically the same but smoothly distinct.
1.A. Different viewpoint: exotic satellite maps

For a satellite map $P : \{ \text{knot} \} \rightarrow \{ \text{knot} \}$ and $n \in \mathbb{Z}$, we define a 4-dimensional $n$-framed satellite map $P^{(n)} : \{ \text{knot in } S^3 \} \rightarrow \{ \text{smooth 4-mfd} \}$ by $P^{(n)}(K) =$ 4-manifold represented by $n$-framed $P(K)$.

$P^{(n)}$ and $Q^{(n)}$ are called topologically the same, if $P^{(n)}(K)$ and $Q^{(n)}(K)$ are homeo for any knot $K$.

New difference between smooth and topological categories:

**Theorem (Y)**

$\forall n \in \mathbb{Z}, \exists 4$-dim $n$-framed satellite maps which are topologically the same but smoothly distinct.
1.B. Application to knot concordance

\( n \)-surgery on a knot \( K \) in \( S^3 := \text{boundary of the 4-mfd} \)

represented by \( n \)-framed \( K \).

Two oriented knots \( K_0, K_1 \) are \textbf{concordant} if

\[
\exists S^1 \times I \hookrightarrow S^3 \times I \quad \text{s.t.} \quad S^1 \times i = K_i \times i \quad (i = 0, 1).
\]

\textbf{Conjecture} (Akbulut-Kirby 1978)

If 0-surgeries on two knots in \( S^3 \) give the same 3-mfd, then the knots (with relevant ori) are concordant.

\textbf{Remark}. Quotation from Kirby’s problem list (’97):

all known concordance invariants of the two knots are the same.
1.B. Application to knot concordance

**Conjecture** (Akbulut-Kirby 1978)

If 0-surgeries on two knots in $S^3$ give the same 3-mfd, then the knots (with relevant ori) are concordant.

**Theorem** (Cochran-Franklin-Hedden-Horn 2013)

∃ infinitely many pairs of non-concordant knots with homology cobordant 0-surgeries.

**Theorem** (Abe-Tagami)

If the slice-ribbon conjecture is true, then the Akbulut-Kirby conjecture is false.
1.B. Application to knot concordance

**Conjecture** (Akbulut-Kirby 1978)

If 0-surgeries on two knots in $S^3$ give the same 3-mfd, then the knots (with relevant ori) are concordant.

**Theorem** (Y)

$\exists$ infinitely many counterexamples to AK conjecture.

In fact, our exotic 0-framed knots are counterexamples.

**Corollary** (Y)

Knot concordance invariants $g_4, \tau, s$ are NOT invariants of 3-manifolds given by 0-surgeries on knots.
1.B. Application to knot concordance

**Conjecture (Akbulut-Kirby 1978)**

If 0-surgeries on two knots in $S^3$ give the same 3-mfd, then the knots (with relevant ori) are concordant.

**Simple counterexample**

\[ P_0(T_{2,3}) \quad \text{and} \quad Q_0(T_{2,3}) \]
1.B. Application to knot concordance

**Conjecture** (Akbulut-Kirby 1978)

If 0-surgeries on two knots in $S^3$ give the same 3-mfd, then the knots (with relevant ori) are concordant.

**Question.**
If two 0-framed knots in $S^3$ give the same smooth 4-mfd, are the knots (with relevant ori) concordant?

**Remark**
Abe-Tagami’s proof shows the answer is no, if the slice-ribbon conjecture is true.
2. Brief review of corks

$C$: cpt contractible 4-mfd, $\tau: \partial C \to \partial C$: involution,

**Definition**

$(C, \tau)$ is a **cork** $\iff \tau$ extends to a self-homeo of $C$, but cannot extend to any self-diffeo of $C$.

Suppose $C \subset X^4$.

The following operation is called a **cork twist** of $X$:

$$X \rightsquigarrow (X - C) \cup_{\tau} C.$$
2. Brief review of corks

**Theorem** (Curtis-Freedman-Hsiang-Stong ’96, Matveyev ’96)

Let $X, Y :$ simp. conn. closed ori. smooth 4-mfds

If $Y$ is an exotic copy of $X$,
then $Y$ is obtained from $X$ by a cork twist.

Smooth structures are determined by corks !!

**Remark**

Cork twists do NOT always produce exotic smooth structures.
2. Brief review of corks: examples

**Definition** \( L = K_0 \sqcup K_1 \) is a symmetric Mazur link if

- \( K_0 \) and \( K_1 \) are unknot, \( lk(K_0, K_1) = 1 \).
- \( \exists \) involution of \( S^3 \) which exchanges \( K_0 \) and \( K_1 \).

A symmetric Mazur link \( L \) gives a contractible 4-mfd \( C_L \) and an involution \( \tau_L : \partial C_L \to \partial C_L \).

![Diagram](image)
2. Brief review of corks: examples

**Definition** $L = K_0 \sqcup K_1$ is a symmetric Mazur link if

- $K_0$ and $K_1$ are unknot, $lk(K_0, K_1) = 1$.
- $\exists$ involution of $S^3$ which exchanges $K_0$ and $K_1$.

**Theorem** (Akbulut ’91) There exists a cork.
2. Brief review of corks: examples

**Theorem** (Akbulut-Matveyev ’97, cf. Akbulut-Karakurt ’12)
For a symmetric Mazur link $L$, $(C_L, \tau_L)$ is a cork if $C_L$ becomes a Stein handlebody in a ‘natural way’.

**Theorem** (Akbulut ’91, Akbulut-Y ’08).
$(W_n, f_n)$ is a cork for $n \geq 1$.

\[ W_n := n \rightarrow n+1 \]

**Theorem(Y)**
For a symmetric Mazur link $L$, $(C_L, \tau_L)$ is NOT a cork if $L$ becomes a trivial link by one crossing change.
2. Brief review of corks: examples

**Theorem** (Akbulut-Matveyev ’97, cf. Akbulut-Karakurt ’12)
For a symmetric Mazur link $L$, $(C_L, \tau_L)$ is a cork if $C'_L$ becomes a Stein handlebody in a ‘natural way’.

**Theorem (Y)**
For a symmetric Mazur link $L$, $(C_L, \tau_L)$ is NOT a cork if $L$ becomes a trivial link by one crossing change.

![Cork vs Non-Cork Diagrams](image-url)
2. Brief review of corks: applications

**Theorem** (Akbulut ’91, Akbulut-Matveyev 97’)

\[ \exists \text{ exotic pair of simp. conn. 4-manifold with } b_2 = 1. \]

\[ X_1 \quad \quad X_2 \]

Stein

minimal

non-minimal

cork twist

exotic

non-Stein
2. Brief review of corks: applications

2-handlebody := handlebody consisting of 0-, 1-, 2-handles.

**Thm** (Akbulut-Y ’13)

∀X: 4-dim cpt ori 2-handlebody with $b_2(X) \neq 0$, $\forall n \in \mathbb{N}$,

∃$X_1, X_2, \ldots, X_n$: 4-mfds admitting Stein str. s.t.

- $X_1, X_2, \ldots, X_n$ are pairwise exotic.
- $H_*(X_i) \cong H_*(X)$, $\pi_1(X_i) \cong \pi_1(X)$, $Q_{X_i} \cong Q_X$,
  \[ H_*(\partial X_i) \cong H_*(\partial X). \]
- Each $X_i$ can be embedded into $X$.

**Cor** (Akbulut-Y ’13)

For a large class of 4-manifolds with $\partial$, their topological invariants are realized as those of arbitrarily many pairwise exotic 4-mfds.
2. Brief review of corks: applications

**Thm** (Akbulut-Y '13)

$Z$, $Y$: cpt conn. ori. 4-mfds, $Y \subset Z$.

$Z - \text{int } Y$ is a 2-handlebody with $b_2 \neq 0$.

Then $\forall n \in \mathbb{N}$, $\exists Y_1, Y_2, \ldots, Y_n \subset Z$: cpt 4-mfds s.t.

- $Y_i$ is diffeo to $Y_j$ ($\forall i \neq j$).
- $(Z, Y_i)$ is homeo but non-diffeo to $(Z, Y_j)$ ($i \neq j$).
- $H_*(Y_i) \cong H_*(Y)$, $\pi_1(Y_i) \cong \pi_1(Y)$, $Q_{Y_i} \cong Q_Y$, $H_*(\partial Y_i) \cong H_*(\partial Y_i)$.

**Cor** (Akbulut-Y '13) Every cpt. ori. 4-manifold $Z$ admits arbitrarily many pairwise exotic embedding of a 4-mfd into $Z$. 
3. Proof: new presentations of cork twists

**Lemma (Y).** \((V_m, g_m)\) is a cork for \(m \geq 0\).

\[
\begin{array}{c}
\text{Remark. } (V_{-1}, g_{-1}) \text{ is NOT a cork.}
\end{array}
\]

**Definition**

\[
\begin{array}{c}
\text{Definition}
\end{array}
\]

\[
\begin{array}{c}
\text{Definition}
\end{array}
\]
**Theorem (Y) [hook surgery]**

There exists a diffeomorphism $g_m^* : \partial V_m \to \partial V_m^*$ s.t.

- $g_m^*$ sends the knot $\gamma_K$ to $\gamma_K^*$ for any knot $K$ in $S^3$.
- $g_m^* \circ g_m^{-1} : \partial V_m \to \partial V_m^*$ extends to a diffeo $V_m \to V_m^*$. 

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![Diagram](image)
Theorem (Y) [hook surgery]

There exists a diffeomorphism \( g_m^* : \partial V_m \to \partial V_m^* \) s.t.

- \( g_m^* \) sends the knot \( \gamma_K \) to \( \gamma_K^* \) for any knot \( K \) in \( S^3 \).
- \( g_m^* \circ g_m^{-1} : \partial V_m \to \partial V_m^* \) extends to a diffeo \( V_m \to V_m^* \).

Corollary \( X : 4\text{-}mfd, V_m \subset X \).

The cork twist \( (X - V_m) \cup_{g_m} V_m \) is diffeomorphic to the hook surgery \( (X - V_m) \cup_{g_m^*} V_m^* \).
3. Proof: satellite maps

Machines producing vast examples:

**Main Theorem (Y)**

\[ \forall n \in \mathbb{Z}, \exists \text{ satellite maps } P_n, Q_n \text{ s.t.} \]

for any knot \( K \) in \( S^3 \) with

\[ 2g_4(K) - 2 = \overline{ad}(K) \text{ and } n \leq \hat{tb}(K), \]

\( n \)-framed \( P_n(K) \) and \( Q_n(K) \) are an exotic pair.
3. Proof: satellite maps

\( P_{m,n}, Q_{m,n} : \) (pattern) knots in \( S^1 \times D^2 \)

The case \( m = 0 \):

\( P_{0,n}, Q_{0,n} : \)
3. Proof: satellite maps

\( P_{m,n}, \ Q_{m,n} \) : (pattern) knots in \( S^1 \times D^2 \)

**Remark.**

- \( Q_{m,n}(K) \) is concordant to \( K \).
- \( g_4(Q_{m,n}(K)) = g_4(K), \quad g_4(P_{m,n}(K)) \leq g_4(K) + 1 \).

**Definition.**

\( P_{m,n}^{(n)}(K) := 4\text{-manifold represented by } n\text{-framed } P_{m,n}(K) \).

\( Q_{m,n}^{(n)}(K) := 4\text{-manifold represented by } n\text{-framed } Q_{m,n}(K) \).
Lemma.

\[
\binom{P_{m,n}(K)}{Q_{m,n}(K)} \cong \binom{K}{K}
\]

Therefore \( P_{m,n}^{(n)}(K) \) is homeo to \( Q_{m,n}^{(n)}(K) \)
\( \mathcal{L}(K) := \{ \text{Legendrian knot isotopic to } K \} \)
\( \overline{ad}(K) := \max \{ \text{ad}(\mathcal{K}) := \text{tb}(\mathcal{K}) - 1 + |r(\mathcal{K})| \mid \mathcal{K} \in \mathcal{L}(K) \} \)
\( \hat{\text{tb}}(K) := \max \{ \text{tb}(\mathcal{K}) \mid \mathcal{K} \in \mathcal{L}(K), \text{ad}(\mathcal{K}) = \overline{ad}(K) \} \)
\( g_s^{(n)}(K) := \min \{ g(\Sigma) \mid [\Sigma] \text{ is a generator of } H_2(K^{(n)}) \} \)

**Fact** (adjunction inequality).
For \( n < \hat{\text{tb}}(K) \), \( \overline{ad}(K) \leq 2g_s^{(n)}(K) - 2 \).

**Main Theorem** (Y)
Fix \( m \geq 0 \). Assume a knot \( K \) and \( n \in \mathbb{Z} \) satisfies
\[ 2g_4(K) - 2 = \overline{ad}(K) \text{ and } n \leq \hat{\text{tb}}(K). \]
Then \( P_{m,n}^{(n)}(K) \) and \( Q_{m,n}^{(n)}(K) \) are homeo but not diffeo.
Main Theorem (Y)

Fix $m \geq 0$. Assume a knot $K$ and $n \in \mathbb{Z}$ satisfies

$$2g_4(K) - 2 = \overline{ad}(K) \text{ and } n \leq \hat{tb}(K).$$

Then $P_{m,n}^{(n)}(K)$ and $Q_{m,n}^{(n)}(K)$ are homeo but not diffeo.

By finding Legendrian realization of $P_{m,n}(K)$, we see

$$\overline{ad}(P_{m,n}(K)) \geq \overline{ad}(K) + 2, \quad \hat{tb}(P_{m,n}(K)) \geq n + 2.$$

$$\implies g_s^{(n)}(P_{m,n}(K)) = g_4(K) + 1$$

Since $g_s^{(n)}(Q_{m,n}(K)) \leq g_4(K), \quad P_{m,n}^{(n)}(K) \not\cong Q_{m,n}^{(n)}(K)$. 