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Smooth models of singular $K3$-surfaces

According to a recent paper by Shimada and Shioda, the famous *Fermat quartic*

$$X_{48}\colon \quad z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$$

has another smooth projective model $X_{56} \subset \mathbb{P}^3$, which is Cremona, but not projectively equivalent to X_{48}. We will start with showing that X_{48} and X_{56} are *all* smooth spatial models of X_{48}, a fact especially surprising considering that there are thousands of *singular* models. Generalizing, we will discover that there are relatively few smooth quartics of the form $X(T) \subset \mathbb{P}^3$, where $X(T)$ is a singular $K3$-surface and $\det T \leq 80$. This classification sheds new light on the problem of counting lines in smooth quartics: all models found have many lines, and the champion, Schur's quartic

$$X_{64}\colon \quad z_0(z_0^3 - z_1^3) = z_2(z_2^3 - z_3^3)$$

containing 64 lines, can alternatively be characterized as the singular $K3$-surface of the smallest discriminant (which equals 48) admitting a smooth embedding to $\mathbb{P}^3$.

In spite of the previous result, the number of smooth spatial models of a given singular $K3$-surface $X(T)$ may grow very fast as $\det T \to \infty$; a few examples will be considered. In particular, if $\det T = 163$ (a case important for counting lines defined over $\mathbb{Q}$), there are over 3200 distinct models.

We conclude by obtaining a similar classification for a few other commonly studied polarizations ($h^2 = 2$, 6, or 8) and stating conjectures concerning the number of lines:

- a smooth sextic curve in $\mathbb{P}^2$ has at most 72 tritangents;
- a smooth sextic surface in $\mathbb{P}^4$ has at most 42 lines;
- a smooth octic surface in $\mathbb{P}^5$ has at most 36 lines.

These numbers are those realized by the singular $K3$-surfaces of the smallest discriminant (108, 39, and 32, respectively) admitting corresponding models.

All results are obtained by embedding the definite part $h^\perp$ of the Néron–Severi lattice of X to an appropriate Niemeier lattice.