A quandle approach to Hoste's conjecture

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Conjecture (Hoste, 2002)

K: an alternating knot

$$\Delta_K(t) = 0 \quad \Rightarrow \quad \operatorname{Re} t > -1.$$

$t \in \mathbb{C}$: a root of $\Delta_K(T)$

- alternating property (Murasugi ('58), Crowell ('59))
  $\rightsquigarrow \quad t \in \mathbb{R} \Rightarrow t > 0.$

For the 2-bridge knots,

- Lyubich-Murasugi ('12): $-3 < \operatorname{Re} t < 6.$
- Stoimenow (('13)), Koseleff-Pecker ('15):

$$|t^{1/2} - t^{-1/2}| < 2. \rightsquigarrow \operatorname{Re} t > -\frac{3}{2}.$$

Why quandle?

Diagram

Topology

"alternating"

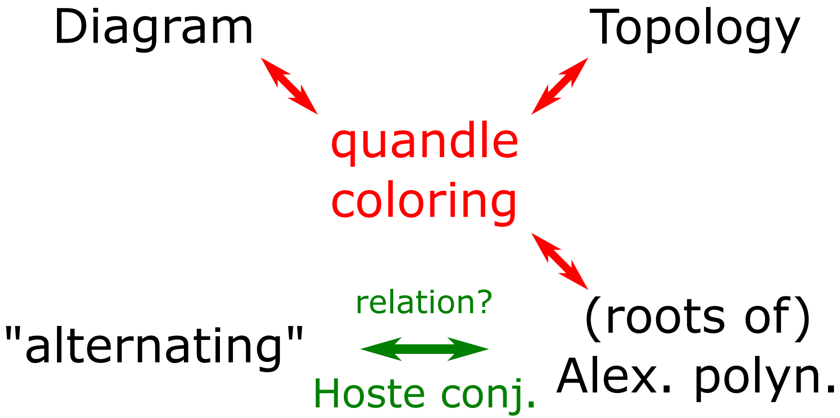
relation?

⟷

Hoste conj.

(roots of)
Alex. polyn.

Why quandle?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Results

Conjecture (Hoste, 2002)

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Thm A Hoste's conjecture holds for

$$\begin{cases} \text{the 2-bridge knots (links),} \\ \text{the alternating-pretzel knots (links).} \end{cases}$$

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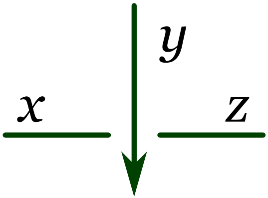
Thm B Hoste's conjecture does NOT hold for infinitely many alternating Montesinos knots.

· a $\mathbb{C}_t$-**coloring** is $\mathcal{C}$: {arcs} $\to \mathbb{C}$ s.t.



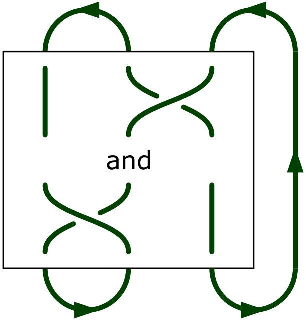
$$\begin{aligned} \mathcal{C}(z) &= t\mathcal{C}(x) + (1-t)\mathcal{C}(y). \\ (\Leftrightarrow \mathcal{C}(x) &= t^{-1}\mathcal{C}(z) + (1-t^{-1})\mathcal{C}(y).) \end{aligned}$$

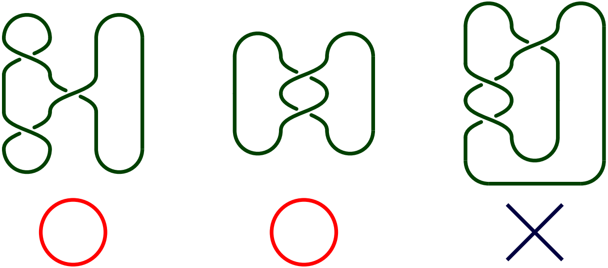
Rem $\mathbb{C}_t$ is a quandle.

- (Inoue, '01) $^\exists$non-triv. $\mathbb{C}_t$-col. $\Leftrightarrow \Delta_K(t) = 0$.

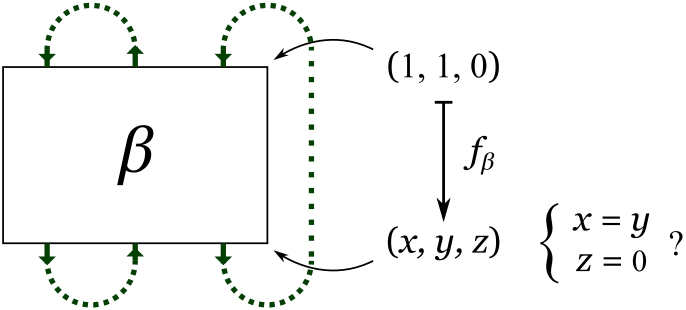
(a diagram of)
a **2-bridge knot/link**:

(cf. Conway's normal form)

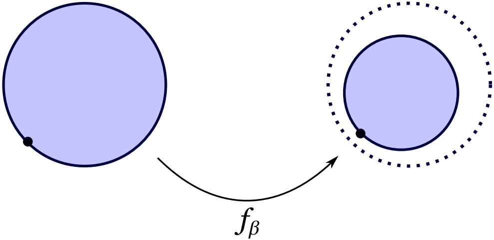




Idea of Proof

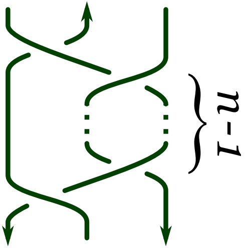


$(\mathrm{Re}\, t \leq -1)$

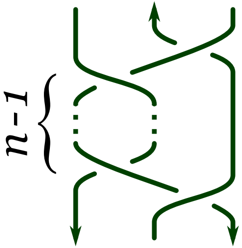


Lem β is decomposed into β_n's:

$$\beta = \beta_{n_1} \cdots \beta_{n_k}.$$

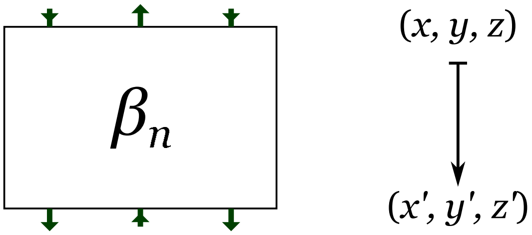


β_n



β_{-n}

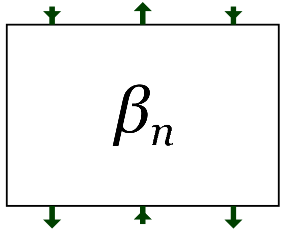
$(n = 1, 2, \cdots)$



$$\left(\text{e. g.} \quad x' = \frac{t(x-y+z) + x + t^{-1}(y-x) + (-t)^n(z-x)}{1+t}. \right)$$

- $x - y + z = x' - y' + z'$.

  $\rightsquigarrow x - y + z = 0 \quad \Rightarrow \quad x' - y' + z' = 0.$



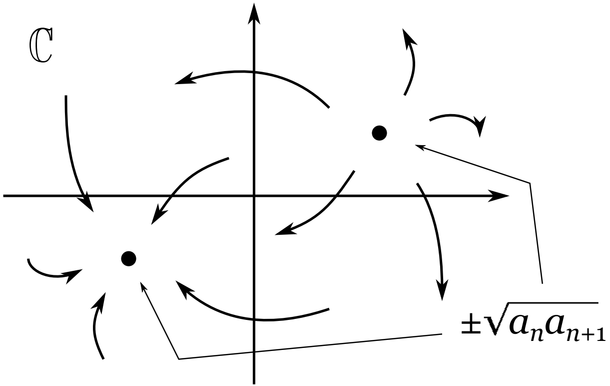
$$(x, y, z) \mapsto (x', y', z') \qquad \xi := z/x \overset{f_n}{\dashrightarrow} \xi' := z'/x'$$

<u>Lem</u> $\xi' = \dfrac{\xi - a_n}{-\xi/a_{n+1} + 1} \; (=: f_n(\xi))$,

where $\quad a_n := \dfrac{s^n - s}{s^n - 1}, \quad s := -t$.

Idea of proof II

$$f_n(\xi) = \frac{\xi - a_n}{-\xi/a_{n+1} + 1} \qquad a_n = \frac{s^n - s}{s^n - 1} \qquad (\mathrm{Re}\, s \geq 1)$$



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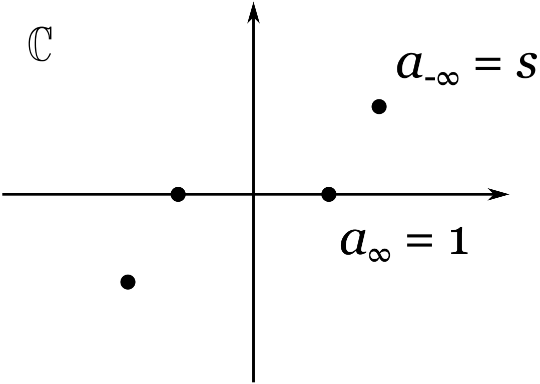

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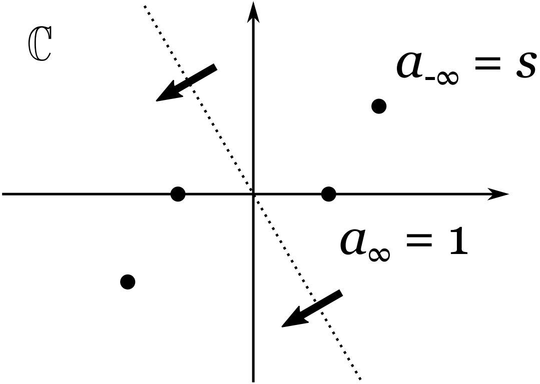
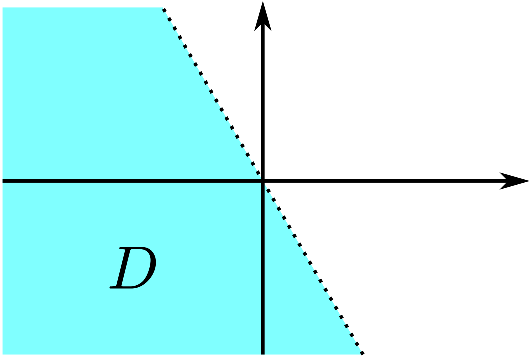


Assume $\operatorname{Re} s \geq 1,\ (\operatorname{Im} s \geq 0,)\ s \neq 1.$

<u>Lem</u> $^\exists D \subset \mathbb{C}$: an open half plane s.t.

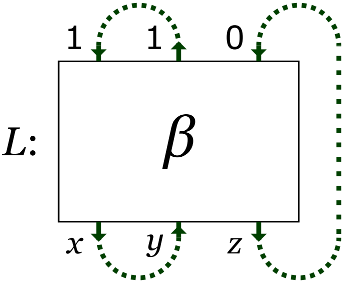
$$\begin{cases} f_n(\overline{D}) \subset D\ (n \in \mathbb{Z} \backslash \{0,1\}), \\ f_1(D) \subset D. \end{cases}$$



Summary of proof

$\beta = \beta_{n_1} \cdots \beta_{n_k}$

$\rightsquigarrow \; z/x = f_{n_k} \circ \cdots \circ f_{n_1}(0/1)$



Suppose $\mathrm{Re}\, t \leq -1$.

(i) $n_1 = \cdots = n_k = 1$

$\Rightarrow L$: trivial link.

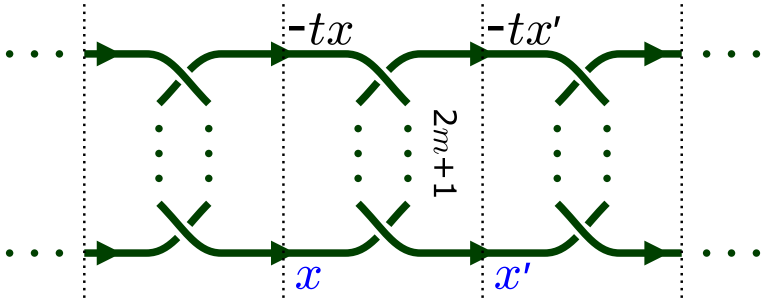
(ii) otherwise: $z/x \in D \quad \therefore z/x \neq 0$.

Theorem A1

L: a 2-bridge knot or a nontrivial 2-bridge link

$$\Delta_L(t) = 0 \quad \Rightarrow \quad \mathrm{Re}\, t > -1.$$

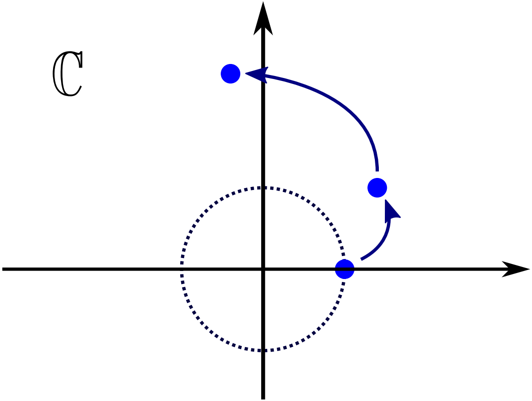
Case of pretzel links



<u>Lem</u> $x' = b_m x$, where $b_m = \dfrac{m(t-1)+t}{m(t-1)-1}$.

<u>Lem</u> $|b_m| > 1$.

Case of pretzel links

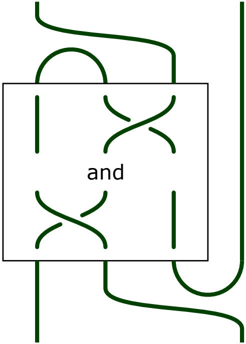


Theorem A2

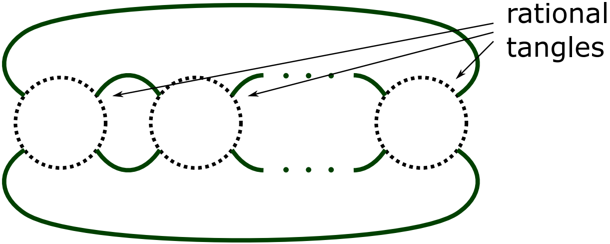
L: an alternating-pretzel link

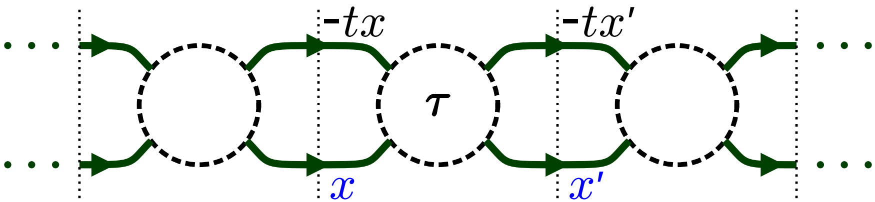
$$\Delta_L(t) = 0 \quad \Rightarrow \quad \mathrm{Re}\, t > -1.$$

(a diagram of)
a **rational tangle**:



a **Montesinos knot/link**:



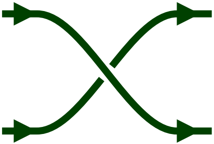


<u>Lem</u> $^{\exists}c_\tau \in \mathbb{C}$ s.t. $x' = c_\tau x$.

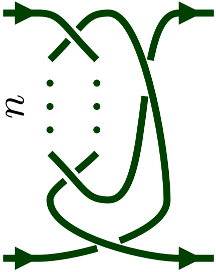
(e.g.) $^{\forall}\tau \ |c_\tau| > 1$?

Ex.

τ_0



τ_n

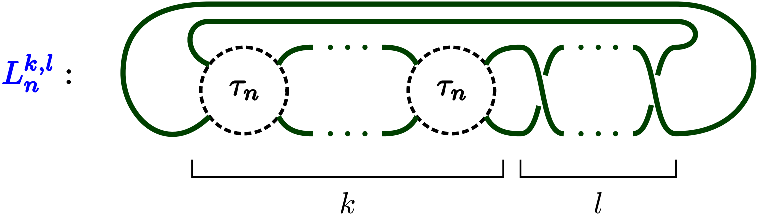


$$c_0 = s$$

$$c_n = \frac{s^{n+1} + s^n - 2}{2s^n - 1 - s^{-1}}$$

Lem $^\exists n, s\ (\mathrm{Re}\, s \geq 1)$ s.t. $|c_n| < 1$.

Fix n, s_0 ($\mathrm{Re}\, s_0 \geq 1$) s.t. $|c_n(s_0)| < 1$.



$$\log c_n(s_0) \in \log s_0 \mathbb{R} + 2\pi i \mathbb{R}$$

$$\overset{\text{perturb}}{\rightsquigarrow} \quad \log c_n(s) \in \log s \mathbb{Q} + 2\pi i \mathbb{Q}$$

$$\rightsquigarrow \quad k \log c_n(s) \in -l \log s + 2\pi i \mathbb{Z} \quad (k, l \in \mathbb{Z}_{>0})$$

$$\rightsquigarrow \quad (c_n(s))^k s^l = 1$$

i.e. $L_n^{k,l}$ is a counter example.

Theorem B

The closure of the set

$$\left\{ t \in \mathbb{C} \;\middle|\; \begin{array}{l} \mathrm{Re}\, t < -1, \\ {}^{\exists}\text{a Montesinos knot } K \text{ s.t. } \Delta_K(t) = 0 \end{array} \right\}$$

contains the nonempty open set

$$\bigcup_{n>0} \{ t \in \mathbb{C} \mid \mathrm{Re}\, t < -1,\ |c_n(-t)| < 1 \}.$$

Future problems

- Is there a (lower) bound of (the real parts of) the roots of the Alexander polynomials for the alternating knots? If there is, find it.
- Find a lower bound of the real parts of the roots for the alternating Montesinos knots.
- Characterize (the closure of the set of) the roots for the 2-bridge knots.
- Can we say something about the roots of other polynomial invariants (e.g. Jones polyn.)?