Sigma function of $y^3 = x^2(x - b_1)(x - b_2)$

Branched Coverings, Degenerations, and Related Topics 2018

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March 5, 2018

Menu

1. Purpose of study of sigma function
2. History of sigma function
3. Review of elliptic sigma function of $g = 1$
4. Generalization of sigma function to general affine curves
5. Sigma function of $y^3 = x(x - b_1)(x - b_2)(x - b_3)$ of $g = 3$,
6. Sigma function of $y^3 = x^2(x - b_1)(x - b_2)$ of $g = 2$,
7. An observation:
   Sigma function of $\lim_{s \to 0}\{y^3 = x(x - s)(x - b_2)(x - b_3)\}$

Purpose of study of sigma functions

Purpose of study of sigma functions:

In the theory of elliptic functions, the concrete and explicit descriptions of the elliptic functions have a power to have effects on various fields including physics, number theory, and industrial problems.
Following such properties of the elliptic functions, our purpose is to reconstruct the theory of Abelian functions for general curves such that it has concrete and explicit descriptions, and thus influences various fields.

History of sigma function

History of sigma function

1. 1832-44 **Jacobi:(Jacobi inversion problem)**: Posing "inverse functions of Abelian integrals for curves of genus two".
2. 1840 **Weierstrass(1815-1897)**:(publication in 1889): Study on **Al (σ) functions of elliptic curves**
3. 1854 **Weierstrass**: Study on **Al (σ) functions of hyperelliptic curves of general genus** and Jacobi inversion problem.
4. 1856 **Riemann**: Construction of Abelian functions for general compact Riemann surfaces.
5. 1882 **Weierstrass**: **Renaming his Al function σ function** of genus one by refining it.
6. 1886 年:**Klein**: Refining Weierstrass' Al function of hyperelliptic curves and calling it **hyperelliptic sigma function**.
7. 1903 **Baker** (Hodge's supervisor): Discovery of **KdV and KP hierarchy** using **bilinear form**. Posing the problem: to find **whether KdV hierarchy characterizes the sigma functions**.

However the Abelian function theory in XIX century came up against the brick wall which was the rigorousness of algebraic geometry.
The modern algebraic geometry in XX cenury developed to overcome the problems.
Then the theory of sigma function was buried in oblivion in XX century.!

History of sigma function

1. **Nonlinear integrable differential equations** had been studied 1950-1990.
2. Krichever, Novikov, Mumford and so on **rediscovered** the algebro-geometric solutions of the KdV and KP hierarchies 1970's.
3. **Hirota** rediscovered the bilinear operator and bilinear equations and developed the study of the bilinear equations.
4. Sato constructed **Sato-theory of UGM** around 1980.
5. Novikov gave a conjecture that the KP hierarchy characterizes the Jacobi varieties in the Abelian varieties, which is closely related to Baker's problem.
6. **Mulase and Shiota proved Novikov's conjecture 1987**.
   It means the settlement of Baker's problem; in the settlement, Baker's theory on the differential equations plays an important role, which was prepared for his problem.

History of sigma function: 1990-

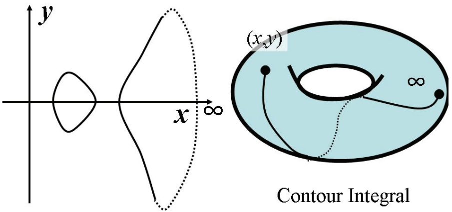
1. 1990: **Grant(Number theory)**: Sigma function of genus two: Linear dependence relation of differentials of σ / σ^ℓ as $\mathcal{O}(\mathcal{J}, n\Theta)$.
2. 1995- **Yoshihiro Ônishi (Meijo Univ. Number theory)**: Structure of Jacobian and addition structures of its strata using hyperelliptic σ function.
3. 1997- **Buchsterber-Enolskii-Leykin**: Investigation on the **integrable system** using hyperelliptic σ function.
4. 2000 Eilbeck-Enolskii-Leykin: Generalization of sigma function of hyperelliptic curve to (n, s)-**type plan curve** $(y^n + x^s + \dots.)$
5. 2008-: **Nakayashiki (Tsudajuku univ.)**: Show the expression of sigma functions of (n, s) in terms of **Fay's results and tau functions of Sato theory**.
6. 1997- **M, Previato, Onishi, Enolskii, Kodama**, Reconstruction of the Abelian function theory
7. 2010- **M, Previato, Komeda: Extension to space curves.**

Review of elliptic sigma function

σ of genus one

Elliptic curve

$$X_1 := \left\{(x,y) \;\middle|\; \begin{aligned} y^2 &= x^3+\lambda_2 x^2+\lambda_1 x+\lambda_0 \\ &= (x-e_1)(x-e_2)(x-e_3) \end{aligned}\right\} \cup \infty.$$



Commutative Ring

$$R = \mathbb{C}[x,y]/(y^2 - x^3 - \lambda_2 x^2 - \lambda_1 x - \lambda_0)$$

Differentials

Differentials of the first kind (holo. one-from = Canonical form)

$$du := \nu^{\mathrm{I}} := \frac{dx}{2y},$$

Differential of the second kind (Basis of $\mathcal{H}^1(X_1 \setminus \{\infty\})$)

$$\nu^{\mathrm{II}} := \frac{xdx}{2y},$$

Differential of the 3rd kind (holo. except Q and Q' s.t. res. is ± 1)

$$\Pi = \Sigma(P(x,y); Q(x_1,y_1)) - \Sigma(P(x,y); Q'(x_2,y_2))$$

where $\Sigma(P(x,y), Q(x',y')) := \dfrac{(y+y')dx}{2(x-x')y}$

σ of genus one: Abelian map & Period

Covering space of X_1

$\tilde{X}_1$: Abelian covering of X_1
(Abelianization of quotient space of path space of X_1)

$$\kappa' : \tilde{X}_1 \to X_1$$

Abel integral (Elliptic incomplete integral of the first kind)

$$w : \tilde{X}_1 \to \mathbb{C}:\ u = w(x,y) \equiv \int_\infty^{(x,y)} \nu^{\mathrm{I}},\ \nu^{\mathrm{I}} = du = \frac{dx}{2y},$$

Half period integrals (Elliptic complete integrals of 1st kind)

$$\omega_i := w(e_i, 0) \equiv \int_\infty^{(e_i,0)} \nu^{\mathrm{I}}, \quad \nu^{\mathrm{I}} := \frac{dx}{2y},$$

where $(e_i, 0)$ $(i = 1, 2, 3)$ and ∞ are branch pints

Elliptic Jacobi variety & Legendre's relation

Lattice

$$\Gamma = 2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_3 \subset \mathbb{C}. \quad \hat{\Gamma} = \mathbb{Z} + \mathbb{Z}\tau, \quad (\tau = \omega_3/\omega_1)$$

Jacobi variety

$$\kappa : \mathbb{C} \to \mathcal{J} = \mathbb{C}/\Gamma,$$

Complete Elliptic integral of 2nd kind

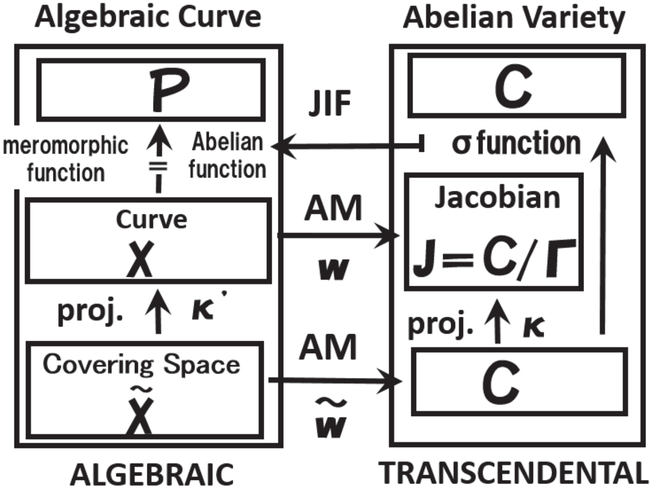
$$\eta_i := \int_\infty^{(e_i,0)} \nu^{\mathrm{II}}, \quad \nu^{\mathrm{II}} := \frac{x dx}{2y},$$

Legendre's relation: (Symplectic Str.($\approx$ Hodge Str.))

$$\omega_3\eta_1 - \omega_1\eta_3 = \frac{\pi}{2}.$$

The Roles of Abelian Map and Jacobi inversion formulae

The Roles of Abelian Maps and Jacobi inversion formulae



The Roles of Abelian Map and Jacobi inversion formulae

Algebraic space and Analytic space

Natural Projection

Covering space $\tilde{X}_1 =$ Abelianization of quotient space of Path space of X_1

$\kappa' : \tilde{X}_1 \to X_1$

$\tilde{X}_1 \supset LX_1(\infty) \cong \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$

SL(2,$\mathbb{Z}$) $\curvearrowright$ this expression

Natural Projection

Covering space $\mathbb{C} =$ Abelianization of quotient space of Path space of $\mathcal{J}$

$\kappa : \mathbb{C} \to \mathcal{J} = \mathbb{C}/\Gamma$

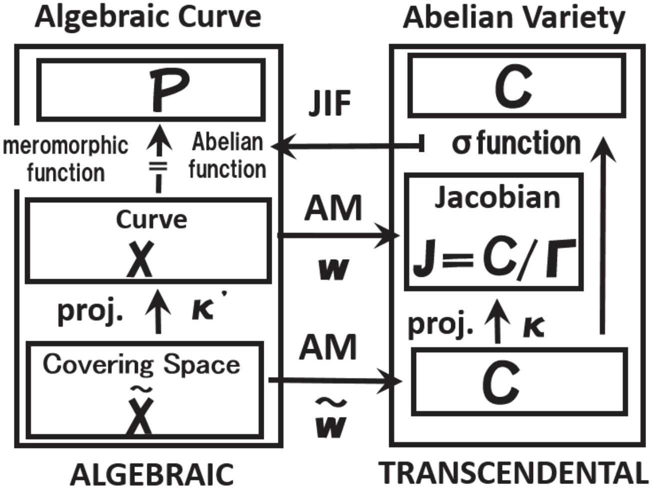
$\Gamma = \mathbb{Z}(2\omega') + \mathbb{Z}(2\omega'')$

SL(2,$\mathbb{Z}$) $\curvearrowright \Gamma$, $\mathcal{J}$

In order to obtain Metamorphic functions over X_1 and $\mathcal{J}$, the entire function of $\mathbb{C}$ should be invariant for SL(2,$\mathbb{Z}$).

σ is modular invariant for SL(2,$\mathbb{Z}$)

The Roles of Abelian Map and Jacobi inversion formulae



Elliptic Weierstrass' σ function

Weierstrass' σ function over $\mathbb{C}$ as an entire function:

$$\sigma(u) = 2\omega_1 \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \frac{\theta_1(\frac{u}{2\omega_1})}{\theta_1'^0}$$

Translation formula: $\Omega_{m,n} := 2m\omega_1 + 2n\omega_3$:

$$\sigma(u + \Omega_{m,n}) = (-1)^{m+n+mn} \exp\left((m\eta_1 + n\eta_3)(2u + \Omega_{m,n})\right) \sigma(u).$$

Zero of σ function:

$$\{\text{zeros of } \sigma\} \equiv 0 \text{ mod. } \Gamma, \qquad \lim_{u \to 0} \frac{\partial \sigma(u)}{\partial u} = 1.$$

$\mathrm{SL}(2, \mathbb{Z})$

Modular invariance for $\mathrm{SL}(2, \mathbb{Z})$.

Elliptic Weierstrass' $\wp$ function / $\mathcal{J} = \mathbb{C}/\Gamma$

$$\zeta(u) = \frac{d}{du}\log\sigma(u). \qquad \wp(u) = -\frac{d^2}{du^2}\log\sigma(u).$$

Theorem (Jacobi inversion formula): for $u = w(x, y)$

$$(x, y) = (\wp(u), \wp_u(u)), \quad \wp_u(u) := \frac{d}{du}\wp(u).$$

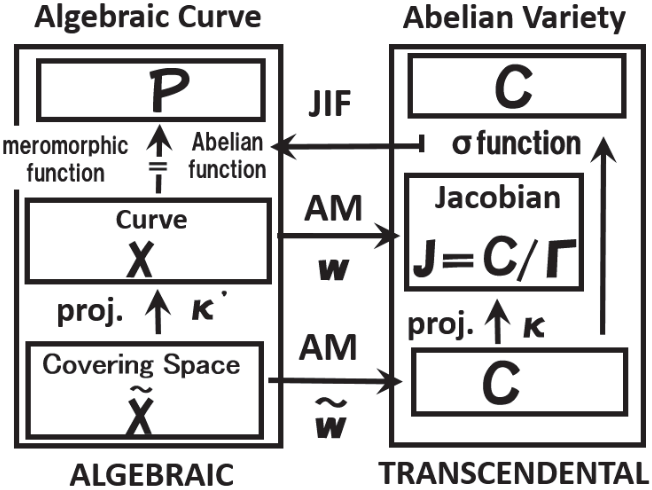
JIF recovers the governing equation $y^2 = x^3 + \cdots$

JIF recovers the governing equation $y^2 = x^3 + \cdots$, i.e.,

$$\wp_u(u)^2 = \wp(u)^3 + \lambda_2\wp(u)^2 + \lambda_1\wp(u) + \lambda_0.$$

The Roles of Abelian Map and Jacobi inversion formulae

The Roles of Abelian Map and Jacobi inversion formulae



The σ function is invariant for $\mathrm{SL}(2,\mathbb{Z})$

Review:
How to get the Jacobi inversion formulae?

Review: how to get Jacobi inversion formulae

1. Define the second fundamental from:

Idea($\mathbb{P}^1$) : Fundamental Second form

$$\frac{dw_1 dw_2}{(w_1 - w_2)^2}:$$

2. Express the Riemann fundamental formulae:

Idea($\mathbb{P}^1$)

$$\int_{z_2}^{z_1}\int_{z_4}^{z_3} \frac{dw_1 dw_2}{2(w_1 - w_2)^2} = \log\left(\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}\right)$$

Algebraic $\Leftrightarrow$ Transcendental

Review: how to get Jacobi inversion formulae

2. Express the Riemann fundamental formulae:

Idea($\mathbb{P}^1$)

$$\int_{z_2}^{z_1}\int_{z_4}^{z_3} \frac{dw_1 dw_2}{2(w_1 - w_2)^2} = \log\left(\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}\right)$$

Algebraic $\Leftrightarrow$ Transcendental

3. Differentiate both sides twice:

Idea($\mathbb{P}^1$)

$$\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_3}\int_{z_2}^{z_1}\int_{z_4}^{z_3} \frac{dw_1 dw_2}{(w_1 - w_2)^2} = \frac{\partial}{\partial z_1}\frac{\partial}{\partial z_3}\log\left(\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}\right)$$

The second fundamental form

The second fundamental form in terms of $\mathcal{Q}(R \otimes R)$

For $P_1, P_2 \in X_1$ and $u_a = w(P_a)$,

$$\begin{aligned}\Omega(P_1, P_2) &:= \wp(u_1 - u_2)\nu^{\mathrm{I}}(P_1) \otimes \nu^{\mathrm{I}}(P_2),\\ &= \frac{F(P_1, P_2) dx_1 \otimes dx_2}{(x_1 - x_2)^2 4 y_{P_1} y_{P_2}}\\ &= \frac{dt_1 \otimes dt_2}{(t_1 - t_2)^2}(1 + \text{holo.}),\end{aligned}$$

where

$$F(P_1, P_2) := f(P_1, P_2) + y_1 y_2 \qquad \in R \otimes R$$

$$f(P_1, P_2) := x_1 x_2 (x_1 + x_2 + 2\lambda_2) + \lambda_1 (x_1 + x_2) + 2\lambda_0$$

1. 1-form w.r.t P_1 is sing. at P_2 with order 2 and is holo. over $C \setminus P_2$,
2. 1-form w.r.t P_2 is sing. at P_1 with order 2 and is holo. over $C \setminus P_2$,
3. symmetric for P_1 and P_2.

Riemannian fundamental relation (RFR)

Theorem (RFR)

For $(P, Q, P', Q') \in \tilde{X}_1^4$,

$$u := w(P), \quad u' := w(P'), \quad v := w(Q), \quad v' := w(Q'),$$

$$\exp\left(\int_Q^P \int_{Q'}^{P'} \Omega(P_1, P_2)\right) = \frac{\sigma(u-u')\sigma(v-v')}{\sigma(u-v')\sigma(v-u')}.$$

Merom. func/X_1 or $\tilde{X}_1$ $\Leftrightarrow$ Merom. func/J or $\mathbb{C}$

Jacobi inversion formula

Idea($\mathbb{P}^1$)

By acting $\partial/\partial u$ and $\partial/\partial u'$ onto RFR, we obtain the JIF of $\wp(u_1 - u_2)$

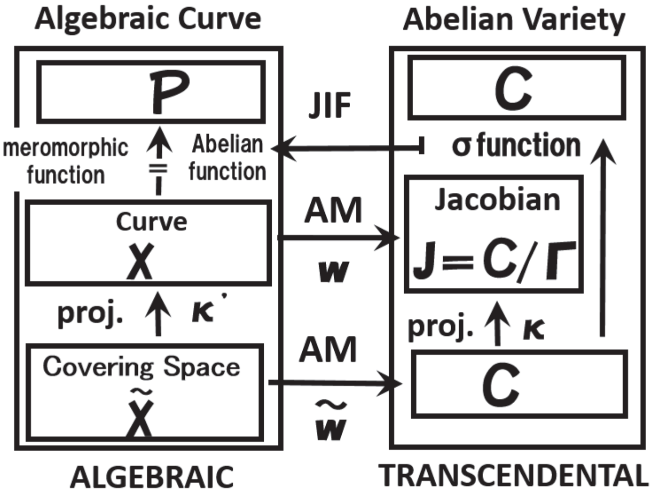
JIF

$$\frac{F(P_1, P_2)}{(x_1 - x_2)^2} = \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\log\sigma(u_1 - u_2)$$

$$x = \wp\left(\int_\infty^{(x,y)} \nu^{\mathrm{I}}\right)$$

The Roles of Abelian Map and Jacobi inversion formulae

The second fundamental form



Using σ function, consider the both spaces.

Generalization of sigma function to general affine curves

D. Mumford "Tata Lecture of Theta II" (1982)

In order to express concrete hyperelliptic solutions of non-linear integrable differential equations, Mumford unified the studies of algebraic geometry of XIX and XX.

Mumford's program(as in Tata II):

For the application of the Abelian function theory of Hyperelliptic curves, repeat the following:

1. Refine the Jacobi inversion formulae for **hyperelliptic curves**
2. Find algebraic and geometric properties of the Abelian functions and the θ **functions** of **hyperelliptic curves**,
3. Apply the results of (1) and (2) to integrable system, and give the feedback to (1) and (2).

Prototype of study of σ for affine curves

Extend the Mumford's program

For the application of the Abelian function theory of affine curves, repeat the following:

1. Refine the Jacobi inversion formulae for **affine curves**
2. Find algebraic and geometric properties of the Abelian functions and σ **functions** of the **affine curves**,
3. Apply the results of (1) and (2) to integrable system and other problems and give these feedback to (1) and (2).

Purpose of study

Purpose of this study:

In the theory of elliptic functions, the concrete and explicit descriptions of the elliptic functions have a power to have effects on various fields including physics, number theory, and industrial problems.
Following such properties of the elliptic functions, our purpose is to reconstruct the theory of Abelian functions for general curves such that it has concrete and explicit descriptions, and thus influences various fields.

XIX Concrete	$\to$	XX Abstract	$\to$	XXI Unification

Generalization

Weierstrass curves:

An affine curve as a pointed compact Riemann surface (X, ∞) whose Weierstrass gap at ∞ is given by numerical semigroup.

It is given by the normalization of **the Weierstrass normal form (Kato, Baker, Weierstrass)**:

$$\boxed{y^r + \mu_{r-1}(x) y^{r-1} + \cdots + \mu_0(x) = 0}$$

$$\left(\mu_i \in \mathbb{C}[x] \text{ of } \left\lceil \frac{(r-i)s}{r} \right\rceil \text{-th polynomial, } (r, s) = 1, r < s\right)$$

Especially we focus on **the cyclic Weierstrass normal form**:

$$\boxed{y^r = x^s + \lambda_{s-1} x^{s-1} + \cdots + \lambda_1 x + \lambda_0}$$

Curves for the study

Weierstrass curves:

The Weierstrass normal form (Kato, Baker, Weierstrass).

$$\boxed{y^r + \mu_{r-1}(x)y^{r-1} + \cdots + \mu_0(x) = 0}$$

$$\left(\mu_i \in \mathbb{C}[x] \text{ of } \left\lceil \frac{(r-i)s}{r} \right\rceil \text{-th polynomial, } (r,s)=1, r<s\right)$$

1. (r,s)-curve (non-singular plane curve case)
   (2,3)-curve $=$ elliptic curve, $(2, 2g+1) =$ hyperelliptic curve
2. Space curves:
   1. cyclic trigonal curves $y^3 = x^s + \cdots \lambda_0$ (KMP)
   2. Telescopic curves (Ayano)

Weierstrass curves (X, ∞)

(r,s)-curve: $y^r = x^s + \lambda_{s-1} x^{s-1} + \cdots \lambda_1 x + \lambda_0$ $(r,s) = 1$

∞ as a branch point: $x = \dfrac{1}{t^r}$, $y = \dfrac{1}{t^s}(1 + t^{\geq 1})$

t:local parameter at ∞,

Weight $\mathrm{wt}(x) = r$, $\mathrm{wt}(y) = s$

Curves

Weierstrass curves (X, ∞)

(r,s)-curve: $y^r = x^s + \lambda_{s-1}x^{s-1} + \cdots \lambda_1 x + \lambda_0$ $(r,s)=1$

∞ as a branch point: $x = \dfrac{1}{t^r}$, $y = \dfrac{1}{t^s}(1 + t^{\geq 1})$

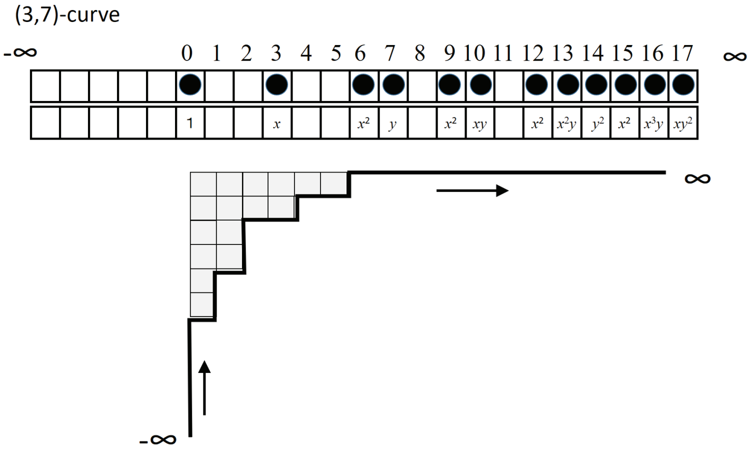
t:local parameter at ∞, Weight $\mathrm{wt}(x) = r$, $\mathrm{wt}(y) = s$

Numerical semigroup: $H = \mathbb{N}_0 r + \mathbb{N}_0 s \subset \mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$.
$\# N_0 \setminus H < \infty$.

(r,s)	g	0	1	2	3	4	5	6	7	8	9	10	11
(2,3)	1	1	-	x	y	x^2	xy	x^3	x^2y	x^4	x^3y	x^5	x^4y
(2,5)	2	1	-	x	-	x^2	y	x^3	xy	x^4	x^2y	x^5	x^3y
(2,7)	3	1	-	x	-	x^2	-	x^3	y	x^4	xy	x^6	x^2y
(3,4)	3	1	-	-	x	y	-	x^2	xy	y^2	x^3	x^2y	xy^2
(3,5)	4	1	-	-	x	-	y	x^2	-	xy	x^3	y^2	x^2y
(3,7)	6	1	-	-	x	-	-	x^2	y	-	x^3	xy	-

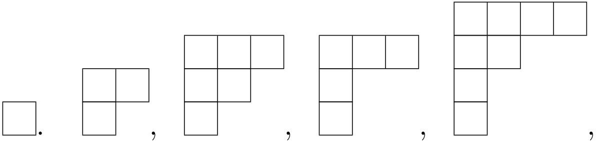
Pointed compact Riemannian surface

For a gap, upward by a box, for a non-gap, step to the right by a box:



Weierstrass curves (X, P)

For a gap, upward by a box, for a non-gap, step to the right by a box:



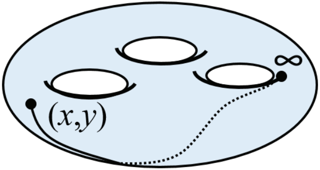
$(2,3)$ $(2,5)$ $(2,7)$ $(3,4)$ $(3,5)$

These Young diagrams are symmetric!

Sigma function of $y^3 = (x - b_0)(x - b_1)(x - b_2)(x - b_3)$: a cyclic trigonal curve of genus three

Cyclic trigonal curve (A non-singular plane curve of $g = 3$)

$$X_3 := \left\{ (x, y) \;\middle|\; \begin{aligned} y^3 &= x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ &= (x - b_0)(x - b_1)(x - b_2)(x - b_3) \end{aligned} \right\} \cup \infty.$$



$$R = \mathbb{C}[x, y]/(y^3 - x^4 - \cdots - \lambda_0).$$

Weierstrass gap at ∞

Weierstrass gap at ∞

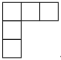
$$wt(x) = 3,\ wt(y) = 4,$$
$$\phi_0 = 1,\ \phi_1 = x,\ \phi_2 = y,\ \phi_3 = x^2,\ \cdots$$

wt	0	1	2	3	4	5	6	7	8	9	10
ϕ	1	-	-	x	y	-	x^2	xy	y^2	x^3	x^2y

weighted $\mathbb{C}$-vector space

$R = \oplus_{i=0}\mathbb{C}\phi_i$ weighted $\mathbb{C}$-vector space,

Numerical Semigroup $H := \{3a + 4b\}_{a,b\in\mathbb{Z}_{\geq 0}} := \langle 3, 4\rangle$

$H = \{0, 3, 4, 6, 7, \cdots\}$, $L = \mathbb{Z} \setminus H = \{1, 2, 5\}$.  .

Differentials

differentials of the 1st kind(= canonical form)

$$\nu^{\mathrm{I}} := \begin{pmatrix} \nu^{\mathrm{I}}{}_1 \\ \nu^{\mathrm{I}}{}_2 \\ \nu^{\mathrm{I}}{}_3 \end{pmatrix} := \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} := {}^t\left(\frac{dx}{3y^2}, \frac{xdx}{3y^2}, \frac{dx}{3y}\right) = {}^t\left(\frac{\phi_0 dx}{3y^2}, \frac{\phi_1 dx}{3y^2}, \frac{\phi_2 dx}{3y^2}\right)$$

$\mathrm{Div}(\nu^{\mathrm{I}}{}_i) \sim (2g-2)\infty$ as a linear equivalent.

differentials of the 2nd kind(holo. except ∞)

$$\nu^{\mathrm{II}} := \begin{pmatrix} \nu^{\mathrm{II}}{}_1 \\ \nu^{\mathrm{II}}{}_2 \\ \nu^{\mathrm{II}}{}_3 \end{pmatrix},$$

$$\nu^{\mathrm{II}}{}_1 = -\frac{5x^2 + 2\lambda_3 x + \lambda_2}{3y^2}, \quad \nu^{\mathrm{II}}{}_2 = -\frac{2x}{3y}, \quad , \nu^{\mathrm{II}}{}_3 = -\frac{x^2}{3y^2},$$

differentials of the 3rd kind

$$\Pi = \Sigma(P(x,y);(x_1,y_1)) - \Sigma(P(x,y);(x_2,y_2))$$

where $\Sigma(P(x,y);Q(x',y')) := \dfrac{(y^2 + y'y + y'^2)dx}{3(x-x')y^2}$

Abelian map (AM) of cyclic trigonal curve

AM (Abelian map)

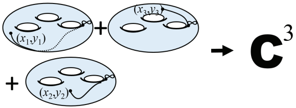
$\tilde{X}_3$: Abelianization of quotient space of Path(X_3)

$$\tilde{w} : \tilde{X}_3 \to \mathbb{C}^3; \quad \left(w(P) = \int_\infty^P \nu^{\mathrm{I}} = \int_\infty^P \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} \right)$$

$$\tilde{w} : S^3(\tilde{X}_3) \to \mathbb{C}^3; \quad \tilde{w}(P_1, P_2, P_3) := \tilde{w}(P_1) + \tilde{w}(P_2) + \tilde{w}(P_3).$$

$$S^3(X_3) := X_3 \times X_3 \times X_3 / \sim :$$

Symmetric product



Homology basis: $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$

$$\langle \alpha_i, \beta_j \rangle = \delta_{ij}, \quad \langle \alpha_i, \alpha_j \rangle = 0, \quad \langle \beta_i, \beta_j \rangle = 0,$$

Half period

$$(\omega'_{ij}) := \frac{1}{2}\left(\int_{\alpha_i} \nu^{\mathrm{I}}{}_j\right),$$

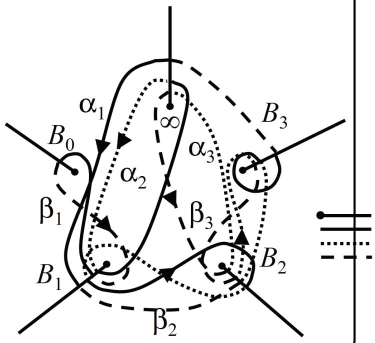
$$(\omega''_{ij}) := \frac{1}{2}\left(\int_{\beta_i} \nu^{\mathrm{I}}{}_j\right)$$

Abelian integrals of the 2nd kind

$$(\eta'_{ij}) := \frac{1}{2}\left(\int_{\alpha_i} \nu^{\mathrm{II}}{}_j\right),$$

$$(\eta''_{ij}) := \frac{1}{2}\left(\int_{\beta_i} \nu^{\mathrm{II}}{}_j\right)$$

$y^3 = x^4 + \lambda_3 x^3 + \cdots + \lambda_0$



Legendre relation of non-hyperelliptic curve

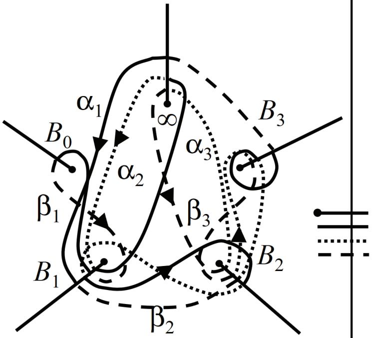
For $\omega_a := \int_\infty^{B_a} \nu^{\mathrm{I}}, (a = 0, 1, 2, 3)$

$$\omega_1' = \frac{1}{2}\Big((1-\hat\zeta_3^2)\omega_1 + \hat\zeta_3^2(1-\hat\zeta_3^2)\omega_2 + \hat\zeta_3(1-\hat\zeta_3)\omega_3\Big),$$

$$\omega_a' = \frac{1}{2}\hat\zeta_3^{a-2}(\hat\zeta_3 - 1)\omega_{a-1}, \quad (a = 2, 3),$$

$$\omega_c'' = \frac{1}{2}\hat\zeta_3^{c-2}(\hat\zeta_3 - 1)(\omega_{c-1} - \omega_c), \quad (c = 1, 2, 3).$$

$y^3 = x^4 + \lambda_3 x^3 + \cdots + \lambda_0$



Jacobi variety

Lattice

$$\Gamma := \langle \omega', \omega'' \rangle_{\mathbb{Z}} = \left\{ \omega \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \omega' \begin{pmatrix} a_4 \\ a_5 \\ a_6 \end{pmatrix} \middle| a_i \in 2\mathbb{Z} \right\} \subset \mathbb{C}^3$$

Jacobi variety

$$\kappa : \mathbb{C}^3 \to \mathcal{J} = \mathbb{C}^3/\Gamma,$$

Legendre relation (Symplectic Str.($\approx$ Hodge Str.))

$${}^t\omega'\eta'' - {}^t\omega''\eta' = \frac{\pi}{2} I_3.$$

The σ function of X_3

σ function $u \in \mathbb{C}^3$

$$\sigma(u) := \gamma_0 \mathrm{e}^{-\frac{1}{2} u^t {\omega'}^{-1\,t} \eta' u} \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \left(\frac{1}{2} {\omega'}^{-1} u; {\omega'}^{-1} \omega'' \right).$$

Here γ_0 is a constant factor, δ **is** θ **characteristics**

$$\delta' \in (\mathbb{Z}/2)^3, \quad \delta'' \in (\mathbb{Z}/2)^3, \quad \Leftrightarrow \quad \textbf{Riemann const. } \xi$$

θ is **the Riemann theta function**,

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z; \tau) = \sum_{n \in \mathbb{Z}^2} \exp \left(\pi \sqrt{-1} ((n+a)^t \tau (n+a) - (n+a)^t (z+b)) \right).$$

Klein's σ function

Translational formula: $u,\ v \in \mathbb{C}^3$, and $\ell\ (= 2\omega'\ell' + 2\omega''\ell'') \in \Lambda$

$$L(u, v) := 2\ {}^t u(\eta' v' + \eta'' v''),$$

$$\chi(\ell) := \exp[\pi\sqrt{-1}(2({}^t\ell'\delta'' - {}^t\ell''\delta') + {}^t\ell'\ell'')]\ (\in \{1, -1\})$$

$$\Rightarrow \quad \sigma(u + \ell) = \sigma(u)\exp(L(u + \frac{1}{2}\ell, \ell))\chi(\ell).$$

Properties of σ

1. Entire function over $\mathbb{C}^3$
2. Zeros: $\{\mathrm{div}\sigma\} \equiv \Theta = w(X_3^2)$
3. Modular invariant for $\mathrm{Sp}(3, \mathbb{Z})$
4. Expansion $\sigma(u) = s_{(3,1,1)} + \cdots$, $s_{(3,1,1)}$: Schur polynomial of $(3,1,1)$.
5. Expansion of $\sigma(u)$ is explicitly determined by UGM.

Properties of σ

1. Expansion $\sigma(u) = s_{(3,1,1)} + \cdots,$

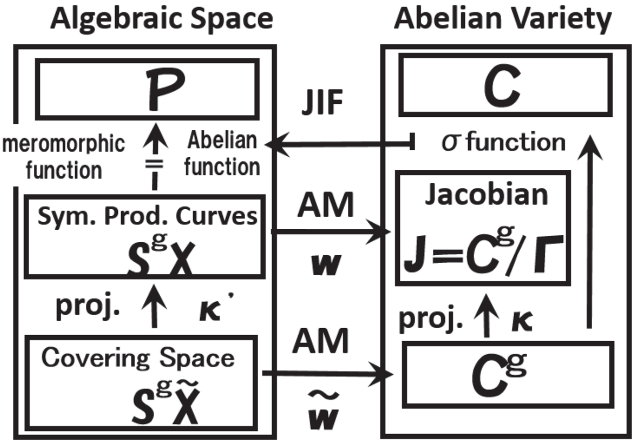
$$s_{(3,1,1)}(u_1, u_2, u_3) = \left.\frac{\begin{vmatrix} t_1^5 & t_2^5 & t_3^5 \\ t_1^2 & t_2^2 & t_3^2 \\ t_1^1 & t_2^1 & t_3^1 \end{vmatrix}}{\begin{vmatrix} t_1^2 & t_2^2 & t_3^2 \\ t_1 & t_2 & t_3 \\ 1 & 1 & 1 \end{vmatrix}}\right|_{u_1:=\frac{1}{5}\sum_a t_a^5, u_2:=\frac{1}{3}\sum_a t_a^3, u_3:=\sum_a t_a}$$

$$= u_1 - u_2^2 u_3$$

2. Expansion of $\sigma(u)$ is explicitly determined by UGM. (Nakayashiki 2010)

The Roles of Abelian Map and Jacobi inversion formulae

The Roles of Abelian Map and Jacobi inversion formulae



The second fundamental form

For $P_1=(x_1,y_1), P_2=(x_2,y_2)\in X_3$,

$\Omega(P_1,P_2)$: $$\Omega(P_1,P_2)=\frac{F(P_1,P_2)dx_1\otimes dx_2}{(x_1-x_2)^2 4y_{P_1}y_{P_2}} = \frac{dt_1\otimes dt_2}{(t_1-t_2)^2}(1+\text{holo.}),$$

where F is an element of $R\otimes R$,

$$F(P_1,P_2)=f(P_1,P_2)+f(P_2,P_1)+y_1^2y_2^2,$$
$$f(P_1,P_2)=x_1^2x_2^{+}2x_1x_2^3+\lambda_2(x_2^2+2x_1x_2)$$
$$\lambda_1(x_1+2x_2)+3\lambda_0).$$

1. 1-form w.r.t P_1 has singularity at P_2 with order 2 and is holomorphic except $C\setminus P_2$,
2. 1-form w.r.t P_2 has singularity at P_1 with order 2 and is holomorphic except $C\setminus P_2$,
3. symmetric for P_1 and P_2.

Riemannian fundamental relation (RFR)

Theorem (RFR)

For $(P, Q, P_1, P_2, P_3, P'_1, P'_2, P'_3) \in X_3^2 \times S^3(X_3) \times S^3(X_3)$,

$$u := w(P_1) + w(P_2) + w(P_3), \quad v := w(P'_1) + w(P'_2) + w(P'_3),$$

$$\exp\left(\sum_{i,j=1}^{2} \int_{P_i}^{P} \int_{P'_j}^{Q} \Omega(P', Q')\right) = \frac{\sigma(w(P) - u)\sigma(w(Q) - v)}{\sigma(w(Q) - u)\sigma(w(P) - v)}$$

Merom. func$/X$ or $\tilde{X} \Leftrightarrow$ Merom. func$/J$ or $\mathbb{C}$

Abelian function, $\wp$

Abelian function, $\wp/\mathcal{J} = \mathbb{C}/\Gamma$

$$\zeta_i(u) = \frac{\partial}{\partial u_i} \log \sigma(u). \qquad \wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u) \in H^0(\mathcal{O}_{\mathcal{J}}(2\Theta)).$$

$$\wp_{ijk\cdots\ell}(u) = -\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}\frac{\partial}{\partial u_k}\cdots\frac{\partial}{\partial u_\ell} \log \sigma(u)$$

$$\sigma_{ijk\cdots\ell}(u) = -\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}\frac{\partial}{\partial u_k}\cdots\frac{\partial}{\partial u_\ell} \sigma(u)$$

Theorem (JIF)

For $P, P_a \in X_3$, $u = w(P_1) + w(P_2) + w(P_3)$,

$$\sum_{i,j=1}^{g} \wp_{i,j}\left(w(P) - u\right) \phi_{i-1}(P)\phi_{j-1}(P_a) = \frac{F(P, P_a)}{(x - x_a)^2},$$

Abelian function, $\wp$ and JIF

Theorem: (JIF): $S^3X_3 \supset S^2X_3 \supset X_3 \iff \mathcal{J} \supset \Theta^{[2]} \supset \Theta^{[1]}$

1) For $u = w((x_1,y_1),(x_2,y_2),(x_3,y_3))$

$$\frac{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \wp_{33}(u), \quad \frac{\begin{vmatrix} 1 & y_1 & x_1^2 \\ 1 & y_2 & x_2^2 \\ 1 & y_3 & x_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \wp_{23}(u), \quad \frac{\begin{vmatrix} x_1 & y_1 & x_1^2 \\ x_2 & y_2 & x_2^2 \\ x_3 & y_3 & x_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \wp_{13}(u),$$

2) For $u = w((x_1,y_1),(x_2,y_2))$

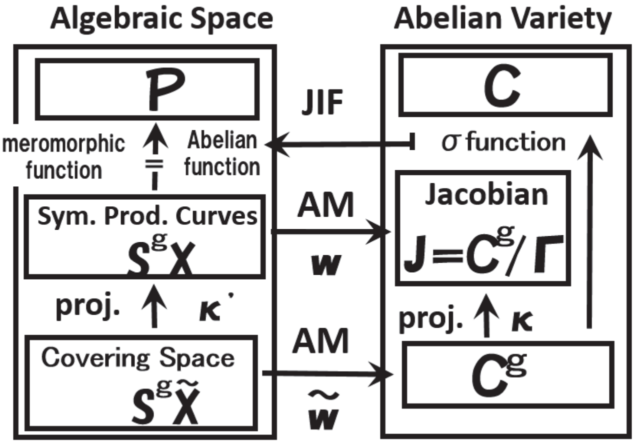
$$\frac{\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}} = \frac{\sigma_2(u)}{\sigma_3(u)}, \quad \frac{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}} = \frac{\sigma_1(u)}{\sigma_3(u)},$$

3) For $u = w((x,y))$

$$x = \frac{\sigma_1(u)}{\sigma_2(u)}$$

The Roles of Abelian Map and Jacobi inversion formulae

The Roles of Abelian Map and Jacobi inversion formulae



Geometric roles of AM & JIF

KP hierarchy (Enolskii-Eilbeck-M-Onishi-Previato 2007)

$$
\begin{aligned}
\wp_{3333} - 6\wp_{33}^2 &= -3\wp_{22} \\
\wp_{2333} - 6\wp_{33}\wp_{23} &= 3\lambda_3\wp_{33} \\
\wp_{3322} - 4\wp_{23}^2 - 2\wp_{33}\wp_{22} &= 4\wp_{13} + 3\lambda_3\wp_{31} + 2\lambda_2 \\
\wp_{3222} - 6\wp_{22}\wp_{32} &= 3\lambda_3\wp_{22} \\
\wp_{2222} - 6\wp_{22}^2 &= -4\wp_{3331} - 2\lambda_3^2\wp_{33} + 12\lambda_2\wp_{33} \\
&\vdots
\end{aligned}
$$

These data recover the governing equation $y^3 = x^4 + \cdots$.

Geometric roles of AM & JIF

Additive str.; Enolskii-Eilbeck-M-Onishi-P

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp_{11}(u) - \wp_{11}(v)$$
$$+ \wp_{12}(u)\wp_{23}(v) - \wp_{12}(v)\wp_{23}(u)$$
$$+ \wp_{13}(u)\wp_{22}(v) - \wp_{13}(v)\wp_{22}(u)$$
$$+ \frac{1}{2}\wp_{33}(u)Q_{1333}(v) - \wp_{33}(v)Q_{1333}(u)$$

where $Q_{1333} := \wp_{1333} - 6\wp_{13}\wp_{33}$

General (r, s) plane curves

Weierstrass gap

Weighted ring:

Commutative ring: $R = \mathbb{C}[x, y]/(y^r - x^s + \cdots + \lambda_0)$

$R = \oplus \mathbb{C}\phi_i$ as a vector space

ϕ_i is given as follows:

(r, s)	0	1	2	3	4	5	6	7	8	9	10	11
(2,3)	1	-	x	y	x^2	xy	x^3	x^2y	x^4	x^3y	x^5	x^4y
(2,5)	1	-	x	-	x^2	y	x^3	xy	x^4	x^2y	x^5	x^3y
(2,7)	1	-	x	-	x^2	-	x^3	y	x^4	xy	x^6	x^2y
(3,4)	1	-	-	x	y	-	x^2	xy	y^2	x^3	x^2y	xy^2
(3,5)	1	-	-	x	-	y	x^2	-	xy	x^3	y^2	x^2y
(3,7)	1	-	-	x	-	-	x^2	y	-	x^3	xy	-
genus 0	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}

Weierstrass gap & σ function

Weighted Ring:

Basis of $R = \mathbb{C}[x, y]/(y^r - x^s + \cdots + \lambda_0)$:

$R = \oplus \mathbb{C}\phi_i$ as a vector space

Differentials of the 1st kind: $du_i = {\nu^{\mathrm{I}}}_i := \dfrac{\phi_i dx}{ry^{r-1}}$, $i = 1, 2, \ldots, g$

Abelian map (AM): $w : S^k\tilde{X} \to \mathbb{C}^g$

Schur function:

σ is modular invariant for $\mathrm{Sp}(g, \mathbb{Z})$.
$\sigma(u) = s_Y(u) + \cdots$:
$s_Y(u)$ is the Schur function of Young diagram Y

Meromorphic function (generalization of Vandermonde/$\mathbb{P}$):

$P_i = (x_i, y_i)$ $(i = 1, \dots, n-1) \in X_g$.

$$\Psi_n(P_n, P_{n-1}, \dots, P_1) := \begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_{n-1}(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_{n-1}(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_{n-1}) & \phi_2(P_{n-1}) & \cdots & \phi_{n-1}(P_{n-1}) \\ 1 & \phi_1(P_n) & \phi_2(P_n) & \cdots & \phi_{n-1}(P_n) \end{vmatrix}$$

Meromorphic function:

$$\mu_n(P; P_1, P_2, \dots, P_n) = \frac{\Psi_{n+1}(P, P_n, P_{n-1}, \dots, P_1)}{\Psi_n(P_n, P_{n-1}, \dots, P_1)} \in R \times S^n \mathcal{Q}(R)$$

$$\mu_n(P; P_1, P_2, \dots, P_n) = \phi_n(P) + \sum_{i=0}^{n-1} \mu_{n,i}(P_1, P_2, \dots, P_n)\phi_i(P)$$

JIF for $\Theta \subset \mathcal{J}$ (M-Previato2008):

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u) = \frac{\sigma \partial_{u_i}\partial_{u_j}\sigma - \partial_{u_i}\sigma\partial_{u_j}\sigma}{\sigma^2}$$

$$\in H^0(\mathcal{O}_{\mathcal{J}}(2\Theta))$$

We have the identity $u = w(P_1, P_2, \ldots, P_g)$

1. $\mu_g(P; P_1, P_2, \ldots, P_g) = \phi_g(P) + \sum_{i=1}^{g} \wp_{g,i}(u)\phi_{i-1}(P)$

   i.e., $\wp_{gi}(u) = (-1)^i \mu_{g,i-1}$, $(i = 1, 2, \ldots, g)$

2. $(k < g)$: $\mu_k(P; P_1, P_2, \ldots, P_k) = \phi_k(P) + \sum_{i=1}^{k} \frac{\sigma_{\sharp_k,i}}{\sigma_{\sharp_k,k}}(u)\phi_{i-1}(P)$

   i.e., $\frac{\sigma_{\sharp_k,i}}{\sigma_{\sharp_k,k}}(u) = (-1)^i \mu_{k,i-1}$, $(i = 1, 2, \ldots, k)$

Sigma function of $y^3 = x^2(x - b_1)(x - b_2)$

a normalized trigonal space curve of genus two

Singular curve X_{sing}

Let us consider the singular curve X_{sing} : $y^3 = x^2 k(x), \quad k(x) = (x - b_1)(x - b_2)$ and its commutative ring,

$$R_{\mathrm{sing}} = \mathbb{C}[x, y]/(y^3 - x^2 k(x))$$

Normalization

Normalization of the ring $R_{\rm sing}$ provides the ring

$$R = \mathbb{C}[x, y, z]/(y^2 - xz, zy - xk(x), z^2 - k(x)y).$$

and a space curve, $X_2 \to X_{\rm sing}$,

$$X = \{(x, y, z) \mid y^2 = xz, zy = x, z^2 = k(x)y\} \cup \{\infty\}$$

Each branch point is given by

$$B_0 = (x = 0, y = 0, z = 0), B_1, B_2.$$

Weierstrass Sequence

Weierstrass Sequence at ∞

wt	0	1	2	3	4	5	6	7	8	9	10	11
ϕ_i	1	-	-	x	y	z	x^2	xy	xz	yz	x^2y	x^2z

$$R = \bigoplus_{i=0} \mathbb{C}\phi_i, \quad \text{as a } \mathbb{C}\text{-vector space.}$$

wt: weight given as the order of singularity at ∞

$$\mathrm{wt}(x) = 3, \mathrm{wt}(y) = 4, \mathrm{wt}(z) = 5$$

genus $g = 2$, $\mathcal{K}_X = (2g)\infty - 2(B_0) \neq (2g-2)\infty$

The fact $\mathcal{K}_X \neq (2g-2)\infty$ causes the difficulty of the construction of σ!

Abelian map (AM) and Riemann constant (RC)

Theorem: a compact Riemann surface(Lewittes 1964):

X: a compact Riemann surface
$w : S^k X \to \mathbb{C}^g$ Abelian map
ξ: Riemann constant
Γ: the lattice
$\mathcal{K}_X$: canonical divisor

1. Relation between the theta divisor $\Theta :=\mathrm{div}(\theta)$ and the standard theta divisor $w(S^{g-1}X)$ is given by

$$\Theta = w(S^{g-1}X) + \xi \mod \Gamma,$$

i.e., for $P_i \in X$, $\theta(w'(P_1, \dots, P_{g-1}) + \xi_s) = 0$.

2. Relation between the canonical divisor $\mathcal{K}_X$ and RC ξ is given by

$$w(\mathcal{K}_X) + 2\xi = 0 \mod \Gamma$$

From Theorem: a compact Riemann surface(Lewittes 196

If $\mathcal{K}_X = (2g-2)\infty$, $2\xi = 0 \bmod \Gamma$ and the Riemann constant ξ corresponds to a half period.
This case has been studied well.

However if $\mathcal{K}_X \neq (2g-2)\infty$, $2\xi \neq 0 \bmod \Gamma$ and the Riemann constant ξ **does not** correspond to a half period.
This case has never been studied yet as far as we know!!

Abelian map (AM) and Riemann constant (RC)

$\mathcal{K}_X = (2g-2-2)\infty + (B_1 + B_2) \neq (2g-2)\infty$

1. $w(\mathcal{K}_X) + 2\xi = 0 \mod \Gamma$ and $w(\mathcal{K}_X) \neq 0 \mod \Gamma$.
   $\to 2\xi \neq 0 \mod \Gamma$
   $\to$ Riemann const ξ is not a half period!
2. Relation between the theta divisor $\Theta :=$div(θ) and the standard theta divisor $w(S^{g-1}X)$:

$$\Theta = w(S^{g-1}X) + \xi \mod \Gamma,$$

   i.e., for $P_i \in X$, $\theta(w'(P_1, \dots, P_{g-1}) + \xi) = 0$.
3. The θ characteristics of a half period **cannot** represent RC ξ!
   $\to \sigma$ function **cannot** be defined well.
   $\to$ JIF may be complicate.

Abelian map (AM) and Riemann constant (RC)

$\mathcal{K}_X = (2g-2-2)\infty + (B_1 + B_2) \neq (2g-2)\infty$

$(B_1 + B_2) - s\infty \sim -2B_0 + 2\infty$ (due to div(y))

1. From $w(\mathcal{K}_X) + 2\xi = 0 \mod \Gamma$ and $w(\mathcal{K}_X) = -w(2B_0)$
   $\rightarrow -2w(B_0) + 2\xi = 0 \mod \Gamma$.
   $\rightarrow$ Riemann const ξ is not a half period but
   $\rightarrow$ $\boxed{\xi_s := \xi - w(B_0)}$ becomes a half period.
2. Relation between the theta divisor $\Theta :=$div(θ) and the standard theta divisor $w(S^{g-1}X)$:

   $$\Theta = w(S^{g-1}X) + \xi \mod \Gamma,$$

   i.e., for $P_i \in X$, $\theta(w'(P_1, \dots, P_{g-1}) + \xi) = 0$.
3. The θ characteristics of a half period **cannot** represent RC ξ!
   $\rightarrow$ σ function **cannot** be defined well.
   $\rightarrow$ JIF may be complicate.

Abelian map (AM) and Riemann constant (RC)

$\mathcal{K}_X = (2g-2-2)\infty + (B_1+B_2) \neq (2g-2)\infty$

$(B_1+B_2) - s\infty \sim -2B_0 + 2\infty$ (due to div(y))

1. From $w(\mathcal{K}_X) + 2\xi = 0 \mod \Gamma$ and $w(\mathcal{K}_X) = -w(2B_0)$
   $\rightarrow -2w(B_0) + 2\xi = 0 \mod \Gamma$.
   $\rightarrow$ Riemann const ξ is not a half period but
   $\rightarrow$ $\boxed{\xi_s := \xi - w(B_0)}$ becomes a half period.
2. Relation between the theta divisor $\Theta :=$div(θ) and the standard theta divisor $w(S^{g-1}X)$: $\Theta = w(S^{g-1}X) + \xi \mod \Gamma \rightarrow$

$$\Theta = \boxed{w(S^{g-1}X) + w(B_0)} + \boxed{\xi - w(B_0)} \mod \Gamma$$

   i.e., for $P_i \in X$, $\theta(w(P_1, \dots, P_{g-1}) + w(B_0) + \xi - w(B_0)) = 0$.
3. The θ characteristics of a half period **cannot** represent RC ξ!
   $\rightarrow$ σ function **cannot** be defined well.
   $\rightarrow$ JIF may be complicate.

Abelian map (AM) and Riemann constant (RC)

$\mathcal{K}_X = (2g-2-2)\infty + (B_1+B_2) \neq (2g-2)\infty$

$(B_1+B_2) - s\infty \sim -2B_0 + 2\infty$ (due to div(y))

1. From $w(\mathcal{K}_X) + 2\xi = 0 \mod \Gamma$ and $w(\mathcal{K}_X) = -w(2B_0)$
   $\to -2w(B_0) + 2\xi = 0 \mod \Gamma$.
   $\to$ Riemann const ξ is not a half period but
   $\to$ $\boxed{\xi_s := \xi - w(B_0)}$ becomes a half period.
2. Relation between the theta divisor $\Theta :=$div(θ) and the standard theta divisor $w(S^{g-1}X)$: $\Theta = w(S^{g-1}X) + \xi \mod \Gamma \to$

$$\Theta = w_s(S^{g-1}X) + \xi_s \mod \Gamma$$

   where $\boxed{w_s(P_1, \dots, P_k) := w(P_1, \dots, P_k) + w(B_0)}$
3. The θ characteristics of a half period **can** represent SRC ξ_s!
   $\to \sigma$ function can be defined well!
   $\to$ JIF may be simple.

Shifted AM (SAM) and shifted RC (SRC)

Theorem

1. *From* $w(\mathcal{K}_X) + 2\xi = 0 \ mod\ \Gamma$ *and* $w(\mathcal{K}_X) = -w(2B_0)$
   $\to -2w(B_0) + 2\xi = 0 \ mod\ \Gamma.$
   $\to$ *Riemann const (RC)* ξ *is not a half period but*
   $\to$ **Shifted RC** $\boxed{\xi_s := \xi - w(B_0)}$ **is a half period**
2. *By using* **the shifted Abelian map**, $w_s(D) := w(D) + w(B_0)$,
   $$\Theta = w_s(S^{g-1}X) + \xi_s \quad mod \quad \Gamma$$
   For $P_i \in X$, $\theta(w_s(P_1, \ldots, P_{g-1}) + \xi_s) = 0$
3. *The* θ *characteristics of a half period* **can** *represent SRC* ξ_s*!*

(KMP Archiv der Mathematik 2016)

Shifted Abelian map

Is the shifted Abelian map natural from algebraic view points?

Is the shifted Abelian map natural from algebraic view points?

Answer is Yes

Is the shifted Abelian map natural from algebraic view points?

Answer is Yes

It is related to the canonical divisor.

Truncated sequence

Truncated sequence $\hat{R} \subset R$

$$\hat{R} := \{h \in R \mid \exists \ell, \text{ such that } (h) - (B_0 + B_1 + B_2) + \ell\infty > 0\}$$

which is decomposed to $\hat{R} = \oplus_{i=0}\mathbb{C}\hat{\phi}_i$ as a $\mathbb{C}$-vect. sp.

Weierstrass sequence at ∞

$s\backslash i$	wt	0	1	2	3	4	5	6	7	8	9	10	11
ϕ_i	1	-	-	x	y	z	x^2	xy	xz	yz	x^2y	x^2z	
$\hat{\phi}_i$	-	-	-	-	y	z	-	xy	xz	yz	x^2y	x^2z	

Differential form

differentials of 1st and 2nd kinds

1. differentials of 1st kind (holomorphic 1-form)

$$\nu^{\mathrm{I}}{}_1 = \frac{\hat\phi_0 dx}{3yz} = \frac{dx}{3z}, \ \nu^{\mathrm{I}}{}_2 = \frac{\hat\phi_1 dx}{3yz} = \frac{dx}{3y} \quad H^0(X_2, \Omega^1) = \mathbb{C}\nu^{\mathrm{I}}{}_1 + \mathbb{C}\nu^{\mathrm{I}}{}_2$$

$((\hat\phi_0 : \hat\phi_1)$ **Canonical embedding of** $X_2)$

$$\mathcal{K}_X = 2g\infty - 2B_0 \sim 2(g-1)\infty + (B_1 + B_2)$$

2. Differentials of 2nd kind

$$\nu^{\mathrm{II}}{}_1 = \frac{-(2x + \lambda_1^{(2)})dx}{3z}, \quad \nu^{\mathrm{II}}{}_2 = \frac{-xdx}{3y}$$

Differentials

differentials of the 3rd kind

$$\Pi = \Sigma(P(x,y,z);(x_1,y_1,z_1)) - \Sigma(P(x,y,z);(x_2,y_2,z_2))$$

where $\Sigma(P(x,y,z);Q(x',y',z')) := \dfrac{(yz + y'z + yz')dx}{3(x-x')yz}$

half Periods

1. half Periods:

$$\omega' := \frac{1}{2}\left(\int_{\alpha_i} \nu^{\mathrm{I}}{}_j\right), \quad \omega'' := \frac{1}{2}\left(\int_{\beta_i} \nu^{\mathrm{I}}{}_j\right),$$

$$\eta' := \frac{1}{2}\left(\int_{\alpha_i} \nu^{\mathrm{II}}{}_j\right), \quad \eta'' := \frac{1}{2}\left(\int_{\beta_i} \nu^{\mathrm{II}}{}_j\right),$$

2. Lattice $\Gamma := \langle \omega', \omega'' \rangle_{\mathbb{Z}}$
3. Jacobi variety: $\kappa : \mathbb{C}^g \to \mathcal{J} := \mathbb{C}^g/\Gamma$
4. Legendre relation (Symplectic Str.)
$${}^t\omega'\eta'' - {}^t\omega''\eta' = \frac{\pi}{2} I_2.$$

Shifted Abelian Map

Abelian Map

$\hat{X}$: Abelianization of Path$X$ $\kappa' : \hat{X} \to X$
Let us assume that **they are unnormalized Abelian map.**

$$\tilde{w}(P) = \int_{\infty}^{P} \kappa'^{*}\nu^{\mathrm{I}}, \quad \tilde{w} : \hat{X} \to \mathbb{C}^g$$

$$w(P) := \kappa\left(\int_{\infty}^{P} \kappa'^{*}\nu^{\mathrm{I}}\right), \quad w : X \to \mathcal{J}$$

Shifted Abelian Map

For $P_1, P_2, \cdots, P_k \in S^k\tilde{X}$

$$\tilde{w}_s(P_1, P_2, \ldots, P_k) = \sum_{i}^{k} \tilde{w}(P_i) + \tilde{w}(B_0).$$

σ-function

σ-function

For $\begin{bmatrix}\delta'' \\ \delta'\end{bmatrix} \in (\mathbb{Z}/2)^{2g}$ which corresponds to ξ_s, we define the σ-function as an entire function over $\mathbb{C}^2$:

$$\sigma(u) = c\mathrm{e}^{-\frac{1}{2}\ {}^t u \eta' \omega'^{-1}\ u} \sum_{n \in \mathbb{Z}^{2g}} \mathrm{e}^{\left[\pi\sqrt{-1}\left\{\ {}^t(n+\delta'')\omega'^{-1}\omega''(n+\delta'') +\ {}^t(n+\delta'')(\omega'^{-1}u+\delta')\right\}\right.}$$

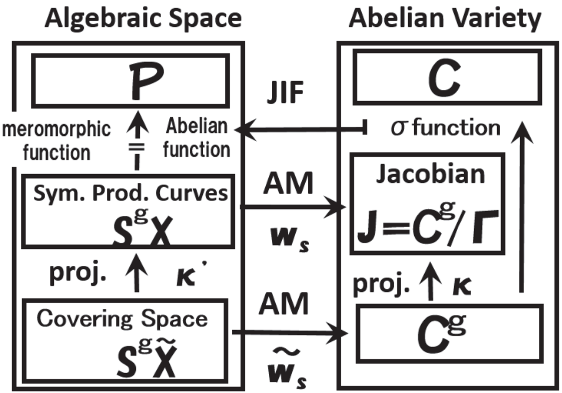
$c(\neq 0)$ is a constant complex number.

Properties of σ

1. Entire function over $\mathbb{C}^3$
2. Zeros: $\{\mathrm{div}\sigma\} \equiv \Theta = w(X_2)$
3. expansion $\sigma(u) = s_{\square\!\square} + \cdots$, $s_{\square\!\square}$: Schur polynomial of $\begin{smallmatrix}\square\\\square\end{smallmatrix}$.
4. Modular invariant for $\mathrm{Sp}(2, \mathbb{Z})$

Shifted AM (SAM) and shifted RC (SRC)

Cyclic singular Weierstrass normal form



Truncated W-Sequence

Meromorphic functions : For $\{\hat\phi_i\} = \hat R$ and $P_i = (x_i, y_i, z_i) \in X_2$

$$\psi_n(P_1, P_2, \dots, P_n) := \begin{vmatrix} \hat\phi_0(P_1) & \hat\phi_1(P_1) & \cdots & \hat\phi_{n-1}(P_1) \\ \hat\phi_0(P_2) & \hat\phi_1(P_2) & \cdots & \hat\phi_{n-1}(P_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat\phi_0(P_n) & \hat\phi_1(P_n) & \cdots & \hat\phi_{n-1}(P_n) \end{vmatrix} \quad \text{and}$$

$$\mu_n(P; P_1, \dots, P_n) = \frac{\psi_{n+1}(P_1, \dots, P_n, P)}{\psi_n(P_1, \dots, P_n)}$$

Additive Structure of shifted Abelian map

Additive Structure truncated Linear system	$\Leftrightarrow$	**Additive Structure shifted Abelian map**

Jacobi inversion formula

Theorem (JIF)

1. For $u = w(P_1, P_2)$

$$\mu_2(P; P_1, P_2) = xy_4 - \wp_{22}(u)y_4 + \wp_{21}(u)y_5.$$

or

$$\wp_{22}(u) = \frac{y_1 x_2 y_2 - y_2 x_1 y_1}{y_1 z_2 - z_1 y_2}, \wp_{21}(u) = \frac{z_1 x_2 y_2 - z_2 x_1 y_1}{y_1 z_2 - z_1 y_2}.$$

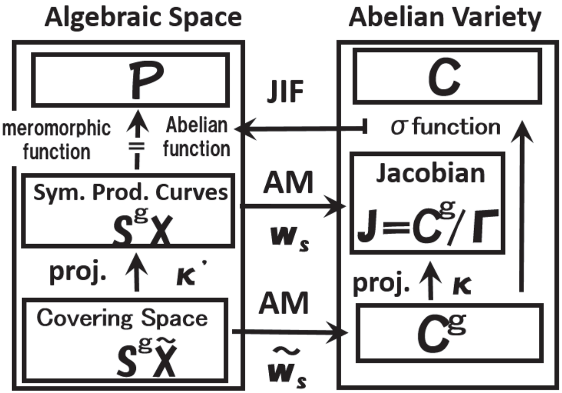
2. For $u = w(P_1)$

$$\mu_2(P; P_1) = y_5 - \frac{\sigma_1(u)}{\sigma_2(u)} y_4, \quad \frac{\sigma_1(u)}{\sigma_2(u)} = \frac{z_1}{y_1}$$

(Komeda-M 2013, Komeda-M-Previato 2014)

JIF

Even for singular curve, we have JIF, which was not written explicitly.



We have constructed that σ functions of X_3 and $X_2 \to X_{\rm sing}$.

$$X_3 : \quad y^3 = (x - b_0)(x - b_1)(x - b_2)(x - b_3)$$
$$X_{\rm sing} : y^3 = x^2(x - b_1)(x - b_2).$$

Thus we could consider the behavior of σ of the limit $\lim_{s\to 0} y^3 = x(x - s)(x - b_1)(x - b_2)$

An observation:

Sigma function of $X_3 \to X_{\rm sing}$

i,e., the limit of $b_3 \to b_0$ of X_3

$$\lim_{s\to 0}\{y^3 = x(x-s)(x-b_1)(x-b_2)\}$$

The holomorphic one-forms (diff. of the 1st kind)

X_3	$\to X_{\rm sing}$ $(s \to 0)$
$\nu^{\rm I}{}_1 = \dfrac{dx}{3y^2}$	$\dfrac{dx}{\sqrt[3]{x^4(x-b_1)^2(x-b_2)^2}}$: not holomorphic
$\nu^{\rm I}{}_2 = \dfrac{xdx}{3y^2}$	$\dfrac{dx}{3z} = \nu^{\rm I}{}_1$ of X_2
$\nu^{\rm I}{}_3 = \dfrac{dx}{3y}$	$\dfrac{dx}{3y} = \nu^{\rm I}{}_2$ of X_2

One form $s \to 0$

Diff. of the 2nd kind

X_3	$\to X_{\mathrm{sing}}$ $(s \to 0)$
${\nu^{\mathrm{II}}}_1 = -\dfrac{(5x^2 + 2\lambda_3 x + \lambda_2)dx}{3y^2}$	$\dfrac{(5x^2 + 2\lambda_3 x + \lambda_2)dx}{\sqrt[3]{x^4(x-b_1)^2(x-b_2)^2}}$
${\nu^{\mathrm{II}}}_2 = \dfrac{-2xdx}{3y}$	$\dfrac{-2xdx}{3y} = {\nu^{\mathrm{II}}}_1$ of X_2
${\nu^{\mathrm{II}}}_3 = -\dfrac{x^2dx}{3y}$	$-\dfrac{xdx}{3z} = {\nu^{\mathrm{II}}}_2$ of X_2

Contours of Period Integral

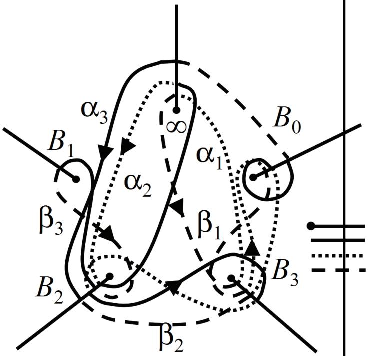
For $\omega_{a,b} := \int_{\infty}^{B_a} \nu^{\mathrm{I}}{}_b, (a = 0, 1, 2, 3)$

$$\omega'_3 = \frac{1}{2}\Big((1-\hat{\zeta}_3^2)\omega_2 + \hat{\zeta}_3^2(1-\hat{\zeta}_3^2)\omega_3 + \hat{\zeta}_3(1-\hat{\zeta}_3)\omega_0\Big),$$

$$\omega'_a = \frac{1}{2}\hat{\zeta}_3^{2-a}(\hat{\zeta}_3 - 1)\omega_{4-a}, \quad (a = 1, 2),$$

$$\omega''_c = \frac{1}{2}\hat{\zeta}_3^{2-c}(\hat{\zeta}_3 - 1) \times(\omega_{4-c} - \omega_{5-c \bmod 4}), \quad (c = 1, 2, 3).$$

$y^3 = x^4 + \lambda_3 x^3 + \cdots + \lambda_0$



Integrals $s \to 0$

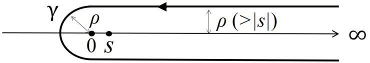
Integrals (Kazuhiko Aomoto, Feb., 2018)

When $\min\{|b_1|, |b_2|\} > s > 0$, $Im(b_1)$ and $Im(b_2) > 0$,

$$\int_\infty^0 \frac{dx}{\sqrt[3]{|x^2(x-s)^2|(x-b_1)^2)(x-b_2)^2}} = s^{-1/3} f(s^{1/3}),$$

$$\int_\gamma \frac{dx}{\sqrt[3]{x^2(x-s)^2(x-b_1)^2(x-b_2)^2}} = g(s),$$

where $f(t)$ and $g(t)$ are regular functions with respect to t around $t = 0$, and the contour γ is given by



Integrals $s \to 0$

$$\omega_{a,b} := \int_{\infty}^{B_a} \nu^{\mathrm{I}}{}_{b}, (a = 0, 1, 2, 3, b = 1, 2, 3)$$

X_3	$\to X_{\mathrm{sing}}$
$\begin{pmatrix} \omega_{0,1} & \omega_{1,1} & \omega_{2,1} & \omega_{3,1} \\ \omega_{0,2} & \omega_{1,2} & \omega_{2,2} & \omega_{3,2} \\ \omega_{0,3} & \omega_{1,3} & \omega_{2,3} & \omega_{3,3} \end{pmatrix}$	$\begin{pmatrix} * & * & * & \infty \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$

*: finite value

X_3	$X_{\mathrm{sing}} \leftarrow X_2$
(ω')	$\begin{pmatrix} \infty & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$

X_3	$X_{\mathrm{sing}} \leftarrow X_2$
(ω'')	$\begin{pmatrix} \infty & \infty & * \\ * & * & * \\ * & * & * \end{pmatrix}$

Integrals: by assuming $\omega_{3,1} = s^{-\ell} A_1(s^\ell)$ $\ell > 0$

X_3	X_3 $(\lvert s \rvert \ll 1)$ $s \longrightarrow 0$
ω'	$\begin{pmatrix} A_1 s^{-\ell} & * & * \\ * & \omega'_{X_2,11} & \omega'_{X_2,12} \\ * & \omega'_{X_2,21} & \omega'_{X_2,22} \end{pmatrix}$
ω'^{-1}	$\begin{pmatrix} A_{11} s^\ell & A_{12} s^\ell & A_{13} s^\ell \\ A_{21} s^\ell & * & * \\ A_{31} s^\ell & * & * \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \omega'^{-1}_{X_2} \end{pmatrix}$
ω''	$\begin{pmatrix} A'_1 s^{-\ell} & A'_2 s^{-\ell} & * \\ * & * & * \\ * & * & * \end{pmatrix} \to \begin{pmatrix} \infty & \infty & * \\ * & \omega''_{X_2,11} & \omega''_{X_2,12} \\ * & \omega''_{X_2,21} & \omega''_{X_2,22} \end{pmatrix}$

Integrals: by assuming $\omega_{3,1} = s^{-\ell} A_1(s^\ell)$ $\ell > 0$

X_3	X_3 $(|s| \ll 1)$ $s \longrightarrow 0$
$\omega'^{-1}\omega''$	$\begin{pmatrix} * & * & A'_{13}s^\ell \\ * & * & * \\ * & * & * \end{pmatrix} \to \begin{pmatrix} * & * & 0 \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}$

*: finite value and $\ell > 0$. $A_i(t)$: const + holo. of t.

Space

$$\mathbb{C}^3 \ni \omega'^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ * \\ * \end{pmatrix} = \omega'^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in \mathbb{C}^2$$

Observation & Conjecture

σ function

$$\sigma(u) := \gamma_0 \mathrm{e}^{-\frac{1}{2} u^t \omega'^{-1} {}^t\eta' u} \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \left(\frac{1}{2} \omega'^{-1} u; \omega'^{-1} \omega'' \right).$$

Observation & Conjecture

For $u = w(P_1, P_2, P_3)$

$$\sigma_{X_3}(u) = u_1 - u_2^2 u_3 + \frac{6}{5!} u_3^5 + \cdots$$

whereas for X_2, ($u_3 \to v_2$ and $u_2 \to v_1$)

$$\sigma_{X_2}(v) = v_1 - v_2^2 + \cdots$$

Observation & Conjecture

Observation & Conjecture around ∞^3

$$u_1 = \frac{1}{5}(t_1^5 + t_2^5 + t_3^5)(1 + O(t_a)),$$
$$u_2 = \frac{1}{2}(t_1^2 + t_2^2 + t_3^2)(1 + O(t_a)),$$
$$u_3 = \frac{1}{1}(t_1 + t_2 + t_3)(1 + O(t_a)),$$

$$\sigma_{X_3}(u) = s_{\yng(3,1,1)} + \cdots,$$

$$s_{\yng(3,1,1)}(u_1, u_2, u_3) = \left.\frac{\begin{vmatrix} t_1^5 & t_2^5 & t_3^5 \\ t_1^2 & t_2^2 & t_3^2 \\ t_1^1 & t_2^1 & t_3^1 \end{vmatrix}}{\begin{vmatrix} t_1^2 & t_2^2 & t_3^2 \\ t_1 & t_2 & t_3 \\ 1 & 1 & 1 \end{vmatrix}}\right|_{u_1 := \frac{1}{5}\sum_a t_a^5, u_2 := \frac{1}{3}\sum_a t_a^3, u_3 := \sum_a t_a}$$

Observation & Conjecture around ∞^2

$$v_1 = \frac{1}{2}(t_1^2 + t_2^2)(1 + O(t_a)),$$
$$v_2 = \frac{1}{1}(t_1 + t_2)(1 + O(t_a)),$$

$$\sigma_{X_2}(u) = s_{\square\atop\square} + \cdots, \quad s_{\square\atop\square}(v_1, v_2) = \frac{\begin{vmatrix} t_1^2 & t_2^2 \\ t_1^1 & t_2^1 \end{vmatrix}}{\begin{vmatrix} t_1 & t_2 \\ 1 & 1 \end{vmatrix}}\Bigg|_{u_2 := \frac{1}{2}(t_1^2 + t_2^2), u_3 := t_1 + t_2}$$

Observation & Conjecture

Observation & Conjecture

$$s_{(3,1,1)} = \frac{\begin{vmatrix} t_1^5 & t_2^5 & t_3^5 \\ t_1^2 & t_2^2 & t_3^2 \\ t_1^1 & t_2^1 & t_3^1 \end{vmatrix}}{\begin{vmatrix} t_1^2 & t_2^2 & t_3^2 \\ t_1 & t_2 & t_3 \\ 1 & 1 & 1 \end{vmatrix}} \qquad s_{(1,1)}(v_1, v_2) = \frac{\begin{vmatrix} t_1^2 & t_2^2 \\ t_1^1 & t_2^1 \end{vmatrix}}{\begin{vmatrix} t_1 & t_2 \\ 1 & 1 \end{vmatrix}}$$

$$s_{(1,1)}(t_1, t_2) = \frac{\partial^3}{\partial t_3^3} s_{(3,1,1)}(t_1, t_2, t_3)\Big|_{t_3=0}$$

Observation & Conjecture

Observation & Conjecture

It is conjectured that

$$\sigma_{X_2}(v) = \lim_{s\to 0}\left(\lim_{P_3\to B_0}\left[\frac{\partial^3}{\partial u_3(P_3)^3}\sigma_{X_3}(u)\right]_{v_1=u_2,v_2=u_3}\right)$$

Thank you for your attention!