Recent progress on topology of plane curves: A quick trip
Part I:
Introduction: Fundamental Group and Braid Monodromy

Enrique ARTAL BARTOLO

Departamento de Matemáticas, IUMA
Universidad de Zaragoza

Branched Coverings in Tokyo - March 7-10, 2011

Contents

Contents

Contents

Contents

Main questions

Knot Theory

Study the relative topology of $(\mathbb{S}^3, K)$ K a link: codimension 2

Main questions

Plane curves

Study the relative topology of $(\mathbb{P}^2, \mathcal{C})$ $\mathcal{C}$ an algebraic curve: codimension 2, $\mathcal{C} := \{[x:y:z] \in \mathbb{P}^2 \mid F(x,y,z) = 0\}$, $F \in \mathbb{C}[x,y,z]$ homogeneous of degree d

Main questions

Plane curves

Study the relative topology of $(\mathbb{P}^2, \mathcal{C})$ $\mathcal{C}$ an algebraic curve: codimension 2, $\mathcal{C} := \{[x:y:z] \in \mathbb{P}^2 \mid F(x,y,z) = 0\}$, $F \in \mathbb{C}[x,y,z]$ homogeneous of degree d

Affine and projective curves

- $\mathcal{C} \subset \mathbb{C}^2$ affine curve $\Longrightarrow (\mathbb{P}^2, \bar{\mathcal{C}} \cup L_\infty)$, $(x,y) \subset [x:y:1]$.
- The completion is not unique!: $xz = 0$ and $(xz - y^2)z = 0$ are completions of the *same* affine curve.
- $\mathcal{C} \subset \mathbb{P}^2$ projective curve $\Longrightarrow (\mathbb{C}^2, \mathcal{C}^{\text{aff}})$, $\mathcal{C}^{\text{aff}} := \mathcal{C} \setminus L_\infty$, $L_\infty \pitchfork \mathcal{C}$, $\mathcal{C} = \{F(x,y,z) = 0\}, \mathcal{C}^{\text{aff}} = \{f(x,y) = 0\}$, $f(x,y) := F(x,y,1) = \sum_{j=0}^{d} f_j(x,y)$, $f_d(x,y)$ product of d distinct linear factors.

Combinatorial Invariants

$\mathcal{C} := \mathcal{C}_1 \cup \ldots \mathcal{C}_r$ (irreducible components), $\mathcal{C}_0 =: L_\infty$ transversal line
$\mathcal{C}^{\text{aff}} := \mathcal{C}_1^{\text{aff}} \cup \ldots \mathcal{C}_r^{\text{aff}}$

- $(d_1, \ldots, d_r)$, $d_i := \deg \mathcal{C}_i$.

Combinatorial Invariants

$\mathcal{C} := \mathcal{C}_1 \cup \ldots \mathcal{C}_r$ (irreducible components), $\mathcal{C}_0 =: L_\infty$ transversal line
$\mathcal{C}^{\text{aff}} := \mathcal{C}_1^{\text{aff}} \cup \ldots \mathcal{C}_r^{\text{aff}}$

- $(d_1, \ldots, d_r)$, $d_i := \deg \mathcal{C}_i$.
- Topological types of singular points

Combinatorial Invariants

$\mathcal{C} := \mathcal{C}_1 \cup \ldots \mathcal{C}_r$ (irreducible components), $\mathcal{C}_0 =: L_\infty$ transversal line
$\mathcal{C}^{\text{aff}} := \mathcal{C}_1^{\text{aff}} \cup \ldots \mathcal{C}_r^{\text{aff}}$

- $(d_1, \ldots, d_r)$, $d_i := \deg \mathcal{C}_i$.
- Topological types of singular points
- $\sigma : X \to \mathbb{P}^2$ minimal resolution of $\mathcal{C}$, Γ dual graph of the divisor $\sigma^{-1}(\mathcal{C})$ with the following weights:

Combinatorial Invariants

$\mathcal{C} := \mathcal{C}_1 \cup \ldots \mathcal{C}_r$ (irreducible components), $\mathcal{C}_0 =: L_\infty$ transversal line
$\mathcal{C}^{\text{aff}} := \mathcal{C}_1^{\text{aff}} \cup \ldots \mathcal{C}_r^{\text{aff}}$

- $(d_1, \ldots, d_r)$, $d_i := \deg \mathcal{C}_i$.
- Topological types of singular points
- $\sigma : X \to \mathbb{P}^2$ minimal resolution of $\mathcal{C}$, Γ dual graph of the divisor $\sigma^{-1}(\mathcal{C})$ with the following weights:
  - Red colors for the strict transforms of $\mathcal{C}_i$

Combinatorial Invariants

$\mathcal{C} := \mathcal{C}_1 \cup \ldots \mathcal{C}_r$ (irreducible components), $\mathcal{C}_0 =: L_\infty$ transversal line
$\mathcal{C}^{\text{aff}} := \mathcal{C}_1^{\text{aff}} \cup \ldots \mathcal{C}_r^{\text{aff}}$

- $(d_1, \ldots, d_r)$, $d_i := \deg \mathcal{C}_i$.
- Topological types of singular points
- $\sigma : X \to \mathbb{P}^2$ minimal resolution of $\mathcal{C}$, Γ dual graph of the divisor $\sigma^{-1}(\mathcal{C})$ with the following weights:
  - Red colors for the strict transforms of $\mathcal{C}_i$
  - Each red vertex E is weighted by (g_E, e_E) where g_E is the genus of E and $e_E := (E \cdot E)_X$.

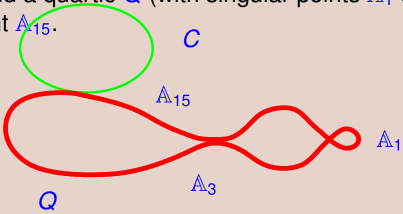
Combinatorial Invariants

$\mathcal{C} := \mathcal{C}_1 \cup \dots \mathcal{C}_r$ (irreducible components), $\mathcal{C}_0 =: L_\infty$ transversal line
$\mathcal{C}^{\text{aff}} := \mathcal{C}_1^{\text{aff}} \cup \dots \mathcal{C}_r^{\text{aff}}$

- $(d_1, \dots, d_r)$, $d_i := \deg \mathcal{C}_i$.
- Topological types of singular points
- $\sigma : X \to \mathbb{P}^2$ minimal resolution of $\mathcal{C}$, Γ dual graph of the divisor $\sigma^{-1}(\mathcal{C})$ with the following weights:
  - Red colors for the strict transforms of $\mathcal{C}_i$
  - Each red vertex E is weighted by (g_E, e_E) where g_E is the genus of E and $e_E := (E \cdot E)_X$.

Example

A smooth conic C and a quartic Q (with singular points $\mathbb{A}_1$ and $\mathbb{A}_3$) intersecting at a point $\mathbb{A}_{15}$.



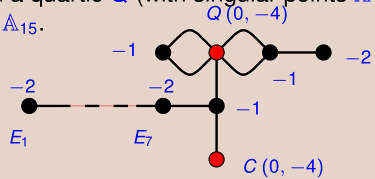
Combinatorial Invariants

$\mathcal{C} := \mathcal{C}_1 \cup \ldots \mathcal{C}_r$ (irreducible components), $\mathcal{C}_0 =: L_\infty$ transversal line
$\mathcal{C}^{\text{aff}} := \mathcal{C}_1^{\text{aff}} \cup \ldots \mathcal{C}_r^{\text{aff}}$

- $(d_1, \ldots, d_r)$, $d_i := \deg \mathcal{C}_i$.
- Topological types of singular points
- $\sigma : X \to \mathbb{P}^2$ minimal resolution of $\mathcal{C}$, Γ dual graph of the divisor $\sigma^{-1}(\mathcal{C})$ with the following weights:
  - Red colors for the strict transforms of $\mathcal{C}_i$
  - Each red vertex E is weighted by (g_E, e_E) where g_E is the genus of E and $e_E := (E \cdot E)_X$.

Example

A smooth conic C and a quartic Q (with singular points $\mathbb{A}_1$ and $\mathbb{A}_3$) intersecting at a point $\mathbb{A}_{15}$.



Combinatorial Invariants

$\mathcal{C} := \mathcal{C}_1 \cup \ldots \mathcal{C}_r$ (irreducible components), $\mathcal{C}_0 =: L_\infty$ transversal line
$\mathcal{C}^{\text{aff}} := \mathcal{C}_1^{\text{aff}} \cup \ldots \mathcal{C}_r^{\text{aff}}$

- $(d_1, \ldots, d_r)$, $d_i := \deg \mathcal{C}_i$.
- Topological types of singular points
- $\sigma : X \to \mathbb{P}^2$ minimal resolution of $\mathcal{C}$, Γ dual graph of the divisor $\sigma^{-1}(\mathcal{C})$ with the following weights:
  - Red colors for the strict transforms of $\mathcal{C}_i$
  - Each red vertex E is weighted by (g_E, e_E) where g_E is the genus of E and $e_E := (E \cdot E)_X$.

Example

A smooth conic C and a quartic Q (with singular points $\mathbb{A}_1$ and $\mathbb{A}_3$) intersecting at a point $\mathbb{A}_{15}$.

The weighted uncolored dual graph Γ codifies the topology of the 3-dimensional graph manifold $\partial T(\mathcal{C})$, where $T(\mathcal{C})$ is a closed regular neighbourhood of $\mathcal{C}$ in $\mathbb{P}^2$

Combinatorial Invariants

$\mathcal{C} := \mathcal{C}_1 \cup \ldots \mathcal{C}_r$ (irreducible components), $\mathcal{C}_0 =: L_\infty$ transversal line
$\mathcal{C}^{\text{aff}} := \mathcal{C}_1^{\text{aff}} \cup \ldots \mathcal{C}_r^{\text{aff}}$

- $(d_1, \ldots, d_r)$, $d_i := \deg \mathcal{C}_i$.
- Topological types of singular points
- $\sigma : X \to \mathbb{P}^2$ minimal resolution of $\mathcal{C}$, Γ dual graph of the divisor $\sigma^{-1}(\mathcal{C})$ with the following weights:
  - Red colors for the strict transforms of $\mathcal{C}_i$
  - Each red vertex E is weighted by (g_E, e_E) where g_E is the genus of E and $e_E := (E \cdot E)_X$.

Example

A smooth conic C and a quartic Q (with singular points $\mathbb{A}_1$ and $\mathbb{A}_3$) intersecting at a point $\mathbb{A}_{15}$.

The weighted colored dual graph Γ codifies the topology of the pair $(T(\mathcal{C}), \mathcal{C})$.

Topological Invariants

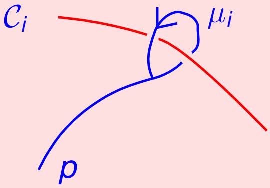
- Any topological invariant of $\mathbb{P}^2 \setminus \mathcal{C}$.

Topological Invariants

- Any topological invariant of $\mathbb{P}^2 \setminus \mathcal{C}$.
- $G_{\mathcal{C}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{C}; p)$.

Topological Invariants

- Any topological invariant of $\mathbb{P}^2 \setminus \mathcal{C}$.
- $G_{\mathcal{C}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{C}; p)$.
- Meridians μ_i of an irreducible component $\mathcal{C}_i$: a conjugacy class.



Topological Invariants

- Any topological invariant of $\mathbb{P}^2 \setminus \mathcal{C}$.
- $G_{\mathcal{C}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{C}; p)$.
- $G_{\mathcal{C}}$ with peripheral structure (conjugacy class of meridians of the irreducible components).
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ (A a coefficient ring)

Topological Invariants

- Any topological invariant of $\mathbb{P}^2 \setminus \mathcal{C}$.
- $G_{\mathcal{C}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{C}; p)$.
- $G_{\mathcal{C}}$ with peripheral structure (conjugacy class of meridians of the irreducible components).
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ (A a coefficient ring)
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ with peripheral structure

Topological Invariants

- Any topological invariant of $\mathbb{P}^2 \setminus \mathcal{C}$.
- $G_{\mathcal{C}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{C}; p)$.

- $G_{\mathcal{C}}$ with peripheral structure (conjugacy class of meridians of the irreducible components).
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ (A a coefficient ring)
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ with peripheral structure
- Alexander-like invariants: Alexander polynomial, characteristic varieties, twisted Alexander polynomials, Alexander-Oka polynomials

Topological Invariants

- Any topological invariant of $\mathbb{P}^2 \setminus \mathcal{C}$.
- $G_{\mathcal{C}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{C}; p)$.

- $G_{\mathcal{C}}$ with peripheral structure (conjugacy class of meridians of the irreducible components).
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ (A a coefficient ring)
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ with peripheral structure
- Alexander-like invariants: Alexander polynomial, characteristic varieties, twisted Alexander polynomials, Alexander-Oka polynomials
- Specific invariants: lattice embeddings for sextics with simple singularities (Degtyarev, Shimada).

Topological Invariants

- Any topological invariant of $\mathbb{P}^2 \setminus \mathcal{C}$.
- $G_{\mathcal{C}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{C}; p)$.

- $G_{\mathcal{C}}$ with peripheral structure (conjugacy class of meridians of the irreducible components).
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ (A a coefficient ring)
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ with peripheral structure
- Alexander-like invariants: Alexander polynomial, characteristic varieties, twisted Alexander polynomials, Alexander-Oka polynomials
- Specific invariants: lattice embeddings for sextics with simple singularities (Degtyarev, Shimada).
- Braid monodromy.

Topological Invariants

- Any topological invariant of $\mathbb{P}^2 \setminus \mathcal{C}$.
- $G_{\mathcal{C}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{C}; p)$.
- $G_{\mathcal{C}}$ with peripheral structure (conjugacy class of meridians of the irreducible components).
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ (A a coefficient ring)
- $H^*(\mathbb{P}^2 \setminus \mathcal{C}; A)$ with peripheral structure
- Alexander-like invariants: Alexander polynomial, characteristic varieties, twisted Alexander polynomials, Alexander-Oka polynomials
- Specific invariants: lattice embeddings for sextics with simple singularities (Degtyarev, Shimada).
- Braid monodromy.

Definition

Two curves form a Zariski pair if they are combinatorially equivalent but not topologically equivalent.

Affine curves

- $\mathcal{C}^{\text{aff}} := \{f(x,y)=0\} \subset \mathbb{C}^2$ affine curve of degree d, f monic in y:

$$f(x,y) = y^n + \sum_{j=1}^{n} a_j(x) y^{n-j}$$

Affine curves

- $\mathcal{C}^{\text{aff}} := \{f(x,y)=0\} \subset \mathbb{C}^2$ affine curve of degree d, f monic in y:

$$f(x,y) = y^n + \sum_{j=1}^{n} a_j(x) y^{n-j}$$

- Interpret $\mathcal{C}$ as $\tilde{f}: \mathbb{C} \to \mathbb{C}[y]_n := \{g \in \mathbb{C}[y] \mid g \text{ monic, } \deg g = n\}$

Affine curves

- $\mathcal{C}^{\text{aff}} := \{f(x,y)=0\} \subset \mathcal{C}^2$ affine curve of degree d, f monic in y:

$$f(x,y) = y^n + \sum_{j=1}^{n} a_j(x) y^{n-j}$$

- Interpret $\mathcal{C}$ as $\tilde{f} : \mathbb{C} \to \mathbb{C}[y]_n := \{g \in \mathbb{C}[y] \mid g \text{ monic, } \deg g = n\}$
- $\Delta_n := \{g \in \mathbb{C}[y]_n \mid g \text{ has multiple roots}\}, \quad \Delta_f := \tilde{f}^{-1}(\Delta_n).$

Affine curves

- $\mathcal{C}^{\text{aff}} := \{f(x,y)=0\} \subset \mathcal{C}^2$ affine curve of degree d, f monic in y:

$$f(x,y) = y^n + \sum_{j=1}^{n} a_j(x) y^{n-j}$$

- Interpret $\mathcal{C}$ as $\tilde{f} : \mathbb{C} \to \mathbb{C}[y]_n := \{g \in \mathbb{C}[y] \mid g \text{ monic, } \deg g = n\}$
- $\Delta_n := \{g \in \mathbb{C}[y]_n \mid g \text{ has multiple roots}\}, \quad \Delta_f := \tilde{f}^{-1}(\Delta_n)$.
- $\tilde{f}_| : \mathbb{C} \setminus \Delta_f \to \mathbb{C}[y]_n \setminus \Delta_n$ induces
  $\nabla : \mathbb{F}_r := \pi_1(\mathbb{C} \setminus \Delta_f; x_0) \to \pi_1(\mathbb{C}[y]_n \setminus \Delta_n; f(x_0, y)) =: \mathbb{B}_n$,
  $\mathbb{F}_r = \langle \alpha_1, \ldots, \alpha_r \mid - \rangle$

Affine curves

- $\mathcal{C}^{\text{aff}} := \{f(x,y)=0\} \subset \mathcal{C}^2$ affine curve of degree d, f monic in y:

$$f(x,y) = y^n + \sum_{j=1}^{n} a_j(x) y^{n-j}$$

- Interpret $\mathcal{C}$ as $\tilde{f} : \mathbb{C} \to \mathbb{C}[y]_n := \{g \in \mathbb{C}[y] \mid g \text{ monic}, \ \deg g = n\}$
- $\Delta_n := \{g \in \mathbb{C}[y]_n \mid g \text{ has multiple roots}\}, \quad \Delta_f := \tilde{f}^{-1}(\Delta_n)$.
- $\mathbb{B}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \middle| \underset{1 \leq i < n-1}{\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}}, \underset{0 \leq i-1 < j < n}{\sigma_i \sigma_j = \sigma_j \sigma_i}, \right\rangle$ acts on $\mathbb{F}_n = \langle \mu_1, \ldots, \mu_n \mid - \rangle$ by

Affine curves

- $\mathcal{C}^{\text{aff}} := \{f(x,y)=0\} \subset \mathcal{C}^2$ affine curve of degree d, f monic in y:

$$f(x,y) = y^n + \sum_{j=1}^{n} a_j(x) y^{n-j}$$

- Interpret $\mathcal{C}$ as $\tilde{f}: \mathbb{C} \to \mathbb{C}[y]_n := \{g \in \mathbb{C}[y] \mid g \text{ monic, } \deg g = n\}$
- $\Delta_n := \{g \in \mathbb{C}[y]_n \mid g \text{ has multiple roots}\}, \quad \Delta_f := \tilde{f}^{-1}(\Delta_n)$.
- $\mathbb{B}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \middle| \underset{1\leq i<n-1}{\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}}, \underset{0\leq i-1<j<n}{\sigma_i\sigma_j = \sigma_j\sigma_i}, \right\rangle$ acts on $\mathbb{F}_n = \langle \mu_1, \ldots, \mu_n \mid - \rangle$ by

$$\mu_i^{\sigma_j} := \begin{cases} \mu_{i+1} & \text{if } i = j \\ \mu_{i+1}\mu_i\mu_{i+1}^{-1} & \text{if } i = j+1 \\ \mu_i & \text{otherwise} \end{cases}$$

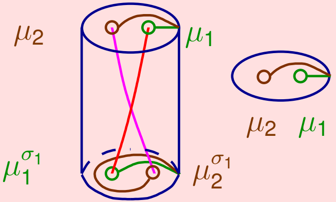
Affine curves

- $\mathcal{C}^{\text{aff}} := \{f(x,y)=0\} \subset \mathcal{C}^2$ affine curve of degree d, f monic in y:

$$f(x,y) = y^n + \sum_{j=1}^{n} a_j(x) y^{n-j}$$

- Interpret $\mathcal{C}$ as $\tilde{f} : \mathbb{C} \to \mathbb{C}[y]_n := \{g \in \mathbb{C}[y] \mid g \text{ monic}, \deg g = n\}$
- $\Delta_n := \{g \in \mathbb{C}[y]_n \mid g \text{ has multiple roots}\}, \quad \Delta_f := \tilde{f}^{-1}(\Delta_n)$.
- $\mathbb{B}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \middle| \underset{1\leq i<n-1}{\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}}, \underset{0\leq i-1<j<n}{\sigma_i\sigma_j = \sigma_j\sigma_i} \right\rangle$ acts on $\mathbb{F}_n = \langle \mu_1, \ldots, \mu_n \mid - \rangle$ by

$$\mu_i^{\sigma_j} := \begin{cases} \mu_{i+1} & \text{if } i = j \\ \mu_{i+1}\mu_i\mu_{i+1}^{-1} & \text{if } i = j+1 \\ \mu_i & \text{otherwise} \end{cases}$$



Versions of Zariski-van Kampen Theorem

Zariski-van Kampen Theorem

The group $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\mathrm{aff}})$ *admits the following finite presentation:*

Generators $\mu_1, \ldots, \mu_n$

Relators $\mu_i^{\nabla(\alpha_j)} = \mu_i$, $1 \leq i < n$, $1 \leq j \leq r$.

Versions of Zariski-van Kampen Theorem

Zariski-van Kampen Theorem

The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ admits the following finite presentation:

Generators $\mu_1, \ldots, \mu_d$

Relators $\mu_i^{\nabla(\alpha_j)} = \mu_i$, $1 \leq i < d$, $1 \leq j \leq r$, $\mu_d \ldots \mu_1 = 1$,

if $\mathcal{C}^{\text{aff}}$ is a generic affine curve associated to $\mathcal{C}$

Versions of Zariski-van Kampen Theorem

Zariski-van Kampen Theorem

The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ *admits the following finite presentation:*

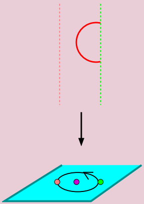
Generators $\mu_1, \dots, \mu_d$

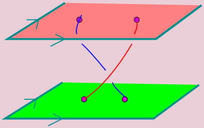
Relators $\mu_i^{\nabla(\alpha_j)} = \mu_i$, $1 \leq i < d$, $1 \leq j \leq r$, $\mu_d \dots \mu_1 = 1$,

if $\mathcal{C}^{\text{aff}}$ *is a generic affine curve associated to* $\mathcal{C}$

Local examples

$y^2 - x = 0$, $\sigma_1 \Longrightarrow \mu_1 = \mu_2$





Versions of Zariski-van Kampen Theorem

Zariski-van Kampen Theorem

The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ *admits the following finite presentation:*

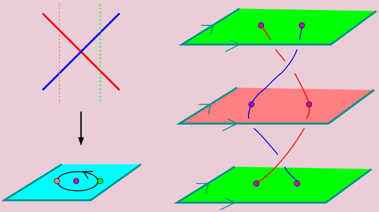
Generators $\mu_1, \ldots, \mu_d$

Relators $\mu_i^{\nabla(\alpha_j)} = \mu_i$, $1 \leq i < d$, $1 \leq j \leq r$, $\mu_d \ldots \mu_1 = 1$,

if $\mathcal{C}^{\text{aff}}$ *is a generic affine curve associated to* $\mathcal{C}$

Local examples

$y^2 - x^2 = 0$, $\sigma_1^2 \Longrightarrow$
$[\mu_1, \mu_2] = 1$



Versions of Zariski-van Kampen Theorem

Zariski-van Kampen Theorem

The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ *admits the following finite presentation:*

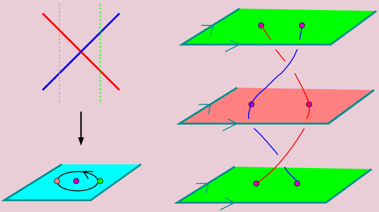
Generators $\mu_1, \ldots, \mu_d$

Relators $\mu_i^{\nabla(\alpha_j)} = \mu_i$, $1 \leq i < d$, $1 \leq j \leq r$, $\mu_d \ldots \mu_1 = 1$,

if $\mathcal{C}^{\text{aff}}$ *is a generic affine curve associated to* $\mathcal{C}$

Local examples

$y^2 - x^3 = 0$, $\sigma_1^3 \Longrightarrow$
$\mu_1\mu_2\mu_1 = \mu_2\mu_1\mu_2$



Versions of Zariski-van Kampen Theorem

Zariski-van Kampen Theorem

The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ *admits the following finite presentation:*

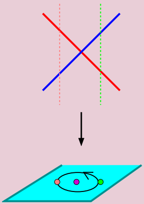
Generators $\mu_1, \dots, \mu_d$

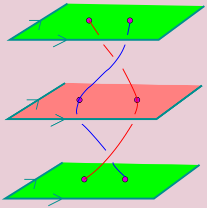
Relators $\mu_i^{\nabla(\alpha_j)} = \mu_i$, $1 \leq i < d$, $1 \leq j \leq r$, $\mu_d \dots \mu_1 = 1$,

if $\mathcal{C}^{\mathrm{aff}}$ *is a generic affine curve associated to* $\mathcal{C}$

Local examples

$$y^2 - x^k = 0,\ \sigma_1^k \Longrightarrow \underbrace{(\mu_1 \mu_2 \dots)}_{k \text{ factors}} = \underbrace{(\mu_2 \mu_1 \dots)}_{k \text{ factors}}$$





Versions of Zariski-van Kampen Theorem

Zariski-van Kampen Theorem

The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ admits the following finite presentation:

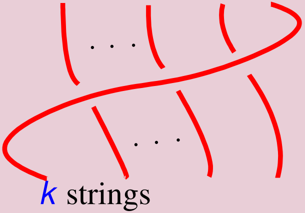
Generators $\mu_1, \dots, \mu_d$

Relators $\mu_i^{\nabla(\alpha_j)} = \mu_i$, $1 \leq i < d$, $1 \leq j \leq r$, $\mu_d \dots \mu_1 = 1$,

if $\mathcal{C}^{\text{aff}}$ is a generic affine curve associated to $\mathcal{C}$

Local examples

$y^k - x = 0$, $\sigma_{k-1} \dots \sigma_1 \Longrightarrow$
$\mu_1 = \mu_2 = \dots = \mu_k$



Consequences and comments

- $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) = \langle \mu_1, \dots, \mu_r \mid \sum_{j=1}^r d_i \mu_i = 0 \rangle \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/e\mathbb{Z}$, where $e := \gcd(d_1, \dots, d_r)$.

Consequences and comments

- $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) = \langle \mu_1, \ldots, \mu_r \mid \sum_{j=1}^{r} d_i \mu_i = 0 \rangle \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/e\mathbb{Z}$, where $e := \gcd(d_1, \ldots, d_r)$.
- $G_{\mathcal{C}_1 \cup \mathcal{C}_2} \twoheadrightarrow G_{\mathcal{C}_i}$ (the kernel is generated by meridians of $\mathcal{C}_i$, following a result from Fujita).

Consequences and comments

- $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) = \langle \mu_1, \ldots, \mu_r \mid \sum_{j=1}^r d_i \mu_i = 0 \rangle \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/e\mathbb{Z}$, where $e := \gcd(d_1, \ldots, d_r)$.
- $G_{\mathcal{C}_1 \cup \mathcal{C}_2} \twoheadrightarrow G_{\mathcal{C}_i}$ (the kernel is generated by meridians of $\mathcal{C}_i$, following a result from Fujita).
- Let $\{\mathcal{C}_t\}_{t \in [0,1]}$ be an equisingular degeneration of curves; then $G_{\mathcal{C}_0} \cong G_{\mathcal{C}_1}$

Consequences and comments

- $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) = \langle \mu_1, \ldots, \mu_r \mid \sum_{j=1}^r d_i \mu_i = 0 \rangle \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/e\mathbb{Z}$, where $e := \gcd(d_1, \ldots, d_r)$.
- $G_{\mathcal{C}_1 \cup \mathcal{C}_2} \twoheadrightarrow G_{\mathcal{C}_i}$ (the kernel is generated by meridians of $\mathcal{C}_i$, following a result from Fujita).
- Let $\{\mathcal{C}_t\}_{t \in [0,1]}$ be an equisingular degeneration of curves; then $G_{\mathcal{C}_0} \cong G_{\mathcal{C}_1}$
- Let $\{\mathcal{C}_t\}_{t \in [0,1]}$ be a degeneration of curves equisingular for $(0, 1]$ and such that $\mathcal{C}_0$ is reduced. Then $G_{\mathcal{C}_0} \twoheadrightarrow G_{\mathcal{C}_1}$.

Consequences and comments

- $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) = \langle \mu_1, \ldots, \mu_r \mid \sum_{j=1}^r d_i \mu_i = 0 \rangle \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/e\mathbb{Z}$, where $e := \gcd(d_1, \ldots, d_r)$.
- $G_{\mathcal{C}_1 \cup \mathcal{C}_2} \twoheadrightarrow G_{\mathcal{C}_i}$ (the kernel is generated by meridians of $\mathcal{C}_i$, following a result from Fujita).
- Let $\{\mathcal{C}_t\}_{t \in [0,1]}$ be an equisingular degeneration of curves; then $G_{\mathcal{C}_0} \cong G_{\mathcal{C}_1}$
- Let $\{\mathcal{C}_t\}_{t \in [0,1]}$ be a degeneration of curves equisingular for $(0, 1]$ and such that $\mathcal{C}_0$ is reduced. Then $G_{\mathcal{C}_0} \twoheadrightarrow G_{\mathcal{C}_1}$.
- $G_{\mathcal{C}} \cong \mathbb{Z}/d\mathbb{Z}$ if $\mathcal{C}$ is smooth.

Consequences and comments

- $H_1(\mathbb{P}^2 \setminus \mathcal{C};\mathbb{Z}) = \langle \mu_1,\ldots,\mu_r \mid \sum_{j=1}^r d_i\mu_i = 0\rangle \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/e\mathbb{Z}$, where $e := \gcd(d_1,\ldots,d_r)$.
- $G_{\mathcal{C}_1\cup\mathcal{C}_2} \twoheadrightarrow G_{\mathcal{C}_i}$ (the kernel is generated by meridians of $\mathcal{C}_i$, following a result from Fujita).
- Let $\{\mathcal{C}_t\}_{t\in[0,1]}$ be an equisingular degeneration of curves; then $G_{\mathcal{C}_0} \cong G_{\mathcal{C}_1}$
- Let $\{\mathcal{C}_t\}_{t\in[0,1]}$ be a degeneration of curves equisingular for $(0,1]$ and such that $\mathcal{C}_0$ is reduced. Then $G_{\mathcal{C}_0} \twoheadrightarrow G_{\mathcal{C}_1}$.
- $G_{\mathcal{C}} \cong \mathbb{Z}/d\mathbb{Z}$ if $\mathcal{C}$ is smooth.
- $G_{\mathcal{C}} \cong \mathbb{Z}^{d-1}$ if $\mathcal{C}$ is an arrangement of lines in general position.

Consequences and comments

- $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) = \langle \mu_1, \dots, \mu_r \mid \sum_{j=1}^r d_i \mu_i = 0 \rangle \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/e\mathbb{Z}$, where $e := \gcd(d_1, \dots, d_r)$.
- $G_{\mathcal{C}_1 \cup \mathcal{C}_2} \twoheadrightarrow G_{\mathcal{C}_i}$ (the kernel is generated by meridians of $\mathcal{C}_i$, following a result from Fujita).
- Let $\{\mathcal{C}_t\}_{t \in [0,1]}$ be an equisingular degeneration of curves; then $G_{\mathcal{C}_0} \cong G_{\mathcal{C}_1}$
- Let $\{\mathcal{C}_t\}_{t \in [0,1]}$ be a degeneration of curves equisingular for $(0, 1]$ and such that $\mathcal{C}_0$ is reduced. Then $G_{\mathcal{C}_0} \twoheadrightarrow G_{\mathcal{C}_1}$.
- $G_{\mathcal{C}} \cong \mathbb{Z}/d\mathbb{Z}$ if $\mathcal{C}$ is smooth.
- $G_{\mathcal{C}} \cong \mathbb{Z}^{d-1}$ if $\mathcal{C}$ is an arrangement of lines in general position.
- If $\mathcal{C}_1^{\text{aff}} \pitchfork \mathcal{C}_2^{\text{aff}}$ (and do not intersect at L_∞) then $\pi_1(\mathbb{C}^2 \setminus (\mathcal{C}_1^{\text{aff}} \cup \mathcal{C}_2^{\text{aff}}) \cong \pi_1(\mathbb{C}^2 \setminus \mathcal{C}_1^{\text{aff}}) \times \pi_1(\mathbb{C}^2 \setminus \mathcal{C}_2^{\text{aff}})$ (Oka).

Consequences and comments

- $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) = \langle \mu_1, \dots, \mu_r \mid \sum_{j=1}^r d_i \mu_i = 0 \rangle \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/e\mathbb{Z}$, where $e := \gcd(d_1, \dots, d_r)$.
- $G_{\mathcal{C}_1 \cup \mathcal{C}_2} \twoheadrightarrow G_{\mathcal{C}_i}$ (the kernel is generated by meridians of $\mathcal{C}_i$, following a result from Fujita).
- Let $\{\mathcal{C}_t\}_{t \in [0,1]}$ be an equisingular degeneration of curves; then $G_{\mathcal{C}_0} \cong G_{\mathcal{C}_1}$
- Let $\{\mathcal{C}_t\}_{t \in [0,1]}$ be a degeneration of curves equisingular for $(0,1]$ and such that $\mathcal{C}_0$ is reduced. Then $G_{\mathcal{C}_0} \twoheadrightarrow G_{\mathcal{C}_1}$.
- $G_{\mathcal{C}} \cong \mathbb{Z}/d\mathbb{Z}$ if $\mathcal{C}$ is smooth.
- $G_{\mathcal{C}} \cong \mathbb{Z}^{d-1}$ if $\mathcal{C}$ is an arrangement of lines in general position.
- If $\mathcal{C}_1^{\text{aff}} \pitchfork \mathcal{C}_2^{\text{aff}}$ (and do not intersect at L_∞) then $\pi_1(\mathbb{C}^2 \setminus (\mathcal{C}_1^{\text{aff}} \cup \mathcal{C}_2^{\text{aff}}) \cong \pi_1(\mathbb{C}^2 \setminus \mathcal{C}_1^{\text{aff}}) \times \pi_1(\mathbb{C}^2 \setminus \mathcal{C}_2^{\text{aff}})$ (Oka).
- $G_{\mathcal{C}}$ is abelian if $\mathcal{C}$ is a nodal curve (Zariski, Fulton, Deligne, Harris).

Affine and projective group

- $\mathcal{C} \subset \mathbb{P}^2$, $\mathcal{C}^{\text{aff}} \subset \mathbb{C}^2$ generic affine associated group, i.e., $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}} = \mathbb{P}^2 \setminus (\mathcal{C} \cup L_\infty)$, $\mathcal{C} \pitchfork L_\infty$

Affine and projective group

- $\mathcal{C} \subset \mathbb{P}^2$, $\mathcal{C}^{\text{aff}} \subset \mathbb{C}^2$ generic affine associated group, i.e., $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}} = \mathbb{P}^2 \setminus (\mathcal{C} \cup L_\infty)$, $\mathcal{C} \pitchfork L_\infty$
- The following diagram is commutative

$$\begin{array}{ccc} G_{\mathcal{C}\cup L_\infty} & \longrightarrow & G_{\mathcal{C}} \\ \downarrow & & \downarrow \varepsilon \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \end{array}$$

The upper map is induced by the inclusion. The vertical maps are defined by $\mu_1 \mapsto 1$ ($\mod d$ in the right-hand case).

Affine and projective group

- $\mathcal{C} \subset \mathbb{P}^2$, $\mathcal{C}^{\text{aff}} \subset \mathbb{C}^2$ generic affine associated group, i.e., $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}} = \mathbb{P}^2 \setminus (\mathcal{C} \cup L_\infty)$, $\mathcal{C} \pitchfork L_\infty$
- The following diagram is commutative

$$\begin{array}{ccc} G_{\mathcal{C}\cup L_\infty} & \longrightarrow & G_{\mathcal{C}} \\ \downarrow & & \downarrow \varepsilon \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \end{array}$$

  The upper map is induced by the inclusion. The vertical maps are defined by $\mu_1 \mapsto 1 \pmod d$ in the right-hand case).
- It is a pull-back diagram, since a meridian of L_∞ is central:

$$G_{\mathcal{C}\cup L_\infty} = \{(t^k, \mu) \in \mathbb{Z} \times G_{\mathcal{C}} \mid \varepsilon(\mu) = t \mod d\}$$

Final remarks

- The transition from Zariski-van Kampen method to braid monodromy is due to Chisini and Moishezon: a stronger invariant than $G_{\mathcal{C}}$.

Final remarks

- The transition from Zariski-van Kampen method to braid monodromy is due to Chisini and Moishezon: a stronger invariant than $G_{\mathcal{C}}$.
- Problems with fundamental group:

Final remarks

- The transition from Zariski-van Kampen method to braid monodromy is due to Chisini and Moishezon: a stronger invariant than $G_{\mathcal{C}}$.
- Problems with fundamental group:
  - Difficult and expensive computations

Final remarks

- The transition from Zariski-van Kampen method to braid monodromy is due to Chisini and Moishezon: a stronger invariant than $G_{\mathcal{C}}$.
- Problems with fundamental group:
  - Difficult and expensive computations
  - Even if it is computed, it is difficult to know its structure.

Final remarks

- The transition from Zariski-van Kampen method to braid monodromy is due to Chisini and Moishezon: a stronger invariant than $G_{\mathcal{C}}$.
- Problems with fundamental group:
  - Difficult and expensive computations
  - Even if it is computed, it is difficult to know its structure.
- Shortcuts

Final remarks

- The transition from Zariski-van Kampen method to braid monodromy is due to Chisini and Moishezon: a stronger invariant than $G_{\mathcal{C}}$.
- Problems with fundamental group:
  - Difficult and expensive computations
  - Even if it is computed, it is difficult to know its structure.
- Shortcuts
  - Find effective invariants, e.g., Alexander like-invariants

Final remarks

- The transition from Zariski-van Kampen method to braid monodromy is due to Chisini and Moishezon: a stronger invariant than $G_{\mathcal{C}}$.
- Problems with fundamental group:
  - Difficult and expensive computations
  - Even if it is computed, it is difficult to know its structure.
- Shortcuts
  - Find effective invariants, e.g., Alexander like-invariants
  - Compute these invariants from the curve, without computing the fundamental group.

Topological properties of braid monodromy

Theorem (Kulikov-Teicher, Carmona)

Let $\mathcal{C}$ be a projective curve and let ∇ be a braid monodromy of its generic $\mathcal{C}^{\text{aff}}$. Then, a topological model of the pair $(\mathbb{P}^2, \mathcal{C})$ can be constructed from ∇. In particular, if two curves have the same braid monodromy then they are topologically equivalent.

Topological properties of braid monodromy

Theorem (Kulikov-Teicher, Carmona)

Let $\mathcal{C}$ be a projective curve and let ∇ be a braid monodromy of its generic $\mathcal{C}^{\mathrm{aff}}$. Then, a topological model of the pair $(\mathbb{P}^2, \mathcal{C})$ can be constructed from ∇. In particular, if two curves have the same braid monodromy then they are topologically equivalent.

Remark

There is partial converse to this statement by –, Carmona and Cogolludo.

Topological properties of braid monodromy

Theorem (Kulikov-Teicher, Carmona)

Let $\mathcal{C}$ be a projective curve and let ∇ be a braid monodromy of its generic $\mathcal{C}^{\mathrm{aff}}$. Then, a topological model of the pair $(\mathbb{P}^2, \mathcal{C})$ can be constructed from ∇. In particular, if two curves have the same braid monodromy then they are topologically equivalent.

Properties and comments

- If two curves are connected by an equisingular deformation, then they have the same braid monodromy.

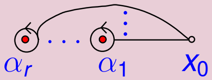
Topological properties of braid monodromy

Theorem (Kulikov-Teicher, Carmona)

Let $\mathcal{C}$ be a projective curve and let ∇ be a braid monodromy of its generic $\mathcal{C}^{\text{aff}}$. Then, a topological model of the pair $(\mathbb{P}^2, \mathcal{C})$ can be constructed from ∇. In particular, if two curves have the same braid monodromy then they are topologically equivalent.

Properties and comments

- If two curves are connected by an equisingular deformation, then they have the same braid monodromy.
- What does it mean? A braid monodromy is determined by an element in $(\mathbb{B}_n)^r$ if we choose a (pseudo)geometric basis of $\mathbb{F}_r$.



Topological properties of braid monodromy

Theorem (Kulikov-Teicher, Carmona)

Let $\mathcal{C}$ be a projective curve and let ∇ be a braid monodromy of its generic $\mathcal{C}^{\text{aff}}$. Then, a topological model of the pair $(\mathbb{P}^2, \mathcal{C})$ can be constructed from ∇. In particular, if two curves have the same braid monodromy then they are topologically equivalent.

Properties and comments

- If two curves are connected by an equisingular deformation, then they have the same braid monodromy.
- What does it mean? A braid monodromy is determined by an element in $(\mathbb{B}_n)^r$ if we choose a (pseudo)geometric basis of $\mathbb{F}_r$.
- There is an action of $\mathbb{B}_n \times \mathbb{B}_r$ (simultaneous conjugation and Hurwitz moves) on $(\mathbb{B}_n)^r$: two braid monodromies are equal if their representatives in $(\mathbb{B}_n)^r$ are in the same orbit.

Non generic and Puiseux braid Monodromy

More properties and comments

- Sometimes it is useful to choose the line at infinity and the vertical direction in a non-generic way: choose $P \in \mathcal{C}$ with only one tangent line L (e.g. a generic smooth point); put L as the line at infinity and P as the point at infinity of the vertical direction. For the resulting $\mathcal{C}^{\text{aff}}$ we obtain a braid monodromy in $\mathbb{B}_n$, $n := d - \text{mult}_{\mathcal{C}}(P)$.

Non generic and Puiseux braid Monodromy

More properties and comments

- Sometimes it is useful to choose the line at infinity and the vertical direction in a non-generic way: choose $P \in \mathcal{C}$ with only one tangent line L (e.g. a generic smooth point); put L as the line at infinity and P as the point at infinity of the vertical direction. For the resulting $\mathcal{C}^{\text{aff}}$ we obtain a braid monodromy in $\mathbb{B}_n$, $n := d - \text{mult}_{\mathcal{C}}(P)$.
- An equisingular deformation $(C_t, P_t)_{t\in[0,1]}$ respects braid monodromy.

Non generic and Puiseux braid Monodromy

More properties and comments

- Sometimes it is useful to choose the line at infinity and the vertical direction in a non-generic way: choose $P \in \mathcal{C}$ with only one tangent line L (e.g. a generic smooth point); put L as the line at infinity and P as the point at infinity of the vertical direction. For the resulting $\mathcal{C}^{\text{aff}}$ we obtain a braid monodromy in $\mathbb{B}_n$, $n := d - \operatorname{mult}_{\mathcal{C}}(P)$.
- An equisingular deformation $(C_t, P_t)_{t\in[0,1]}$ respects braid monodromy.
- In that case, it is more difficult to find the relation we have to add to pass from $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ to $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$.

Non generic and Puiseux braid Monodromy

More properties and comments

- Sometimes it is useful to choose the line at infinity and the vertical direction in a non-generic way: choose $P \in \mathcal{C}$ with only one tangent line L (e.g. a generic smooth point); put L as the line at infinity and P as the point at infinity of the vertical direction. For the resulting $\mathcal{C}^{\mathrm{aff}}$ we obtain a braid monodromy in $\mathbb{B}_n$, $n := d - \operatorname{mult}_{\mathcal{C}}(P)$.

Puiseux monodromy

If each non-transversal vertical line L_i contains only one singular point P_i, then $\rho(\alpha_i) = \beta_i^{-1}\tau_i\beta_i$ where τ_i is a *Puiseux braid* involving only the first $m_i := (\mathcal{C} \cdot L_i)_P$ strings (usually m_i is the multiplicity).

Non generic and Puiseux braid Monodromy

More properties and comments

- Sometimes it is useful to choose the line at infinity and the vertical direction in a non-generic way: choose $P \in \mathcal{C}$ with only one tangent line L (e.g. a generic smooth point); put L as the line at infinity and P as the point at infinity of the vertical direction. For the resulting $\mathcal{C}^{\text{aff}}$ we obtain a braid monodromy in $\mathbb{B}_n$, $n := d - \text{mult}_{\mathcal{C}}(P)$.

Puiseux monodromy

If each non-transversal vertical line L_i contains only one singular point P_i, then $\rho(\alpha_i) = \beta_i^{-1}\tau_i\beta_i$ where τ_i is a *Puiseux braid* involving only the first $m_i := (\mathcal{C} \cdot L_i)_P$ strings (usually m_i is the multiplicity).

Zariski-van Kampen Theorem

The group $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ *admits the following finite presentation:*

Generators $\mu_1, \ldots, \mu_n$

Relators $(\mu_i^{\beta_i})^{\tau_i} = \mu_i^{\beta_i}$, $1 \leq i < m_i$, $1 \leq j \leq r$.

Applications of braid monodromy

Example

Assume that $\nabla(\alpha_i) = \sigma_2^{-1}\sigma_1\sigma_2$. Then, only one relation is needed: $\mu_1 = \mu_3$.

Applications of braid monodromy

Example

Assume that $\nabla(\alpha_i) = \sigma_2^{-1}\sigma_1\sigma_2$. Then, only one relation is needed: $\mu_1 = \mu_3$.

Theorem (Libgober)

The homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$ is the one of the 2-complex associated with the presentation of the fundamental group obtained via a Puiseux monodromy.

Applications of braid monodromy

Example

Assume that $\nabla(\alpha_i) = \sigma_2^{-1}\sigma_1\sigma_2$. Then, only one relation is needed: $\mu_1 = \mu_3$.

Theorem (Libgober)

The homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$ is the one of the 2-complex associated with the presentation of the fundamental group obtained via a Puiseux monodromy.

Theorem (Libgober)

Let $\rho : \mathbb{B}_d \to \mathrm{GL}(m_d; R)$ be a morphism, where R is an UFD. Let $\mathcal{C}$ be a projective curve and consider its generic braid monodromy, represented by braids $\kappa_1, \ldots, \kappa_r$. Consider the matrix $A \in \mathrm{Mat}(m_d r \times m_d; R)$ obtained by joining $\rho(\kappa_i) - I_{m_d}$. Then, the Fitting ideals of this matrix define an invariant of the equisingularity deformation type of $\mathcal{C}$.

Applications of braid monodromy

Example

Assume that $\nabla(\alpha_i) = \sigma_2^{-1}\sigma_1\sigma_2$. Then, only one relation is needed: $\mu_1 = \mu_3$.

Theorem (Libgober)

The homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$ is the one of the 2-complex associated with the presentation of the fundamental group obtained via a Puiseux monodromy.

Theorem (Libgober)

Let $\rho : \mathbb{B}_d \to \mathrm{GL}(m_d; R)$ be a morphism, where R is an UFD. Let $\mathcal{C}$ be a projective curve and consider its generic braid monodromy, represented by braids $\kappa_1, \dots, \kappa_r$. Consider the matrix $A \in \mathrm{Mat}(m_d r \times m_d; R)$ obtained by joining $\rho(\kappa_i) - I_{m_d}$. Then, the Fitting ideals of this matrix define an invariant of the equisingularity deformation type of $\mathcal{C}$.

Challenge

Find specific invariants.

Examples I

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{y = 0\}$ (resp. $y = x^2$)

Braid monodromy: (1)

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu \mid - \rangle \cong \mathbb{Z}$

$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) =$ Trivial (resp. cyclic of order 2)

Homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$: $\mathbb{S}^1$

Examples I

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{y = 0\}$ (resp. $y = x^2$)

Braid monodromy: (1)

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu \mid - \rangle \cong \mathbb{Z}$

$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) =$ Trivial (resp. cyclic of order 2)

Homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$: $\mathbb{S}^1$

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{y^2 = x\}$

Braid monodromy: σ_1

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu_1, \mu_2 \mid \mu_1 = \mu_2 \rangle \cong \mathbb{Z}$

$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) =$ Cyclic of order 2

Homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$: $\mathbb{S}^1$

Examples I

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{y = 0\}$ (resp. $y = x^2$)

Braid monodromy: (1)

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu \mid - \rangle \cong \mathbb{Z}$

$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) =$ Trivial (resp. cyclic of order 2)

Homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$: $\mathbb{S}^1$

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{y^2 = x^2\}$

Braid monodromy: σ_1^2

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu_1, \mu_2 \mid [\mu_1, \mu_2] = 1 \rangle \cong \mathbb{Z}^2$

$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}$

Homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$: $\mathbb{S}^1 \times \mathbb{S}^1$

Examples I

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{y = 0\}$ (resp. $y = x^2$)

Braid monodromy: (1)

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu \mid - \rangle \cong \mathbb{Z}$

$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) =$ Trivial (resp. cyclic of order 2)

Homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$: $\mathbb{S}^1$

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{y^2 = x^3\}$

Braid monodromy: σ_1^3

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu_1, \mu_2 \mid \mu_1\mu_2\mu_1 = \mu_2\mu_1\mu_2 \rangle \cong \mathbb{B}_3$

$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/3\mathbb{Z}$

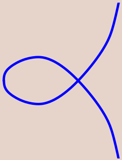
Homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$: Complement of the trefoil knot M_K.

Examples II

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{y^2 = x^2 + x^3\}$

Real picture:



Braid monodromy: (σ_1, σ_1^2)

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu_1, \mu_2 \mid \mu_1 = \mu_2, [\mu_1, \mu_2] = 1 \rangle \cong \mathbb{Z}$

Homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$: $\mathbb{S}^1 \vee \mathbb{S}^2$

Examples II

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{(y^2 - x)^2 - 4x^3\}$

Real picture:

Braid monodromy: $(\sigma_2, \sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3)$

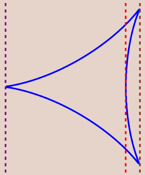
$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu_1, \mu_2 \mid \mu_1\mu_2\mu_1 = \mu_2\mu_1\mu_2, 1 = 1 \rangle$

Homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$: $M_K \vee \mathbb{S}^2$

Examples II

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{(x^2+y^2)^2-48x(x^2+y^2)+72(x^2+y^2)+64x^3-432\}$

Real picture: 

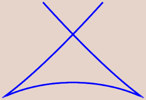
Braid monodromy: $(\sigma_1^3\sigma_3^3, \sigma_2, \sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_1\sigma_3)$

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu_1, \mu_2, \mu_3 \mid \mu_1\mu_2\mu_1 = \mu_2\mu_1\mu_2, \mu_3\mu_2\mu_3 = \mu_2\mu_3\mu_2, \mu_1\mu_3\mu_1 = \mu_3\mu_1\mu_3\rangle$

Examples II

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{4\,x^3 + y^4 - 6\,xy^2 - 3\,x^2 + 4\,y^2\}$

Real picture: 

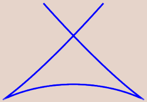
Braid monodromy: $(\sigma_1^2, \sigma_2^3, \sigma_1\sigma_2^3\sigma_1^{-1})$

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu_1, \mu_2 \mid \mu_1\mu_2\mu_1 = \mu_2\mu_1\mu_2, \mu_3\mu_2\mu_3 = \mu_2\mu_3\mu_2, [\mu_1, \mu_3] = 1 \rangle \cong \mathbb{B}_4$

Examples II

Example

Curve $\mathcal{C}^{\text{aff}}$: $\{4x^3 + y^4 - 6xy^2 - 3x^2 + 4y^2\}$



Real picture:

Braid monodromy: $(\sigma_1^2, \sigma_2^3, \sigma_1\sigma_2^3\sigma_1^{-1})$

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \langle \mu_1, \mu_2 \mid \mu_1\mu_2\mu_1 = \mu_2\mu_1\mu_2, \mu_3\mu_2\mu_3 = \mu_2\mu_3\mu_2, [\mu_1, \mu_3] = 1 \rangle \cong \mathbb{B}_4$

Remark

If we *smooth* the node, then σ_1^2 is replaced by σ_1 (twice). The group is $\mathbb{B}_3$ and the homotopy type is the one of $M_K \vee \mathbb{S}^2 \vee \mathbb{S}^2$

Quartics and conics

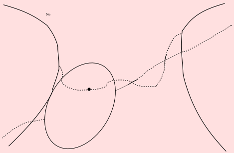
Theorem (–,Carmona,Cogolludo,Tokunaga)

There are two equisingular deformation classes of curves with two irreducible components: a smooth conic C, a quartic Q (with singular points $\mathbb{A}_1$ and $\mathbb{A}_3$) such that they intersect only at one point $\mathbb{A}_{15}$. In one case the common tangent line to $\mathbb{A}_{15}$ passes through $\mathbb{A}_3$.

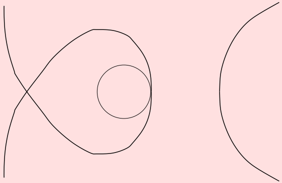
Quartics and conics

Theorem (–,Carmona,Cogolludo,Tokunaga)

There are two equisingular deformation classes of curves with two irreducible components: a smooth conic C, a quartic Q (with singular points $\mathbb{A}_1$ and $\mathbb{A}_3$) such that they intersect only at one point $\mathbb{A}_{15}$. In one case the common tangent line to $\mathbb{A}_{15}$ passes through $\mathbb{A}_3$.



$G_{\mathcal{C}} = \langle a, b \mid a^2(ab)^2 = [a, b^2] = 1 \rangle$



$G_{\mathcal{C}} = \langle a, b \mid b^2 = (ab)^4 \rangle$