On Rational Elliptic Surfaces With Dihedral Group Action

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Outline

1. Background

2. Dihedral group actions on Rational Elliptic Surfaces

3. Sketch of Proof
Let

- $X, Y$: normal algebraic varieties over $\mathbb{C}$.
- $\pi : X \to Y$: surjective finite morphism.

Then $\pi$ induces an inclusion of function fields

$$\pi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$$

**Definition**

$\pi$ is said to be a $G$-cover if

- $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension.
- $\text{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$. 
Fact

If $\pi : X \to Y$ is a $G$-cover then

- $G$ is a finite group.
- $X$ is a $G$-variety (i.e. there exists a $G$-action on $X$.)
- $X/G \cong Y$.
- $\pi$ is the quotient morphism.
- $\mathbb{C}(X)^G \cong \mathbb{C}(Y)$. 
Fact

Conversely given a normal $G$-variety $X$, then the quotient morphism and variety

$$\pi : X \rightarrow X/G$$

is a $G$-cover.
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- To study $G$-covers and to study Galois Theory for function fields are essentially the same.
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- To study $G$-covers and to study Galois Theory for function fields are essentially the same.

$\Rightarrow$ Birational geometry of $G$-varieties
Fundamental Problems

The Inverse Galois Problem

For a given normal variety $Y$ and a given finite group $G$, find a normal variety $X$ and a surjective finite map

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- Give an explicit method to construct such $(X, \pi)$.
  (Not just the existence of $X$.)
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- Give a criterion for $(X, \pi)$ to exist in terms of data on $Y$.
- Give an explicit method to construct such $(X, \pi)$.
  (Not just the existence of $X$.)
- Give a description of the moduli space of such $(X, \pi)$. 
The "pull-back" construction by M. Namba

Given a $G$-cover

$$\pi : X \to Y$$

and a $G$-indecomposable rational map

$$Y' \to Y$$

a $G$-cover cover

$$\pi' : X' \to Y'$$

can be constructed.
Given $\pi : X \to Y$ and $\psi : Y' \to Y$: 
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![Diagram](image_url)
Given $\pi : X \to Y$ and $\psi : Y' \to Y$:
Given $\pi : X \rightarrow Y$ and $\psi : Y' \rightarrow Y$:
The pull back construction allows us to construct new $G$-covers form the data of known Galois covers.

**Difficulties**

- We need to find a simple $G$-cover $\pi : X \rightarrow Y$ to start with.
- The existence of $\psi : Y' \rightarrow Y$ depends on the choice of $\pi : X \rightarrow Y$.

Even if it is possible to construct a $G$-cover over $Y'$, it may not be obtained as a pull-back of $\pi : X \rightarrow Y$. 
Let \( \pi : X \rightarrow Y \) be a \( G \)-cover.

**Definition (Informal)**

\( \pi : X \rightarrow Y \) is said to be a **versal** \( G \)-cover if every \( G \)-cover \( \pi' : X' \rightarrow Y' \) can be obtained by pulling-back \( \pi : X \rightarrow Y \).
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Find versal \( G \)-covers with a simple structure
Known Facts

- Versal $G$-covers exist for all finite groups $G$. 
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- If $G \cong C_n$ or $D_{2n}$ ($n$: odd),
  \[
  \pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 / G \cong \mathbb{P}^1
  \]
  is versal.
Known Facts

- Versal $G$-covers exist for all finite groups $G$.
- If $G \cong C_n$ or $D_{2n}$ ($n$: odd),
  \[ \pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1/G \cong \mathbb{P}^1 \]
  is versal.
- If $G \cong D_{2n}$ ($n$: even), and
  \[ \pi : X \rightarrow Y \]
  is versal, then $\dim X \geq 2$. Further if $\dim X = 2$ then $X, Y$ are rational surfaces.
Known Facts

- Let $\pi: X \to Y$ and $\pi': X' \to Y'$ be birationally equivalent $G$-covers. Then $\pi$ is versal if and only if $\pi'$ is versal.

Definition

$\pi: X \to Y$, $\pi': X' \to Y'$ are said to be birationally equivalent if there exists a $G$-equivariant birational map $\phi: X \dashrightarrow X'$.
Problem

- Classify rational $G$-surfaces up to birational equivalence.
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Today we consider rational elliptic surfaces with relative $D_{2n}$-action.
The birational classification of (minimal) rational $G$-surfaces is known due to Dolgachev-Iskovskikh. It is based on the following facts.

**Theorem (Manin: $G$-equivariant Mori-theory for surfaces)**

Let $G$ be a finite group and $X$ be a minimal $G$-surface. Then one of the following holds:

- $\text{Pic}^G(X) \cong \mathbb{Z}^2$ and $X$ has a $G$-minimal conic bundle structure.
- $\text{Pic}^G(X) \cong \mathbb{Z}$ and $X$ is a $G$-minimal del-Pezzo surface.
Theorem (Iskovskikh: Factorization theorem)

Let $X_1, X_2$ be $G$-surfaces. Then any $G$-equivariant birational map

$$\phi : X_1 \rightarrow X_2$$

can be factored into a finite composition of “Links”.

This is an $G$-equivariant analogue of the famous Noether’s factorization theorem for birational transformations of $\mathbb{P}^2$.

Fact

All “Links” are classified.
The classification is done in the following steps.

1. Find minimal $G$-surfaces.
   - Consider a rational surface.
   - Determine $\text{Aut}(X)$.
   - Find finite subgroups $G$ of $\text{Aut}(X)$ that act minimally on $X$.

2. Use the classification of “Links” to distinguish non-birationally equivalent surfaces.
The classification is “surface” centered, and does not contain much information on non-minimal $G$-surfaces.

It is hard to read off the birational equivalence classes for a fixed group $G$. The following problem is posed in Dolgachev-Iskovskikh.

**Problem (Moduli Problem)**

*Give a finer geometric description of the algebraic variety parametrizing birational equivalence classes of rational $G$-surfaces for fixed $G$.***
Theorem

The algebraic variety parameterizing the birational equivalence classes of rational elliptic surfaces with a relative $D_8$-action is a nodal rational curve.

- The birational equivalence class corresponding to the node is versal.
- There is one non-versal equivalence class.
- It is unknown for the other cases.
Automorphisms of Rational Elliptic Surfaces

Let $S$ be a smooth projective surface, $C$ be a smooth curve.

**Definition**

$S$ is said to be an elliptic surface over $C$ if there is a surjective morphism

$$f : S \to C$$

such that

1. general fibers are smooth curves of genus 1,
2. no fibre contains an exceptional curve of the first kind,
3. $f : S \to C$ has a section,
4. $S$ has at least one singular fiber.
Since \( f : S \rightarrow C \) has a section

\[
O : C \rightarrow S.
\]

Then the generic fiber \( E \) of \( f : S \rightarrow C \) becomes an elliptic curve over \( \mathbb{C}(C) \), and

\[
\{ \text{sections of } f : S \rightarrow C \} \overset{1:1}{\longleftrightarrow} \{ \mathbb{C}(C)\text{-rational points of } E \}
\]

**Definition (Mordell-Weil group)**

The Mordell-Weil group of \( f : S \rightarrow C \) is defined by

\[
\text{MW}(S) := \{ \text{sections of } f : S \rightarrow C \}
\]

The group structure is defined by the elliptic curve structure of \( E \).
Definition

An automorphism $\phi : S \to S$ is said to be a relative automorphism of $f : S \to C$ if it preserves the fibration, i.e.

$$f \circ \phi = f$$

The group of relative automorphisms of $f : S \to C$ is denoted by $\text{Aut}_C(S)$.

Lemma

$$\text{Aut}_C(S) \cong \text{Aut}_O(S) \ltimes \text{MW}(S)$$
Lemma

Let $\text{Aut}_O(E) = \mathbb{Z}/2\mathbb{Z} = \langle \iota, \iota^2 = 1 \rangle$. Then the elements of finite order in $\text{Aut}_C(S)$ consist of

- $\text{MW}(S)_{\text{tor}}$: the torsion elements of $\text{MW}(S)$.
- $\{\iota \circ s | s \in \text{MW}(S)\}$: a translation followed by the involution.

Note that $\iota \circ s$ has order 2.

$$(\iota \circ s)^2 = \iota \circ s \circ \iota \circ s = \iota \circ \iota \circ (s) \circ s = 1$$
Lemma

Let $\tau$ be an $n$-torsion element of $\text{MW}(S)$. Let $\sigma = \iota \circ s$ for some section $s$. Then

$$< \sigma, \tau > \cong D_{2n}.$$  

Conversely any relative $D_{2n}$ action is generated by $\tau, \sigma$ of the above form.

- Conjugate subgroups give rise to isomorphic $D_{2n}$-surfaces.
Lemma

$i \circ s$ and $i \circ s'$ are conjugate in $\text{Aut}_C(S)$ if and only if

$$s - s'$$

is 2-divisible in $MW(S)$.

Let $\tau$ be an $n$-torsion element of $MW(S)$. There are essentially two types of relative $D_{2n}$-actions on $S$.

(i) $D_{2n} = \langle \iota, \tau \rangle$

(ii) $D_{2n} = \langle \iota \circ s, \tau \rangle$ for $s$ that is not 2-divisible
Lemma (Miranda-Persson)

Let $S$ be a rational elliptic surface with $n$ torsion ($n \geq 4$). Then the Mordell-Weil group $MW(S)$ of $S$ is one of the following.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$MW(S)$</th>
<th>Number of surfaces</th>
<th>Type of $S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$\mathbb{Z}/6\mathbb{Z}$</td>
<td>1</td>
<td>No. 66</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{Z}/5\mathbb{Z}$</td>
<td>1</td>
<td>No. 67</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{Z}/4\mathbb{Z}$</td>
<td>2</td>
<td>No. 70, 72</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$</td>
<td>1</td>
<td>No. 74</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$</td>
<td>$\infty$</td>
<td>No. 58</td>
</tr>
</tbody>
</table>

The type of $S$ is the number in Oguiso-Shioda’s list of Mordell-Weil lattices of rational elliptic surfaces.
**Lemma**

The relative automorphism group $\text{Aut}_C(S)$ and the number of conjugacy classes of dihedral subgroups of $\text{Aut}_C(S)$ are as follows:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{MW}(S)$</th>
<th>$\text{Aut}_C(S)$</th>
<th>conj. class. of $D_{2n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$\mathbb{Z}/6\mathbb{Z}$</td>
<td>$D_{12}$</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{Z}/5\mathbb{Z}$</td>
<td>$D_{10}$</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{Z}/4\mathbb{Z}$</td>
<td>$D_8$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$</td>
<td>4*</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})$</td>
<td>2**</td>
</tr>
</tbody>
</table>

\( (*) : < \iota, \tau >, < \iota, \tau' >, < \iota \circ s, \tau >, < \iota \circ s, \tau' > \)

\( (**) : < \iota, \tau >, < \iota \circ s, \tau > \)
In total we have two infinite families and 4 sporadic isomorphism classes of rational elliptic surfaces with relative $D_8$-action.

Infinite families: 58-(i), 58-(ii)

Sporadic cases: 70, 72, 74-(i), 74-(ii)
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Infinite families: 58-(i), 58-(ii)

Sporadic cases: 70, 72, 74-(i), 74-(ii)

**Which of these $D_8$-surfaces are birationally equivalent?**
Theorem

Every rational elliptic surface with relative $D_8$ action is birationally equivalent as a $D_8$-surface to exactly one of the surfaces of the following types.

1. $\mathbb{P}^1 \times \mathbb{P}^1$ with $D_8$-action (I).
   - 74-(i)
2. $\mathbb{P}^1 \times \mathbb{P}^1$ with $D_8$-action (II).
   - 70, 72, 74-(ii), 58-(ii)
3. $D_8$-minimal del Pezzo surface of degree 4.
   - 58-(i)
   
   Each surface of this type is birational equivalent if and only if they are isomorphic as elliptic surfaces.
Let \(([x_0, x_1], [y_0, y_1])\) be homogeneous coordinates of \(\mathbb{P}^1 \times \mathbb{P}^1\).

- **action (I)**
  \[
  \begin{align*}
  \sigma : \quad & ([x_0, x_1], [y_0, y_1]) \\
  & \mapsto ([y_0 - \sqrt{-1}y_1, \sqrt{-1}y_0 - y_1], [x_0 - \sqrt{-1}x_1, \sqrt{-1}x_0 - x_1]) \\
  \tau : \quad & ([x_0, x_1], [y_0, y_1]) \mapsto ([y_1, y_0], [x_0, x_1])
  \end{align*}
  \]

- **action (II)**
  \[
  \begin{align*}
  \sigma : \quad & ([x_0, x_1], [y_0, y_1]) \mapsto ([y_0, y_1], [x_0, x_1]) \\
  \tau : \quad & ([x_0, x_1], [y_0, y_1]) \mapsto ([y_1, y_0], [x_0, x_1])
  \end{align*}
  \]
Sketch of Proof

- **Step 1:**
  For each surface $S$ and $D_{2n}$-action, look for $D_{2n}$-orbits consisting of (-1)-curves, and blow them down to obtain a birationally equivalent minimal $D_{2n}$-surface.
  - The Mordell-Weil lattice of $S$.

- **Step 2:**
  Apply Dolgachev-Iskovskikh's classification.
  - fixed curves
  - classification of links
Case of type No. 58

Step 1
It is known that rational elliptic surfaces of type No. 58 has singular fibres

\[ l_4, l_4, l_2, l_1, l_1 \]

and

\[ MW(S) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \]

Let

- \( O \): the zero section of \( S \).
- \( \iota \): involution of \( S \) (with respect to the zero section).
- \( t \): four torsion element of \( MW(S) \).
- \( s \): generating section. i.e. a section such that

\[ \langle s, t \rangle = MW(S) \]
There are two conjugacy classes of $D_8$ actions (in $\text{Aut}_C(S)$) on $S$ represented by,

- Case 1: $D_8 = \langle \iota, t \rangle$
- Case 2: $D_8 = \langle \iota \circ s, t \rangle$
The orbit of sections differ from case 1 and case 2. Let $s'$ be any section. Then the $D_8$-orbit of $s'$ and $O$ is as follows:

1. $D_8 = \langle \nu, t \rangle$.

   \[
   \text{Orb}_{D_8}(s') = \{ \pm s', \pm (s' + t), \pm (s' + 2t), \pm (s' + 3t) \}
   \]

   \[
   \text{Orb}_{D_8}(O) = \{ O, t, 2t, 3t \}
   \]

2. $D_8 = \langle \nu \circ s, t \rangle$

   \[
   \text{Orb}_{D_8}(s') = \{ s', s' + t, s' + 2t, s' + 3t, -s' - s, -s' - s + t, -s' - s + 2t, -s' - s + 3t \}
   \]

   \[
   \text{Orb}_{D_8}(O) = \{ O, t, 2t, 3t, -s, -s + t, -s + 2t, -s + 3t \}
   \]
Lemma

In both case 1 and 2,

- The sections in \( \text{Orb}_{D_8}(O) \) are mutually disjoint.
- The sections in \( \text{Orb}_{D_8}(s') \) are not mutually disjoint if \( s' \neq O, t \).

By blowing down the 4 (resp. 8) curves in \( \text{Orb}_{D_8}(O) \) we obtain minimal \( D_8 \)-surfaces.

Lemma

The minimal rational \( D_8 \) surface obtained by blowing down \( \text{Orb}_{D_8}(O) \) for each action is:

1. del Pezzo surface of degree 4 with \( D_8 \) action.
2. \( \mathbb{P}^1 \times \mathbb{P}^1 \) with \( D_8 \) action.
Step 2
It remains to determine the corresponding birational equivalence class according to Dolgachev-Iskovskikh’s classification.

- **Case 1**: del Pezzo surface of degree 4 with $D_8$-action.

**Lemma (J. Blanc)**

Let $S, S'$ be a del Pezzo surface of degree 4 with minimal $D_8$ action and let $\phi : S \dasharrow S'$ be a $G$-equivariant birational map. Then $\phi$ is an isomorphism.

**Corollary**

Let $S, S'$ be rational elliptic surfaces of type 58 with $D_8$-action of type (I). Then $S, S'$ are birationally equivalent if and only if they are isomorphic as elliptic surfaces.
• **Case 2:** $\mathbb{P}^1 \times \mathbb{P}^1$ with $D_8$ action.

There are several distinct minimal $D_8$ actions on $\mathbb{P}^1 \times \mathbb{P}^1$. To determine which $D_8$-action we get, we make the following observation:

Define a $D_8$-action on $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$
\begin{align*}
\sigma : ([x_0, x_1], [y_0, y_1]) &\rightarrow ([y_0, y_1], [x_0, x_1]) \\
\tau : ([x_0, x_1], [y_0, y_1]) &\rightarrow ([y_1, y_0], [x_0, x_1])
\end{align*}
$$

Let $f_1, f_2, f_3$ be $D_8$-invariant curves of bi-degree $(2, 2)$ defined by

- $f_1 = x_0 x_1 y_0 y_1$
- $f_2 = x_0^2 y_0^2 + x_0^2 y_1^2 + x_1^2 y_0^2 + x_1^2 y_1^2$
- $f_3 = x_0 x_1 y_0^2 + x_0 x_1 y_1^2 + x_0^2 y_0 y_1 + x_1^2 y_0 y_1$

Let $\Lambda = \{ \alpha f_1 + \beta f_2 + \gamma f_3 \}$ be the $D_8$ invariant linear system of curves generated by $f_1, f_2, f_3$. 
Fact

- A general member of $\Lambda$ is a smooth $D_8$-invariant curve of genus 1.
- A general sub-pencil $\lambda$ of $\Lambda$ determines a $D_8$-invariant pencil of curves of genus 1.
- By blowing up the base points of $\lambda$ we obtain a elliptic surface with $D_8$ action.
Lemma

Every rational elliptic surface $S$ with four torsion can be obtained by blowing up a $D_8$-invariant sub-pencil $\lambda$ of $\Lambda$.

Lemma

$S_\lambda$ and $S_{\lambda'}$ are isomorphic if and only if $p_\lambda$ and $p_{\lambda'}$ lie on the same member of the pencil generated by $C$ and $L_1 + L_2$.

Lemma

The $D_8$ action of $\mathbb{P}^1 \times \mathbb{P}^1$ lifts to $S$ and coincides with the $D_8$-action $\langle i \circ s, t \rangle$ of type (ii).

Corollary

Every elliptic surface with $D_8$-action of type (ii) are birationally equivalent.
Theorem

The algebraic variety parameterizing the birational equivalence classes of rational elliptic surfaces with a relative $D_8$-action is a nodal rational curve.

- The birational equivalence class corresponding to the node is versal.
- There is one non-versal equivalence class.
- It is unknown for the other cases.
$\mathbb{P}^1 \times \mathbb{P}^1$  

$\mathbb{P}^1 \times \mathbb{P}^1 / (\sigma \tau) = \mathbb{P}^1 \times \mathbb{P}^1$  

$\mathbb{P}^1 \times \mathbb{P}^1 / (\sigma \tau, \sigma \tau^3) = \mathbb{P}^1 \times \mathbb{P}^1$  

$\mathbb{P}^1 \times \mathbb{P}^1 / D_8 = \mathbb{P}^2$
\( \mathbb{P}^1 \times \mathbb{P}^1 \)

\[ \xrightarrow{} \]

\( \mathbb{P}^1 \times \mathbb{P}^1 / D_8 = \mathbb{P}^2 \)
$\mathbb{P}^1 \times \mathbb{P}^1$
$P^1 \times P^1 \to P^1 \times P^1 / (\sigma \tau) = P^1 \times P^1$
$P^1 \times P^1$  \hspace{2cm} \xrightarrow{} \hspace{2cm}  P^1 \times P^1/(\sigma \tau) = P^1 \times P^1$
$P^1 \times P^1 \rightarrow P^1 \times P^1 / (\sigma \tau) = P^1 \times P^1$
$P^1 \times P^1 \rightarrow P^1 \times P^1 / (\sigma \tau) = P^1 \times P^1$

$P^1 \times P^1 / (\sigma \tau, \sigma T^3) = P^1 \times P^1$
$P^1 \times P^1$ \rightarrow P^1 \times P^1 / (\sigma \tau) = P^1 \times P^1$

$P^1 \times P^1 / (\sigma \tau, \sigma \tau^3) = P^1 \times P^1$
$P^1 \times P^1$ \rightarrow \quad \text{Diagram with a curve flattening to a line.} \quad \text{Diagram with a line}

$P^1 \times P^1 / (\sigma \tau) = P^1 \times P^1$

$P^1 \times P^1 / (\sigma \tau, \sigma \tau^3) = P^1 \times P^1$
\[
P^1 \times P^1 \xrightarrow{\sigma T, \sigma T^3} P^1 \times P^1 / (\sigma T, \sigma T^3) = P^1 \times P^1
\]
$\mathbb{P}^1 \times \mathbb{P}^1$ \( \xrightarrow{} \) \( \mathbb{P}^1 \times \mathbb{P}^1 / (\sigma \tau) = \mathbb{P}^1 \times \mathbb{P}^1 \)

\[ P^1 \times P^1 / (\sigma \tau) = P^1 \times P^1 \]

\[ P^1 \times P^1 / (\sigma \tau, \sigma \tau^3) = P^1 \times P^1 \]

\[ P^1 \times P^1 / D_8 = P^2 \]
\[ \mathbb{P}^1 \times \mathbb{P}^1 \]
$P^1 \times P^1$
$\mathbb{P}^1 \times \mathbb{P}^1$ 

$\mathbb{P}^2$