On Rational Elliptic Surfaces With Dihedral Group Action

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Outline

G-covers

Let

- X, Y: normal algebraic varieties over $\mathbb{C}$.
- $\pi : X \to Y$: surjective finite morphism.

Then π induces an inclusion of function fields

$$\pi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$$

Definition

π is said to be a G-cover if

- $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension.
- $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$.

Fact

If $\pi : X \to Y$ is a G-cover then

- *G is a finite group.*
- *X is a G-variety (i.e. there exists a G-action on X.)*
- *$X/G \cong Y$.*
- *π is the quotient morphism.*
- *$\mathbb{C}(X)^G \cong \mathbb{C}(Y)$.*

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$$\pi : X \to X/G$$

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- G-covers and G-varieties are essentially the same.
- To study G-covers and to study Galois Theory for function fields are essentially the same.

$\Rightarrow$ **Birational geometry of G-varieties**

Fundamental Problems

The Inverse Galois Problem

For a given normal variety Y and a given finite group G, find a normal variety X and a surjective finite map

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such that $\pi : X \to Y$ is a G-cover.

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- Give an explicit method to construct such (X, π). (Not just the existence of X.)
- Give a description of the moduli space of such (X, π).

Fundamental Problems

The "pull-back" construction by M. Namba

Given a G-cover

$$\pi : X \to Y$$

and a G-indecomposable rational map

$$Y' \dashrightarrow Y$$

a G-cover cover

$$\pi' : X' \to Y'$$

can be constructed.

Given $\pi : X \to Y$ and $\psi : Y' \dashrightarrow Y$:

$$
\begin{tikzcd}
 & X \arrow[d, "\pi"] \\
Y' \arrow[r, dashed, "\psi"] & Y
\end{tikzcd}
$$

Given $\pi : X \to Y$ and $\psi : Y' \dashrightarrow Y$:

$$\begin{tikzcd} & \bar{Y}' \arrow[dl] \arrow[dr] & X \arrow[d, "\pi"] \\ Y' \arrow[rr, dotted, "\psi"] & & Y \end{tikzcd}$$

Given $\pi : X \to Y$ and $\psi : Y' \dashrightarrow Y$:

$$
\begin{tikzcd}
& \bar{Y}' \times_Y X \arrow[d] \arrow[dr] & \\
& \bar{Y}' \arrow[dl] \arrow[dr] & X \arrow[d, "\pi"] \\
Y' \arrow[rr, dotted, "\psi"] & & Y
\end{tikzcd}
$$

Given $\pi : X \to Y$ and $\psi : Y' \dashrightarrow Y$:

$$\begin{tikzcd}
 & \bar{Y}' \times_Y X \arrow[dl] \arrow[d] \arrow[dr] & \\
X' \arrow[d, "\pi'"'] & \bar{Y}' \arrow[dl] \arrow[dr] & X \arrow[d, "\pi"] \\
Y' \arrow[rr, dotted, "\psi"] & & Y
\end{tikzcd}$$

The pull back construction allows us to construct new G-covers form the data of known Galois covers.

Difficulties

- We need to find a simple G-cover $\pi : X \to Y$ to start with.
- The existence of $\psi : Y' \dashrightarrow Y$ depends on the choice of $\pi : X \to Y$.
  Even if it is possible to construct a G-cover over Y', it may not be obtained as a pull-back of $\pi : X \to Y$.

Let $\pi : X \to Y$ be a G-cover.

Definition (Informal)

$\pi : X \to Y$ is said to be a <u>versal</u> G-cover if
every G-cover $\pi' : X' \to Y'$ can be obtained by pulling-back
$\pi : X \to Y$.

Let $\pi : X \to Y$ be a G-cover.

Definition (Informal)

$\pi : X \to Y$ is said to be a versal G-cover if every G-cover $\pi' : X' \to Y'$ can be obtained by pulling-back $\pi : X \to Y$.

Find versal G-covers with a simple structure

Known Facts

- Versal G-covers exist for all finite groups G.

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$$\pi : \mathbb{P}^1 \to \mathbb{P}^1/G \cong \mathbb{P}^1$$

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Known Facts

- Versal G-covers exist for all finite groups G.
- If $G \cong C_n$ or D_{2n} (n: odd),

$$\pi : \mathbb{P}^1 \to \mathbb{P}^1/G \cong \mathbb{P}^1$$

  is versal.
- If $G \cong D_{2n}$ (n: even), and

$$\pi : X \to Y$$

  is versal, then $\dim X \geq 2$. Further if $\dim X = 2$ then X, Y are rational surfaces.

Known Facts

- Let $\pi : X \to Y$ and $\pi' : X' \to Y'$ be birationally equivalent G-covers. Then π is versal if and only if π' is versal.

Definition

$\pi : X \to Y$, $\pi' : X' \to Y'$ are said to be birationally equivalent if there exists a G-equivariant birational map $\phi : X \dashrightarrow X'$.

$$\begin{tikzcd} X \arrow[r, dotted, "\phi"] \arrow[d, "\pi"'] & X' \arrow[d, "\pi'"] \\ Y \arrow[r, dotted] & Y' \end{tikzcd}$$

Problem

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Today we consider rational elliptic surfaces
with relative D_{2n}-action.

Birational classification of rational G-surfaces

The birational classification of (minimal) rational G-surfaces is known due to **Dolgachev-Iskovskikh**. It is based on the following facts.

Theorem (Manin: G-equivariant Mori-theory for surfaces)

Let G be a finite group and X be a minimal G-surface. Then one of the following holds:

- $\mathrm{Pic}^G(X) \cong \mathbb{Z}^2$ *and X has a G-minimal conic bundle structure.*
- $\mathrm{Pic}^G(X) \cong \mathbb{Z}$ *and X is a G-minimal del-Pezzo surface.*

Theorem (Iskovskikh: Factorization theorem)

Let X_1, X_2 be G-surfaces. Then any G-equivariant birational map

$$\phi : X_1 \dashrightarrow X_2$$

can be factored into a finite composition of "Links".

This is an G-equivariant analogue of the famous Noether's factorization theorem for birational transformations of $\mathbb{P}^2$.

Fact

All "Links" are classified.

The classification is done in the following steps.

1. Find minimal G-surfaces.
   - Consider a rational surface.
   - Determine $\mathrm{Aut}(X)$.
   - Find finite subgroups G of $\mathrm{Aut}(X)$ that act minimally on X.
2. Use the classification of "Links" to distinguish non-birationallly equivalent surfaces.

- The classification is "surface" centered, and does not contain much information on non-minimal G-surfaces.
- It is hard to read off the birational equivalence classes for a fixed group G. The following problem is posed in **Dolgachev-Iskovskikh.**

Problem (Moduli Problem)

Give a finer geometric description of the algebraic variety parametrizing birational equivalence classes of rational G-surfaces for fixed G.

Theorem

The algebraic variety parameterizing the birational equivalence classes of rational elliptic surfaces with a relative D_8-action is a nodal rational curve.

- The birational equivalence class corresponding to the node is versal.
- There is one non-versal equivalence class.
- It is unknown for the other cases.

Automorphisms of Rational Elliptic Surfaces

Let S be a smooth projective surface, C be a smooth curve.

Definition

S is said to be an elliptic surface over C if there is a surjective morphism

$$f : S \to C$$

such that

1. general fibers are smooth curves of genus 1,
2. no fibre contains an exceptional curve of the first kind,
3. $f : S \to C$ has a section,
4. S has at least one singular fiber.

Since $f : S \to C$ has a section

$$O : C \to S.$$

Then the generic fiber E of $f : S \to C$ becomes an elliptic curve over $\mathbb{C}(C)$, and

$$\{\text{sections of } f : S \to C\} \xleftrightarrow{1:1} \{\mathbb{C}(C)\text{-rational points of } E\}$$

Definition (Mordell-Weil group)

The Mordell-Weil group of $f : S \to C$ is defined by

$$\mathrm{MW}(S) := \{\text{sections of } f : S \to C\}$$

The group structure is defined by the elliptic curve structure of E.

Definition

An automorphism $\phi : S \to S$ is said to be a relative automorphism of $f : S \to C$ if it preserves the fibration, i.e.

$$f \circ \phi = f$$

The group of relative automorphisms of $f : S \to C$ is denoted by $\mathrm{Aut}_C(S)$.

Lemma

$$\mathrm{Aut}_C(S) \cong \mathrm{Aut}_O(S) \ltimes \mathrm{MW}(S)$$

Lemma

Let $\mathrm{Aut}_O(E) = \mathbb{Z}/2\mathbb{Z} =< \iota, \iota^2 = 1 >$. *Then the elements of finite order in* $\mathrm{Aut}_C(S)$ *consist of*

- $\mathrm{MW}(S)_{tor}$: *the torsion elements of* $MW(S)$.
- $\{\iota \circ s | s \in MW(S)\}$: *a translation followed by the involution.*

Note that $\iota \circ s$ has order 2.

$$(\iota \circ s)^2 = \iota \circ s \circ \iota \circ s = \iota \circ \iota \circ (-s) \circ s = 1$$

Lemma

Let τ be an n-torsion element of $MW(S)$. Let $\sigma = \iota \circ s$ for some section s. Then

$$< \sigma, \tau > \cong D_{2n}.$$

Conversely any relative D_{2n} action is generated by τ, σ of the above form.

- Conjugate subgroups give rise to isomorphic D_{2n}-surfaces.

Lemma

$i \circ s$ and $i \circ s'$ are conjugate in $\mathrm{Aut}_C(S)$ if and only if

$$s - s'$$

is 2-divisible in $MW(S)$.

Let τ be an n-torsion element of $MW(S)$. There are essentially two types of relative D_{2n}-actions on S.

(i) $D_{2n} = \langle \iota, \tau \rangle$

(ii) $D_{2n} = \langle \iota \circ s, \tau \rangle$ for s that is not 2-divisible

Lemma (Miranda-Persson)

Let S be a rational elliptic surface with n torsion ($n \geq 4$). Then the Mordell-Weil group $MW(S)$ of S is one of the following.

n	$MW(S)$	Number of surfaces	Type of S
6	$\mathbb{Z}/6\mathbb{Z}$	1	No. 66
5	$\mathbb{Z}/5\mathbb{Z}$	1	No. 67
4	$\mathbb{Z}/4\mathbb{Z}$	2	No. 70, 72
	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	1	No. 74
	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	∞	No. 58

The type of S is the number in Oguiso-Shioda's list of Mordell-Weil lattices of rational elliptic surfaces.

Lemma

The relative automorphism group $\mathrm{Aut}_C(S)$ *and the number of conjugacy classes of dihedral subgroups of* $\mathrm{Aut}_C(S)$ *are as follows:*

n	$MW(S)$	$\mathrm{Aut}_C(S)$	conj. class. of D_{2n}
6	$\mathbb{Z}/6\mathbb{Z}$	D_{12}	1
5	$\mathbb{Z}/5\mathbb{Z}$	D_{10}	1
4	$\mathbb{Z}/4\mathbb{Z}$	D_8	1
	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$	4^*
	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})$	2^{**}

$$(*) : < \iota, \tau >, < \iota, \tau' >, < \iota \circ s, \tau >, < \iota \circ s, \tau' >$$

$$(**) : < \iota, \tau >, < \iota \circ s, \tau >$$

In total we have two infinite families and 4 sporadic isomorphism classes of rational elliptic surfaces with relative D_8-action.

Infinite families: 58-(i), 58-(ii)

Sporadic cases: 70, 72, 74-(i),74-(ii)

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Infinite families: 58-(i), 58-(ii)

Sporadic cases: 70, 72, 74-(i),74-(ii)

Which of these D_8-surfaces are birationally equivalent?

Theorem

Every rational elliptic surface with relative D_8 action is birationaly equivalent as a D_8-surface to exactly one of the surfaces of the following types.

1. $\mathbb{P}^1 \times \mathbb{P}^1$ *with* D_8*-action (I).*
   - *74-(i)*
2. $\mathbb{P}^1 \times \mathbb{P}^1$ *with* D_8*-action (II).*
   - *70, 72, 74-(ii), 58-(ii)*
3. D_8*-minimal del Pezzo surface of degree* 4*.*
   - *58-(i)*
     Each surface of this type is birational equivalent if and only if they are isomorphic as elliptic surfaces.

Let $([x_0, x_1], [y_0, y_1])$ be homogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$.

- action (I)

$$\begin{cases} \sigma : & ([x_0, x_1], [y_0, y_1]) \\ & \quad \to ([y_0 - \sqrt{-1}y_1, \sqrt{-1}y_0 - y_1], [x_0 - \sqrt{-1}x_1, \sqrt{-1}x_0 - x_1]) \\ \tau : & ([x_0, x_1], [y_0, y_1]) \to ([y_1, y_0], [x_0, x_1]) \end{cases}$$

- action (II)

$$\begin{cases} \sigma : & ([x_0, x_1], [y_0, y_1]) \to ([y_0, y_1], [x_0, x_1]) \\ \tau : & ([x_0, x_1], [y_0, y_1]) \to ([y_1, y_0], [x_0, x_1]) \end{cases}$$

Sketch of Proof

- Step 1:

  For each surface S and D_{2n}-action, look for D_{2n}-orbits consisting of (-1)-curves, and blow them down to obtain a birationaly equivalent minimal D_{2n}-surface.

  - The Mordell-Weil lattice of S.

- Step 2:

  Apply Dolgachev-Iskovskikh's classification.

  - fixed curves
  - classification of links

Case of type No. 58

Step 1

It is known that rational elliptic surfaces of type No. 58 has singular fibres

$$I_4, I_4, I_2, I_1, I_1$$

and

$$MW(S) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$$

Let

- O: the zero section of S.
- ι: involution of S (with respect to the o section).
- t: four torsion element of $MW(S)$.
- s: generating section. i.e. a section such that.

$$< s, t >= MW(S)$$

There are two conjugacy classes of D_8 actions (in $\mathrm{Aut}_C(S)$) on S represented by,

- Case 1: $D_8 =< \iota, t >$
- Case 2: $D_8 =< \iota \circ s, t >$

The orbit of sections differ from case 1 and case 2. Let s' be any section. Then the D_8-orbit of s' and O is as follows:

1. $D_8 = \langle \iota, t \rangle$.

$$\mathrm{Orb}_{D_8}(s') = \{\pm s', \pm(s'+t), \pm(s'+2t), \pm(s'+3t)\}$$
$$\mathrm{Orb}_{D_8}(O) = \{O, t, 2t, 3t\}$$

2. $D_8 = \langle \iota \circ s, t \rangle$

$$\mathrm{Orb}_{D_8}(s') = \{s', s'+t, s'+2t, s'+3t, -s'-s, -s'-s+t, \\ -s'-s+2t, -s'-s+3t\}$$
$$\mathrm{Orb}_{D_8}(O) = \{O, t, 2t, 3t, -s, -s+t, -s+2t, -s+3t\}$$

Lemma

In both case 1 and 2,

- *The sections in* $\mathrm{Orb}_{D_8}(O)$ *are mutually disjoint.*
- *The sections in* $\mathrm{Orb}_{D_8}(s')$ *are not mutually disjoint if* $s' \neq O, t$.

By blowing down the 4 (resp. 8) curves in $\mathrm{Orb}_{D_8}(O)$ we obtain minimal D_8-surfaces.

Lemma

The minimal rational D_8 *surface obtained by blowing down* $\mathrm{Orb}_{D_8}(O)$ *for each action is:*

1. *del Pezzo suraface of degree* 4 *with* D_8 *action.*
2. $\mathbb{P}^1 \times \mathbb{P}^1$ *with* D_8 *action.*

Step 2

It remains to determine the corresponding birational equivalence class according to Dolgachev-Iskovskikh's classification.

- Case 1: del Pezzo surface of degree 4 with D_8-action.

Lemma (J. Blanc)

Let S, S' be a del Pezzo surface of degree 4 with minimal D_8 action and let $\phi : S \dashrightarrow S'$ be a G-equivariant birational map. Then ϕ is an isomorphism.

Corollary

Let S, S' be rational elliptic surfaces of type 58 with D_8-action of type (I). Then S, S' are birationally equivalent if and only if they are isomorphic as elliptic surfaces.

- Case 2: $\mathbb{P}^1 \times \mathbb{P}^1$ with D_8 action.

There are several distinct minimal D_8 actions on $\mathbb{P}^1 \times \mathbb{P}^1$.
To determine which D_8-action we get, we make the following observation:

Define a D_8-action on $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$\begin{cases} \sigma : & ([x_0, x_1], [y_0, y_1]) \to ([y_0, y_1], [x_0, x_1]) \\ \tau : & ([x_0, x_1], [y_0, y_1]) \to ([y_1, y_0], [x_0, x_1]) \end{cases}$$

Let f_1, f_2, f_3 be D_8-invariant curves of bi-degree $(2, 2)$ defined by

$$\begin{aligned} f_1 &= x_0 x_1 y_0 y_1 \\ f_2 &= x_0^2 y_0^2 + x_0^2 y_1^2 + x_1^2 y_0^2 + x_1^2 y_1^2 \\ f_3 &= x_0 x_1 y_0^2 + x_0 x_1 y_1^2 + x_0^2 y_0 y_1 + x_1^2 y_0 y_1 \end{aligned}$$

Let $\Lambda = \{\alpha f_1 + \beta f_2 + \gamma f_3\}$ be the D_8 invariant linear system of curves generated by f_1, f_2, f_3.

Fact

- *A general member of Λ is a smooth D_8-invariant curve of genus 1.*
- *A general sub-pencil λ of Λ determines a D_8-invariant pencil of curves of genus 1.*
- *By blowing up the base points of λ we obtain a elliptic surface with D_8 action.*

Lemma

Every rational elliptic surface S with four torsion can be obtained by blowing up a D_8-invariant sub-pencil λ of Λ.

Lemma

S_λ and $S_{\lambda'}$ are isomorphic if and only if p_λ and $p_{\lambda'}$ lie on the same member of the pencil generated by C and $L_1 + L_2$.

Lemma

The D_8 action of $\mathbb{P}^1 \times \mathbb{P}^1$ lifts to S and coincides with the D_8-action $< i \circ s, t >$ of type (ii).

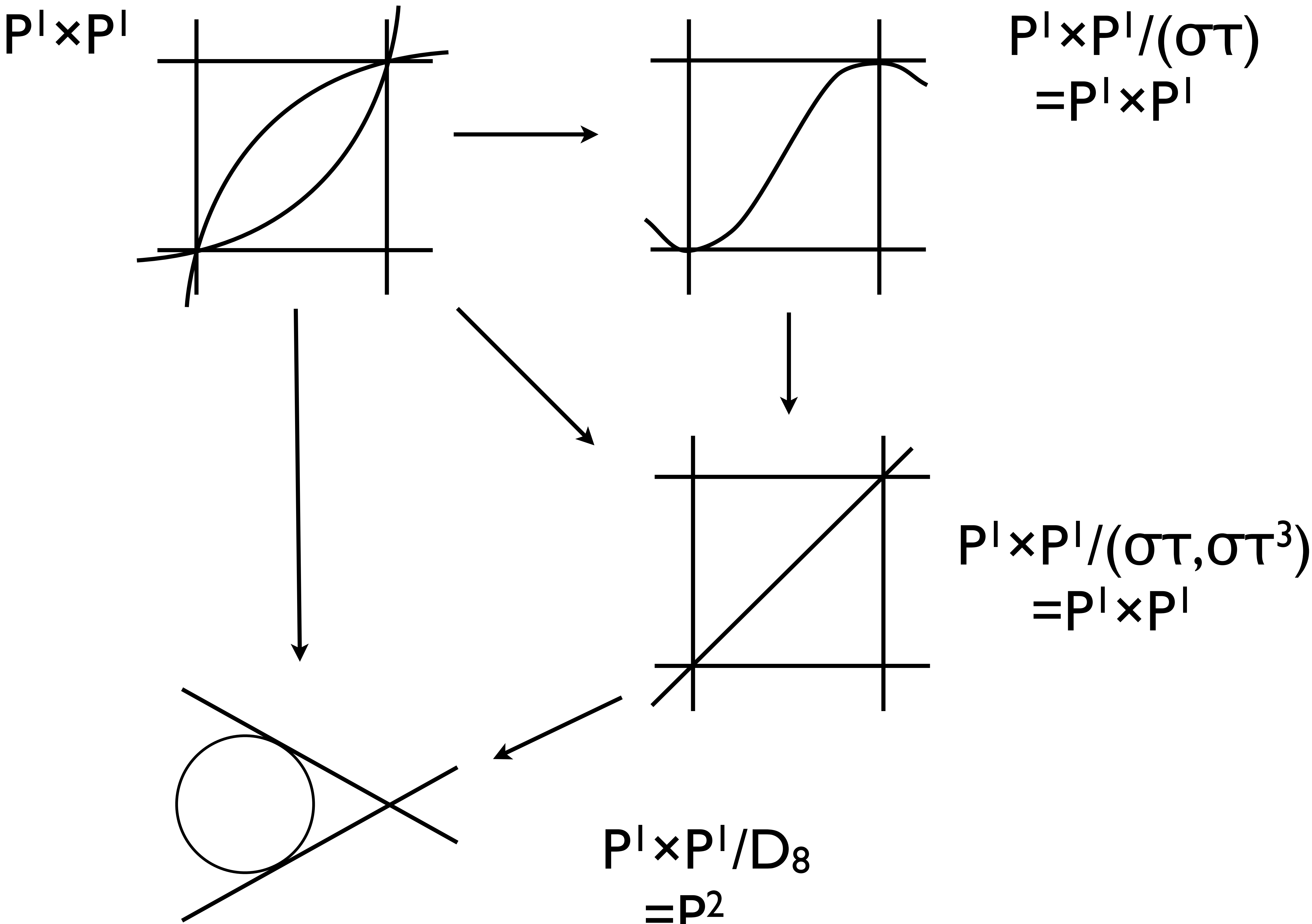
Corollary

Every elliptic surface with D_8-action of type (ii) are birationally equivalent.

Theorem

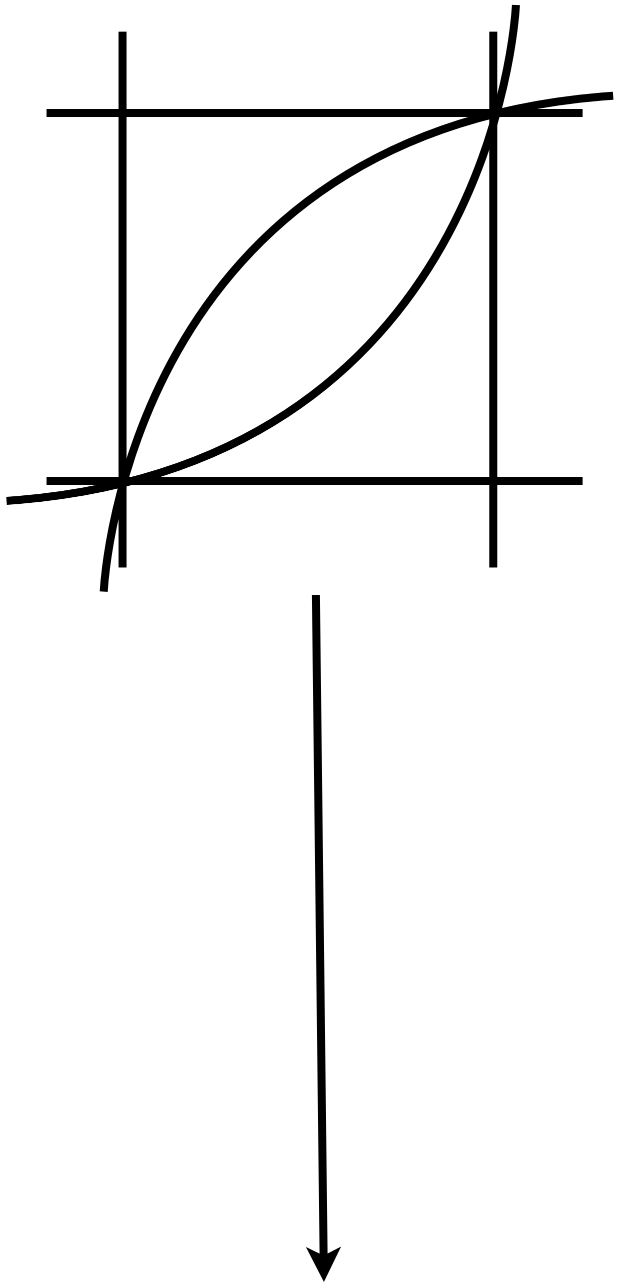
The algebraic variety parameterizing the birational equivalence classes of rational elliptic surfaces with a relative D_8-action is a nodal rational curve.

- The birational equivalence class corresponding to the node is versal.
- There is one non-versal equivalence class.
- It is unknown for the other cases.

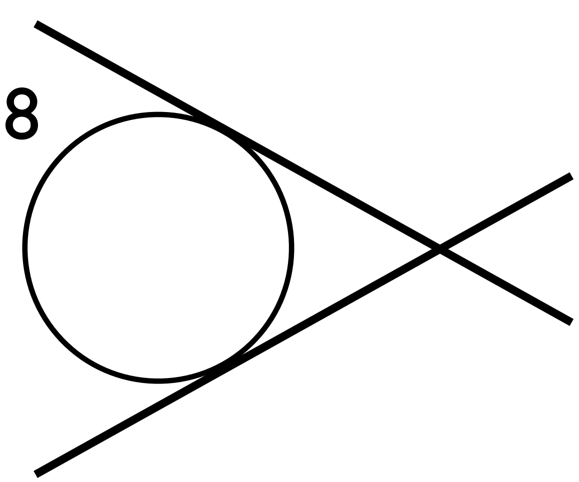


$P^1\times P^1$
$P^1\times P^1/(\sigma\tau)$
$=P^1\times P^1$
$P^1\times P^1/(\sigma\tau,\sigma\tau^3)$
$=P^1\times P^1$
$P^1\times P^1/D_8$
$=P^2$

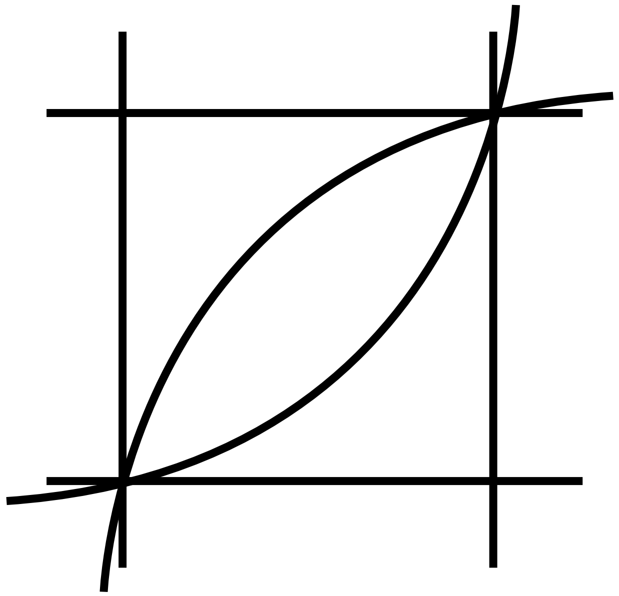
$P^1 \times P^1$



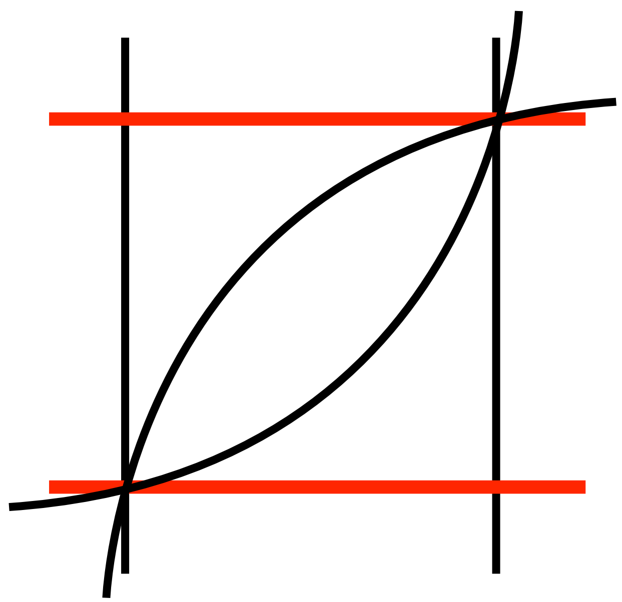
$P^1 \times P^1 / D_8$
$= P^2$



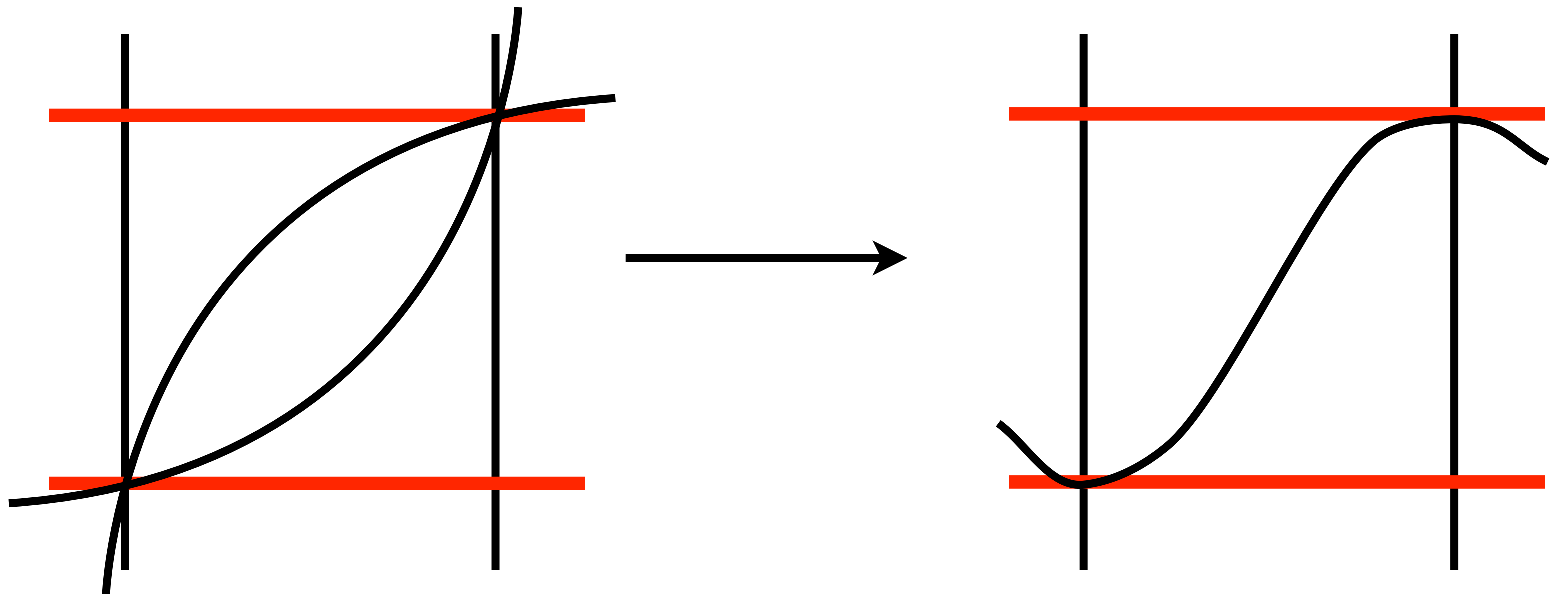
$\mathbf{P}^1 \times \mathbf{P}^1$



$P^1 \times P^1$

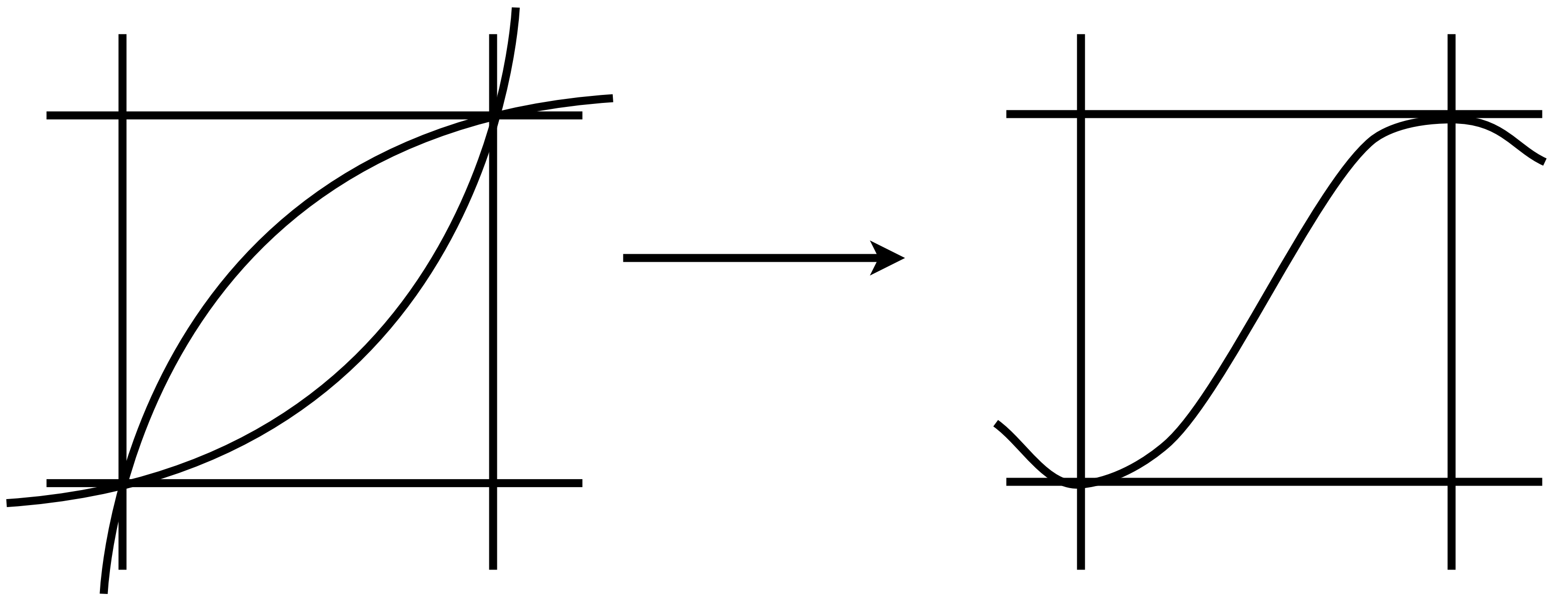


$P^I \times P^I$



$P^I \times P^I / (\sigma\tau)$
$= P^I \times P^I$

$P^I \times P^I$



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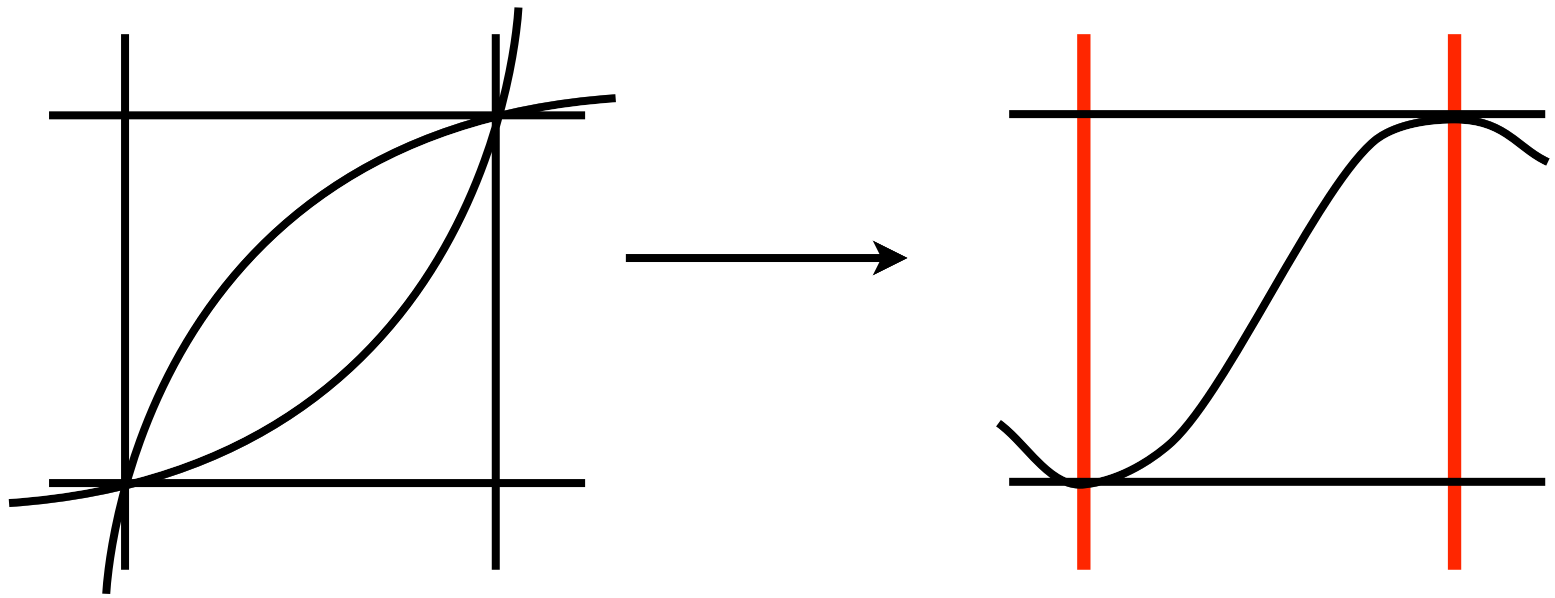
$= P^I \times P^I$

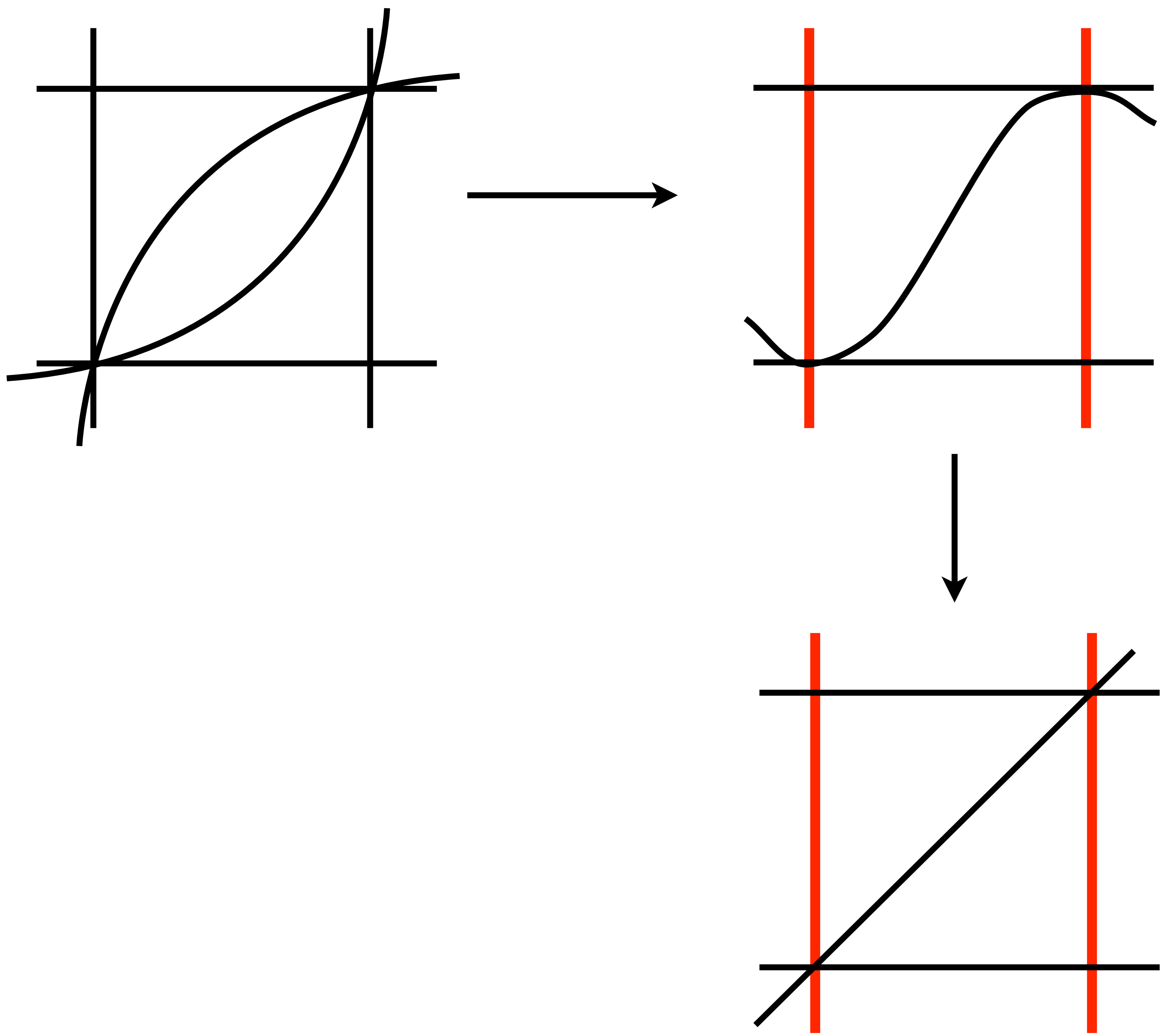
$P^I \times P^I$



$P^I \times P^I / (\sigma\tau)$

$= P^I \times P^I$

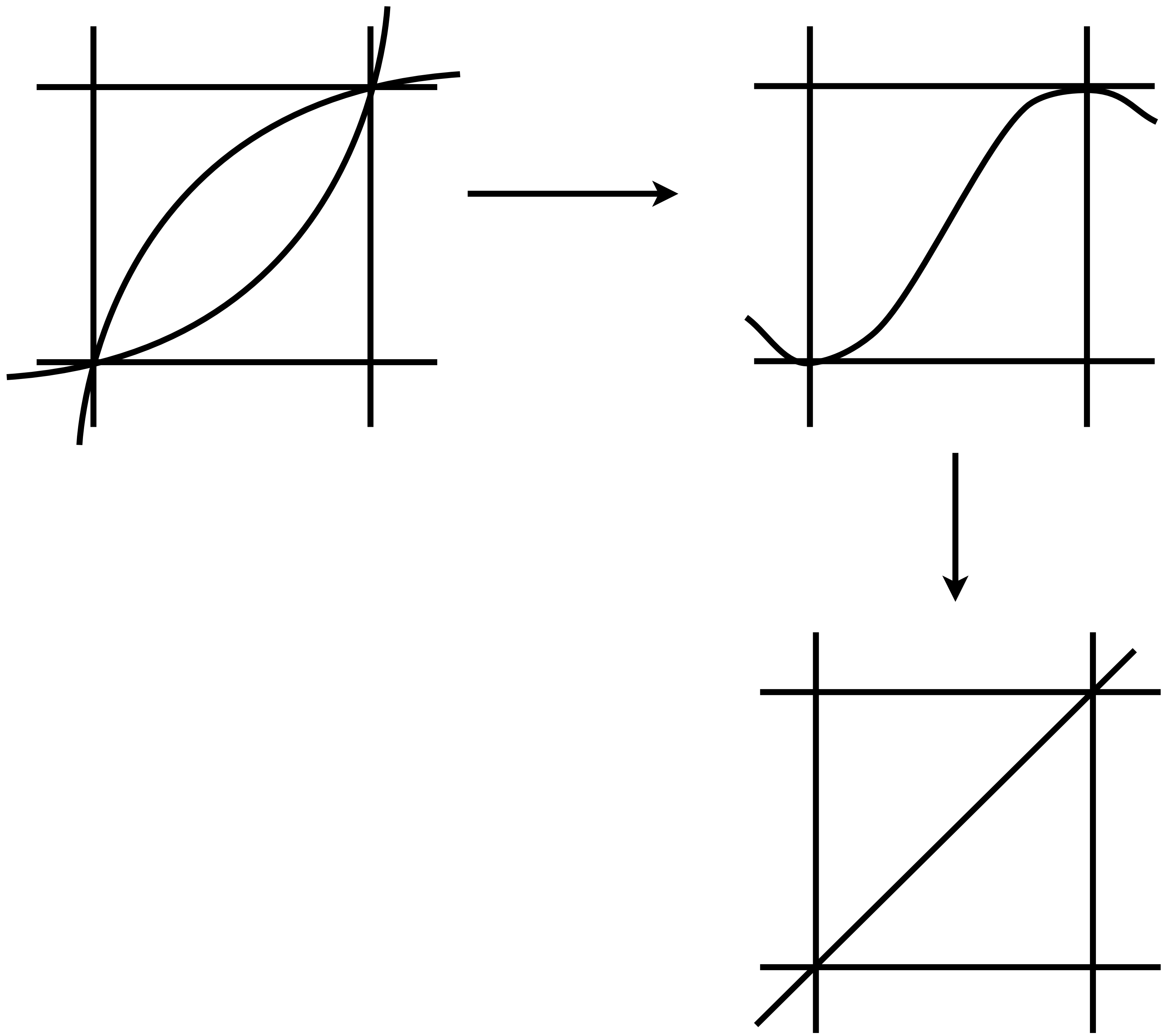
$P^1 \times P^1$



$P^1 \times P^1/(\sigma\tau)$
$= P^1 \times P^1$

$P^1 \times P^1/(\sigma\tau, \sigma\tau^3)$
$= P^1 \times P^1$

$P^1 \times P^1$



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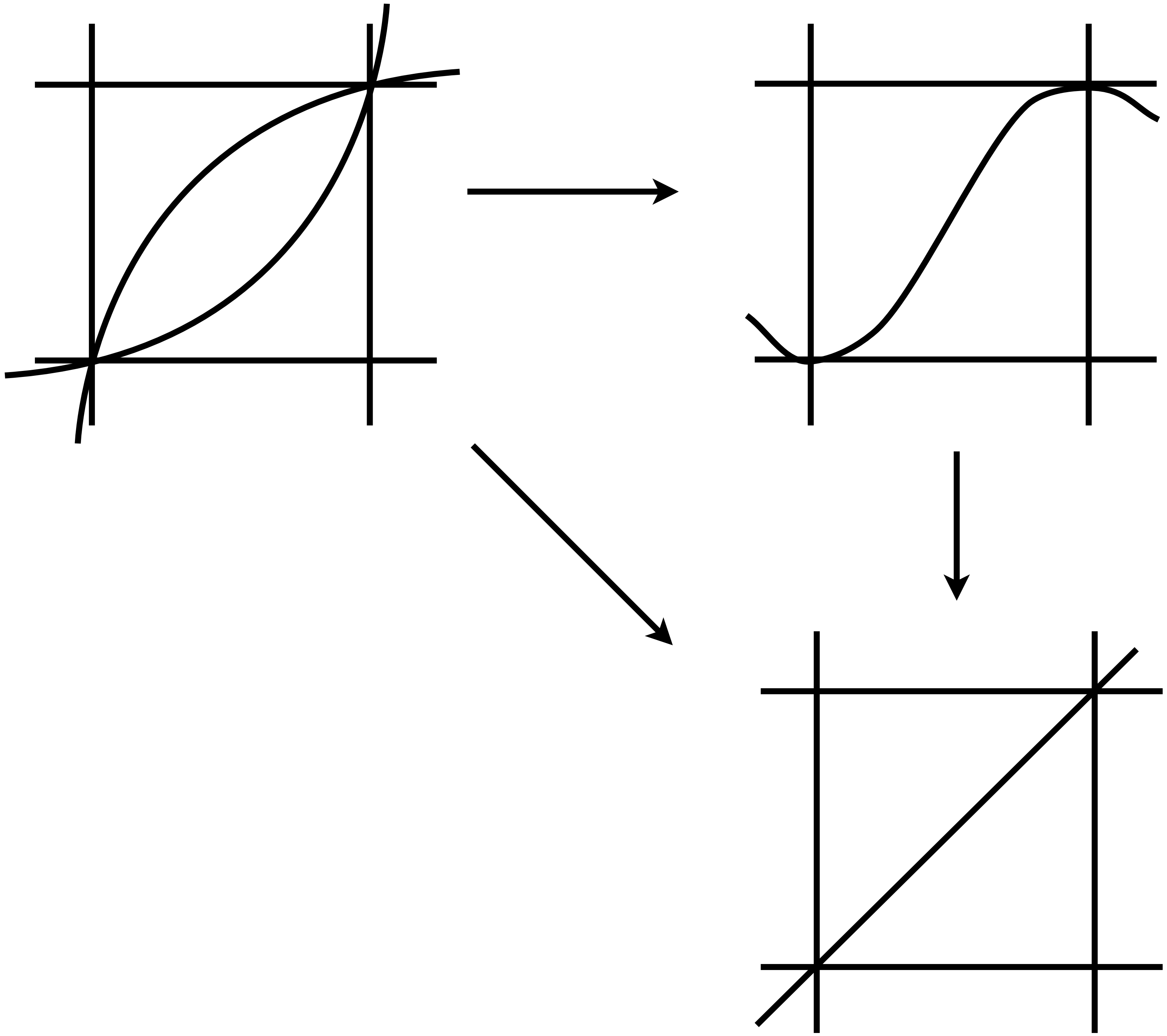

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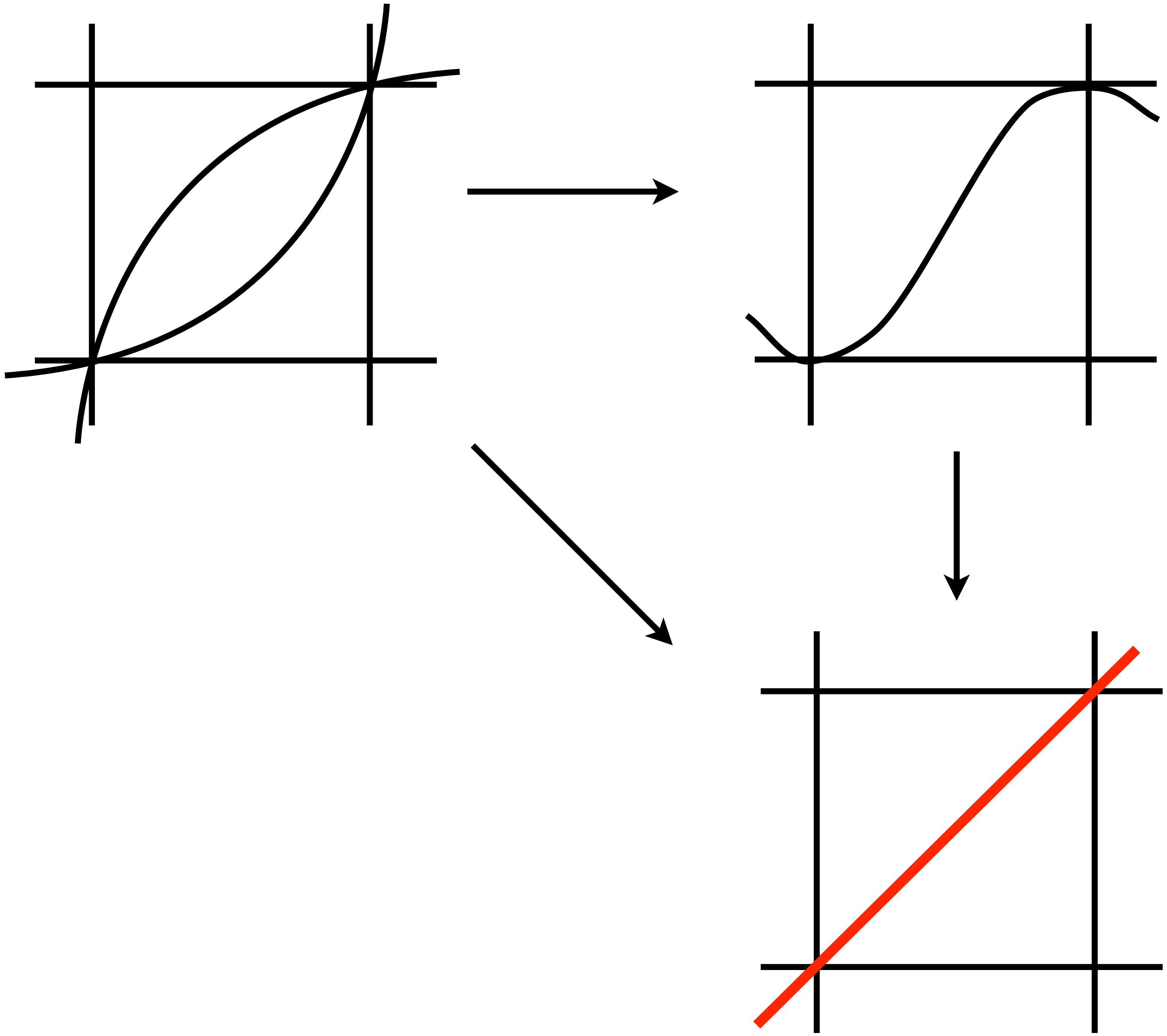
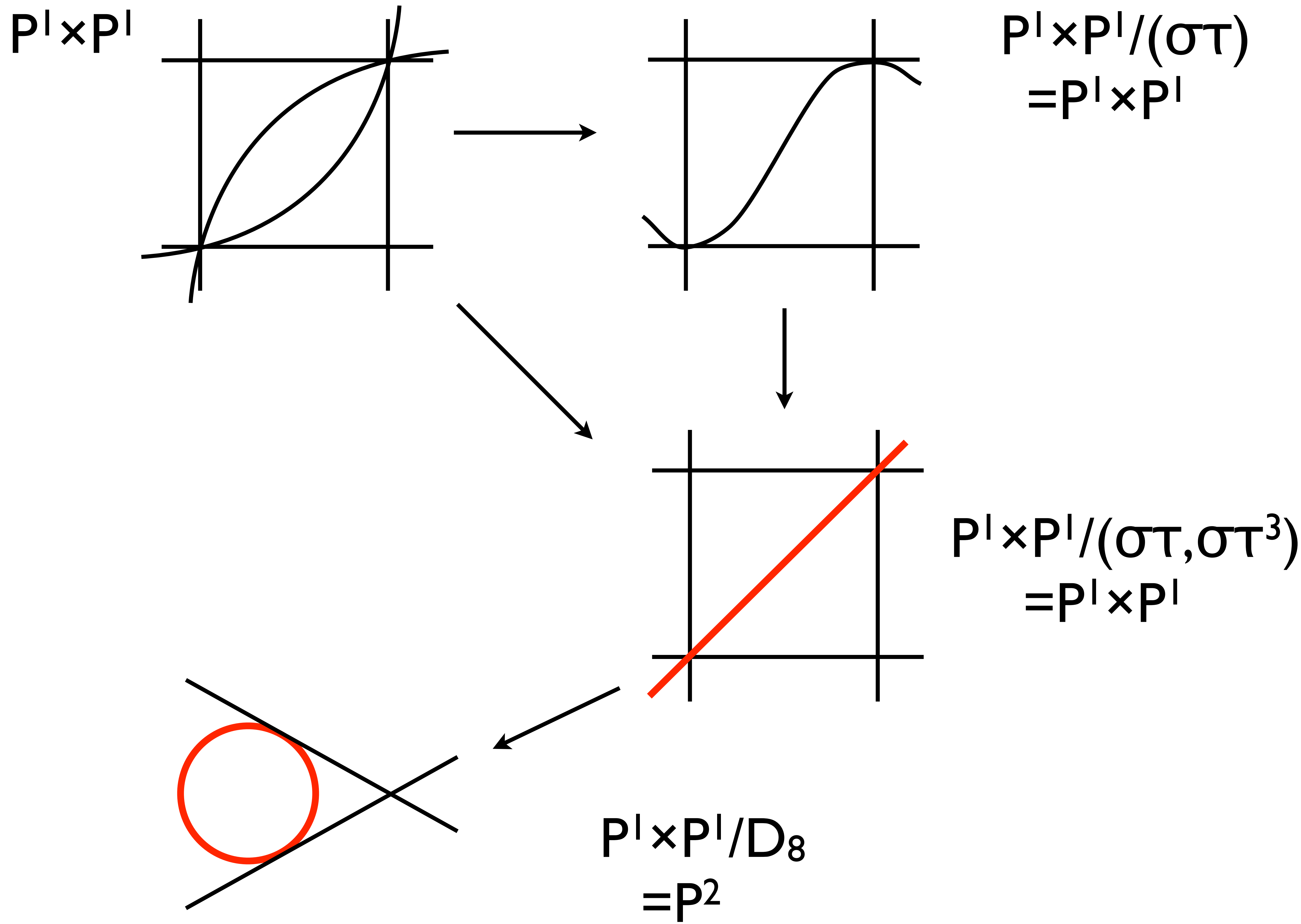
$= P^1 \times P^1$

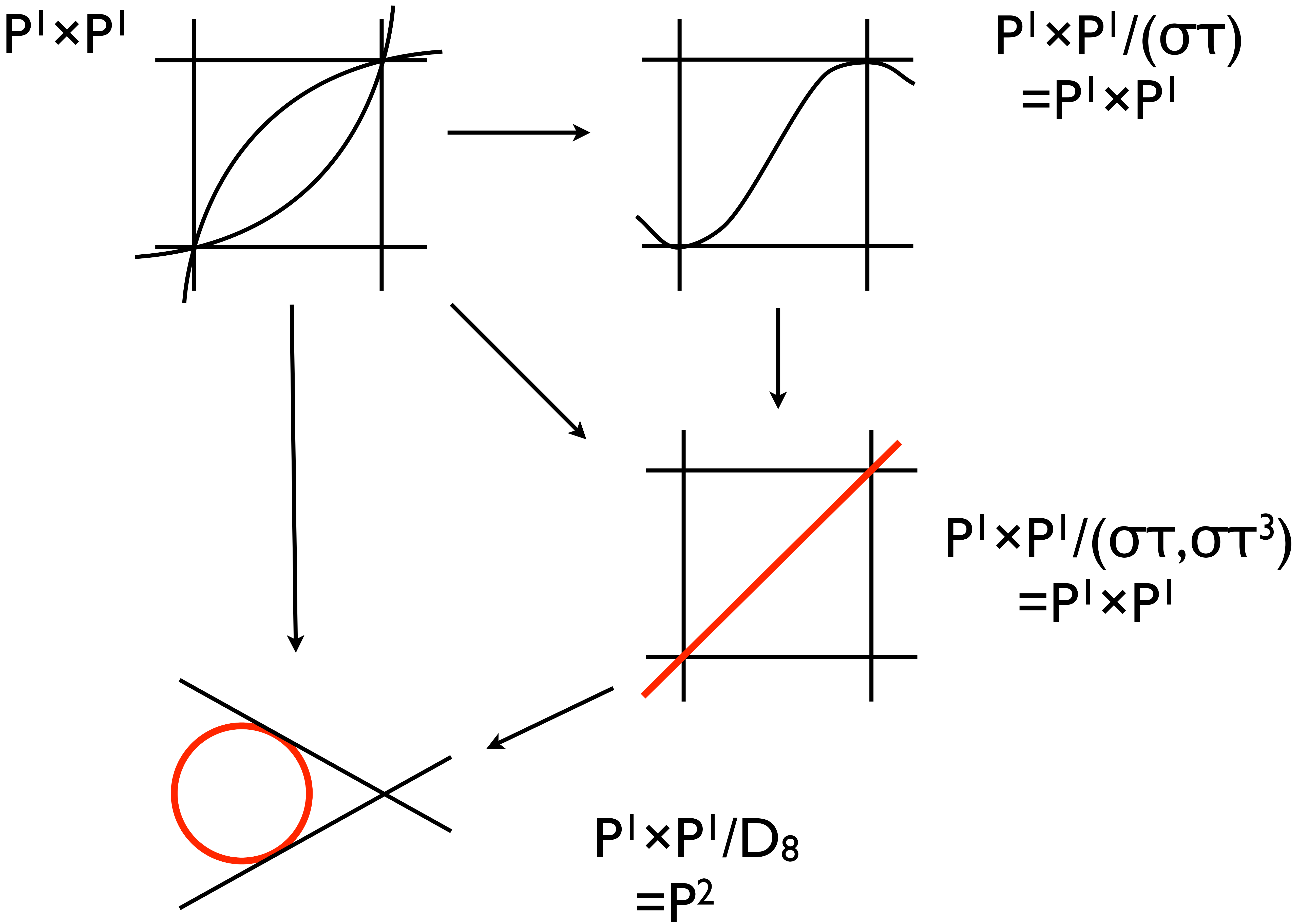
$\mathbb{P}^1 \times \mathbb{P}^1$

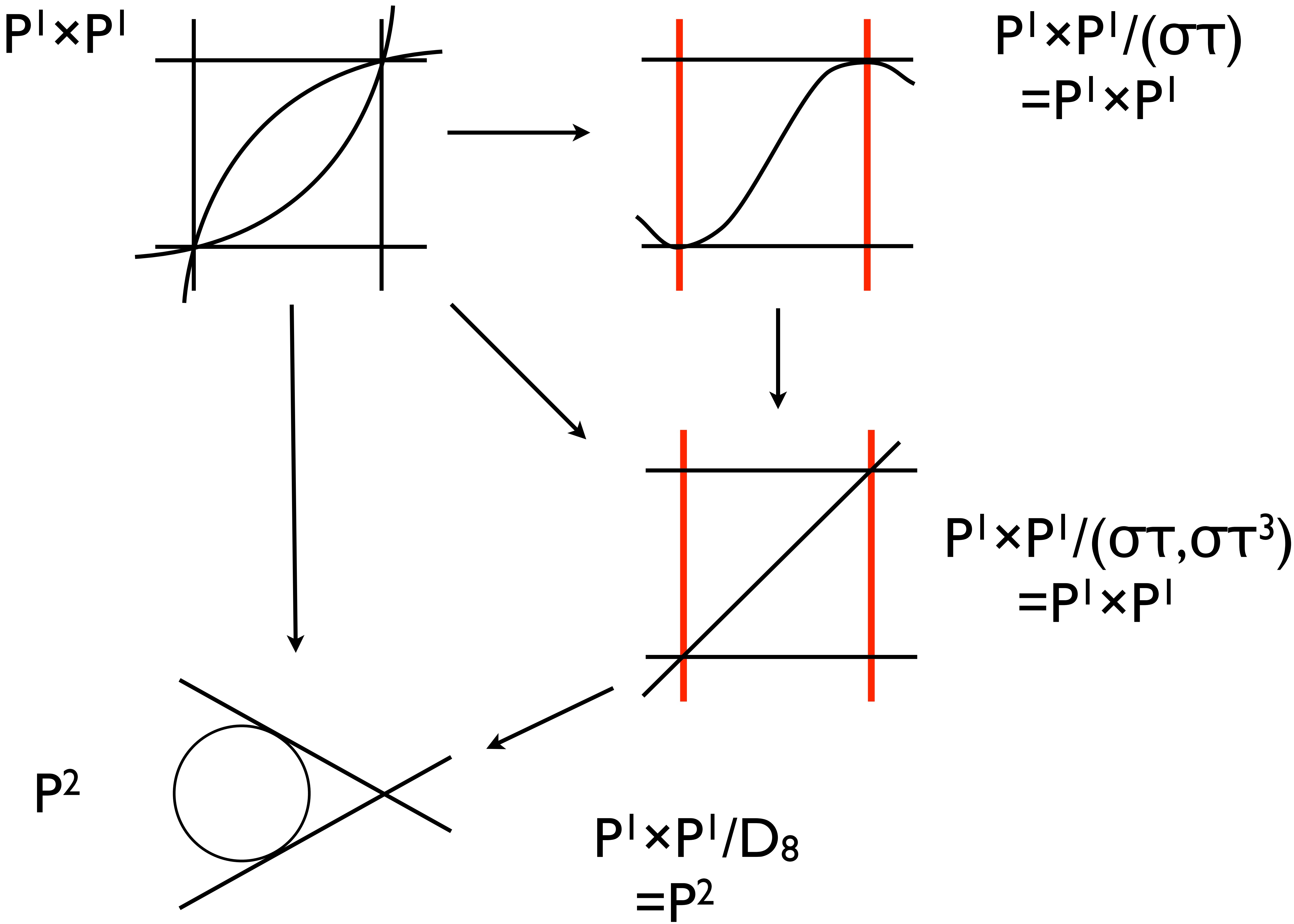


$\mathbb{P}^1 \times \mathbb{P}^1/(\sigma\tau)$
$= \mathbb{P}^1 \times \mathbb{P}^1$

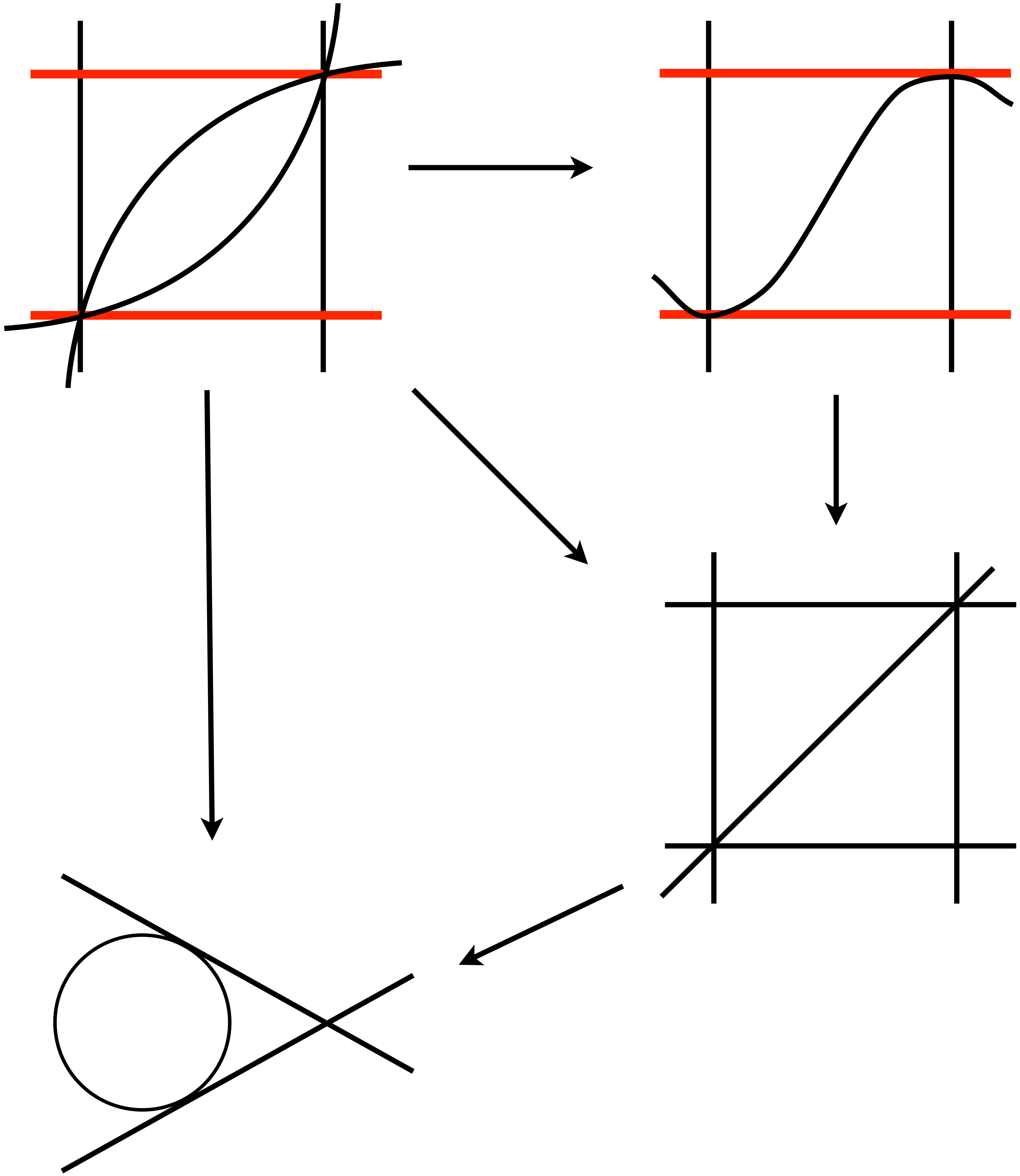
$\mathbb{P}^1 \times \mathbb{P}^1/(\sigma\tau, \sigma\tau^3)$
$= \mathbb{P}^1 \times \mathbb{P}^1$


$P^1 \times P^1$
$P^1 \times P^1/(\sigma\tau)$
$= P^1 \times P^1$
$P^1 \times P^1/(\sigma\tau, \sigma\tau^3)$
$= P^1 \times P^1$
$P^1 \times P^1/D_8$
$= P^2$

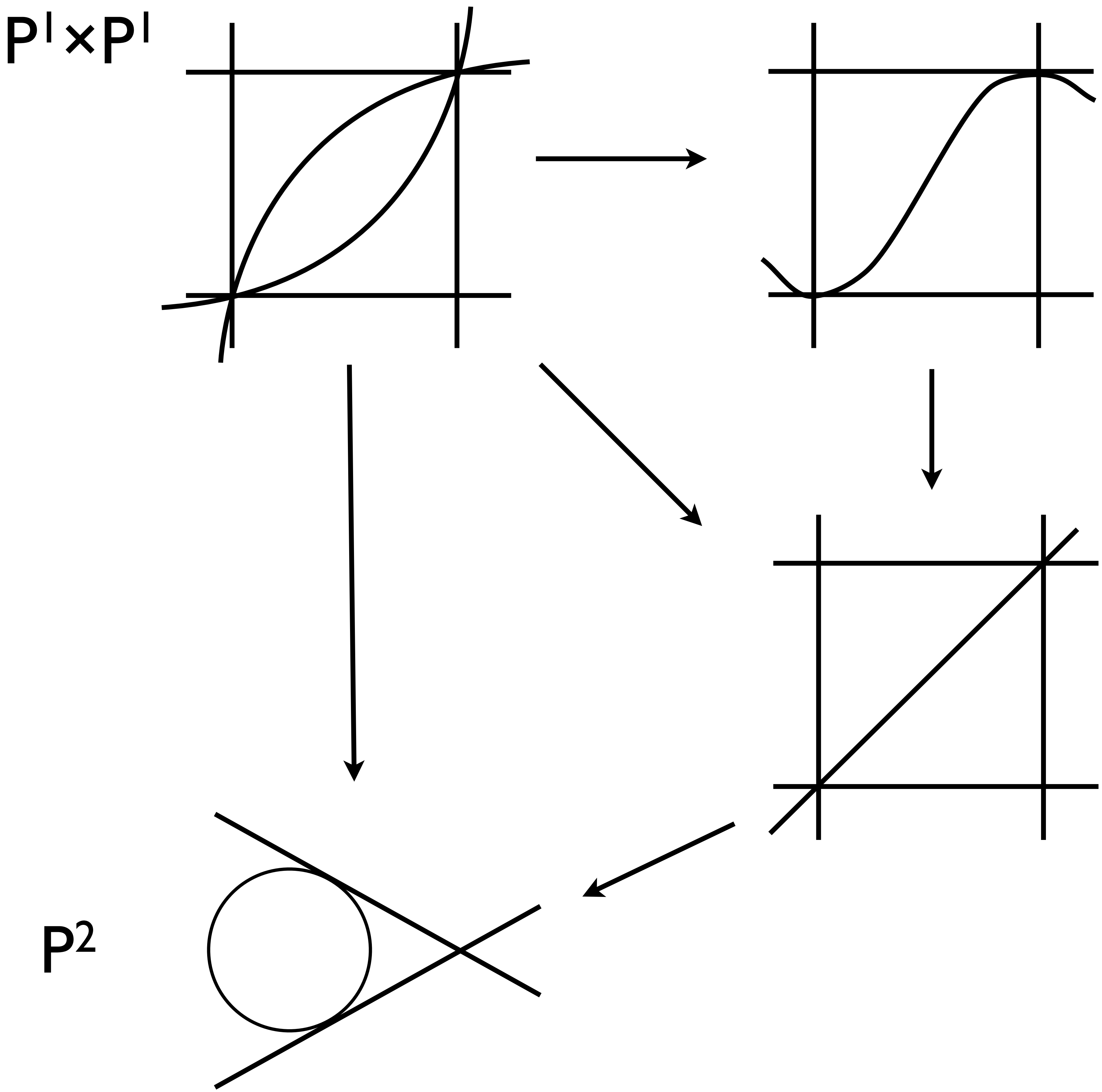

$P^1 \times P^1$
$P^1 \times P^1/(\sigma\tau)$
$= P^1 \times P^1$
$P^1 \times P^1/(\sigma\tau, \sigma\tau^3)$
$= P^1 \times P^1$
$P^1 \times P^1/D_8$
$= P^2$


$P^1\times P^1$
$P^1\times P^1/(\sigma\tau)$
$=P^1\times P^1$
$P^1\times P^1/(\sigma\tau,\sigma\tau^3)$
$=P^1\times P^1$
P^2
$P^1\times P^1/D_8$
$=P^2$

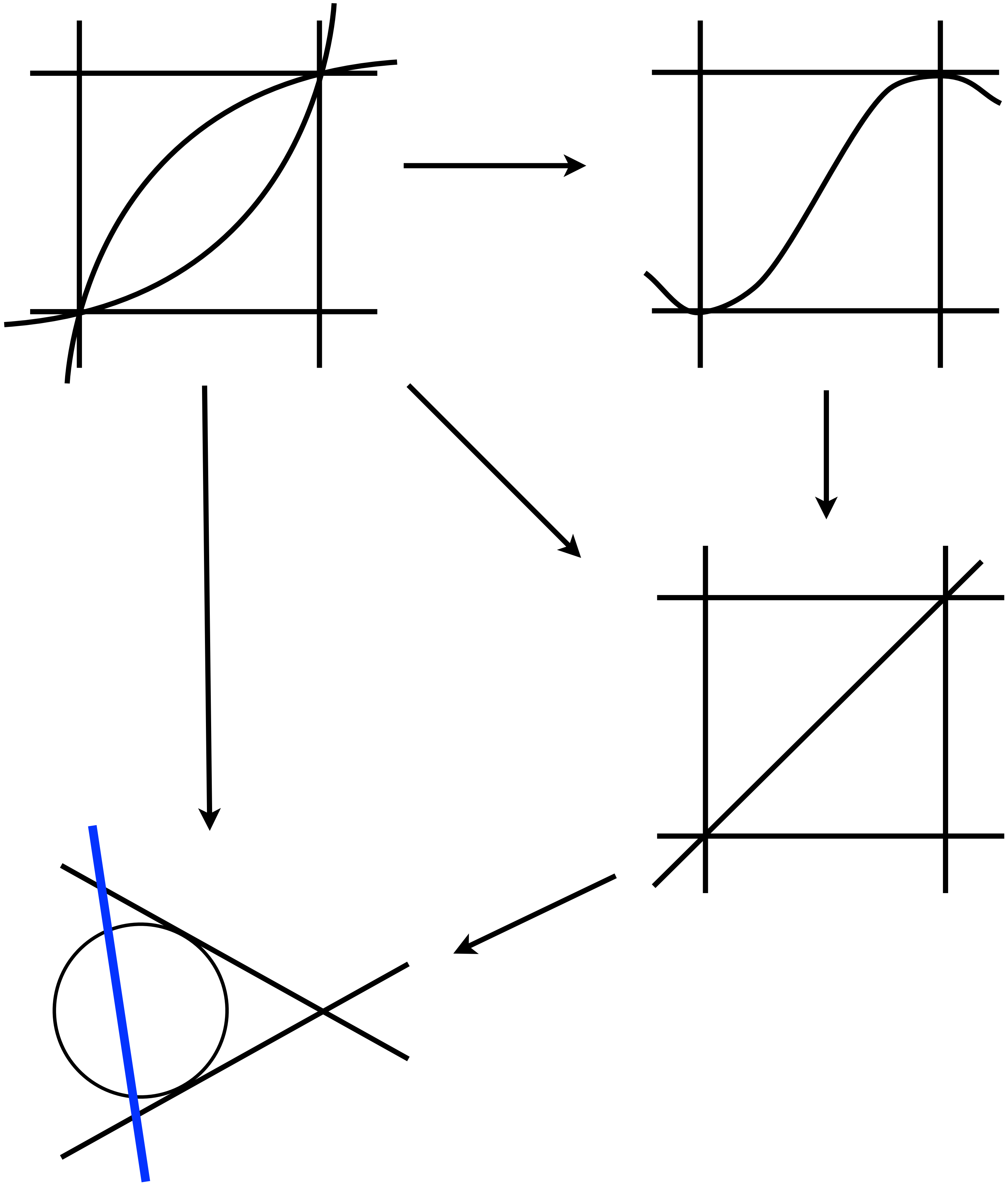
$P^1 \times P^1$



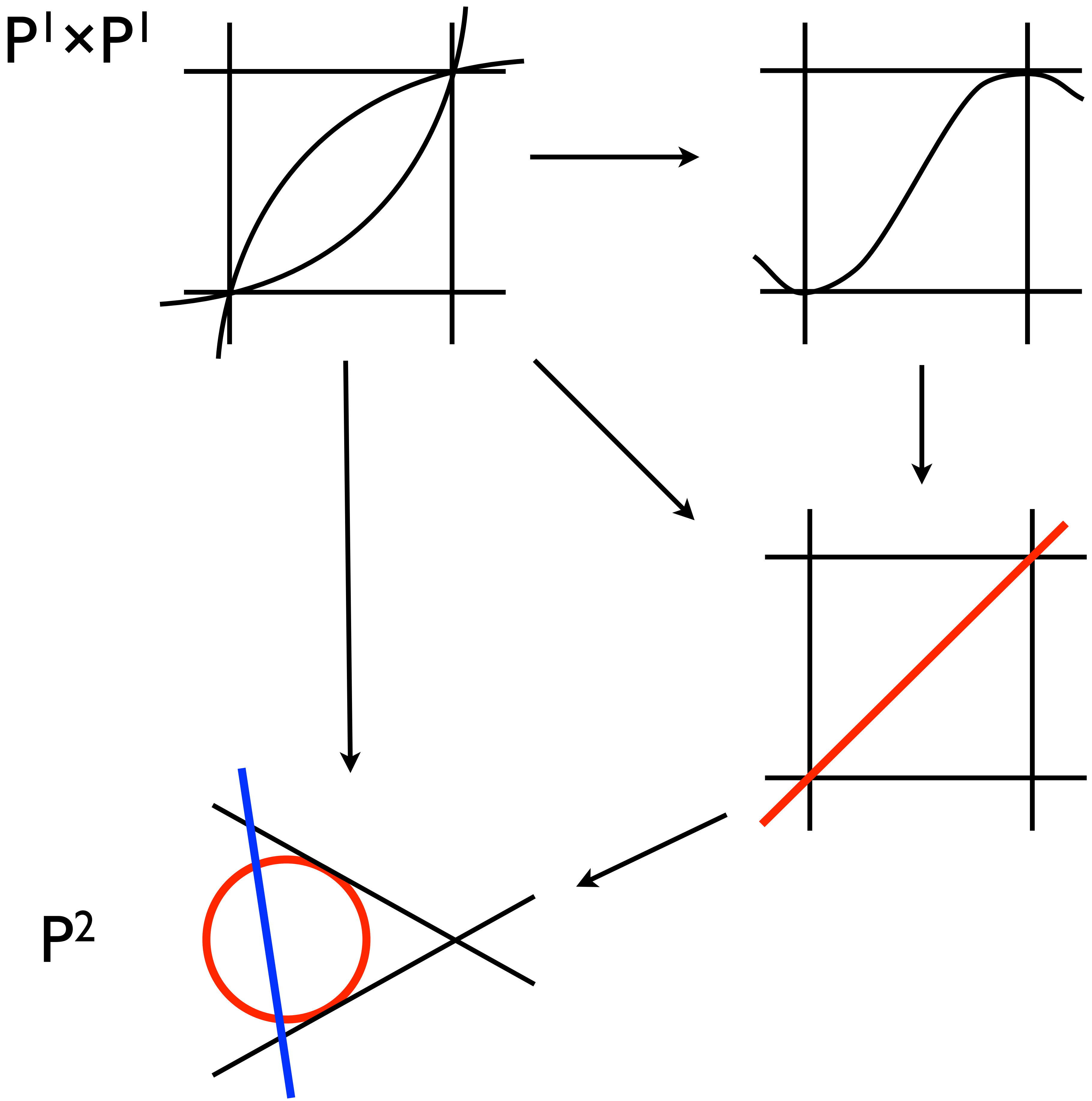
P^2

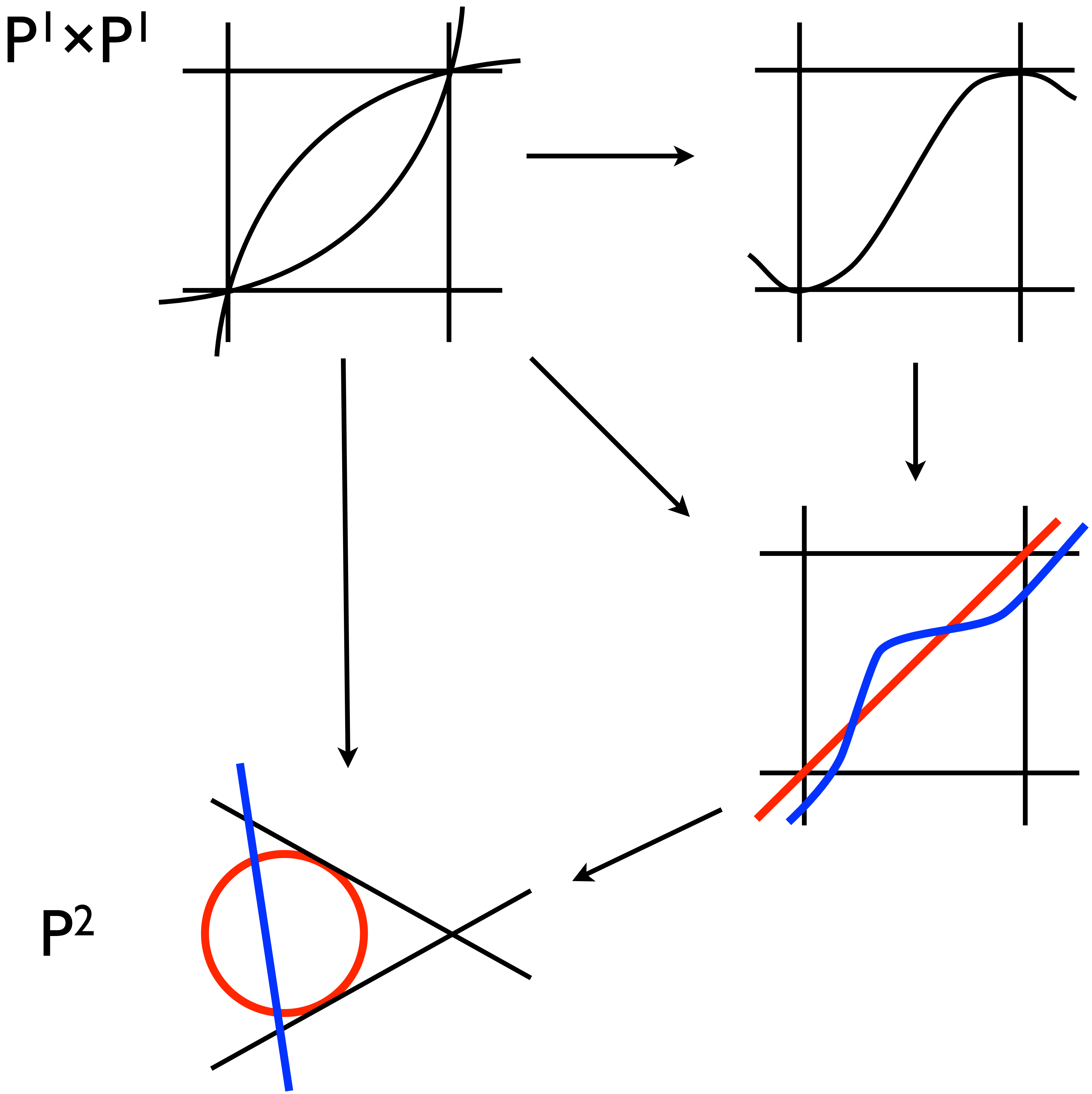

$P^1 \times P^1$
P^2

$P^1 \times P^1$

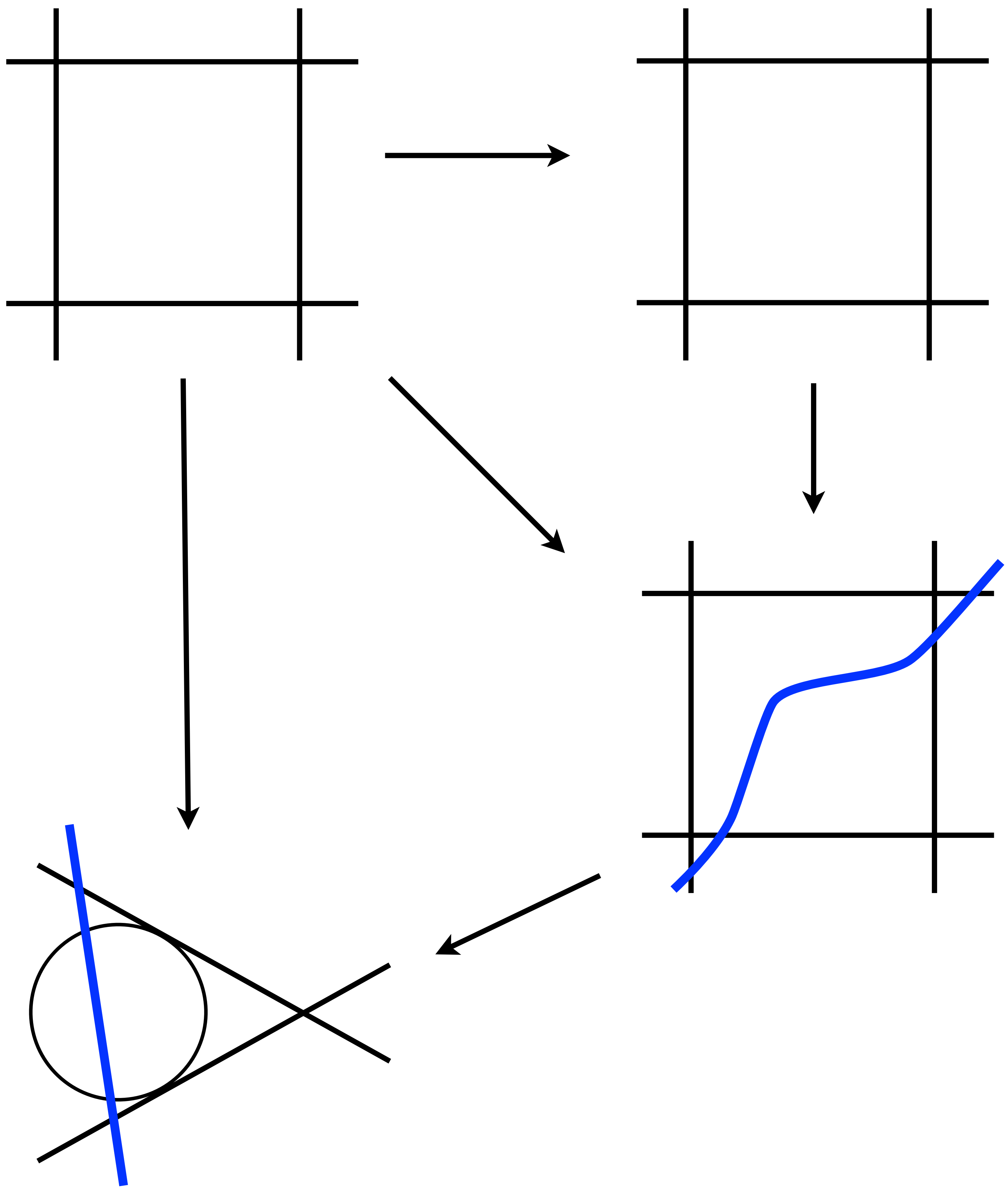


P^2

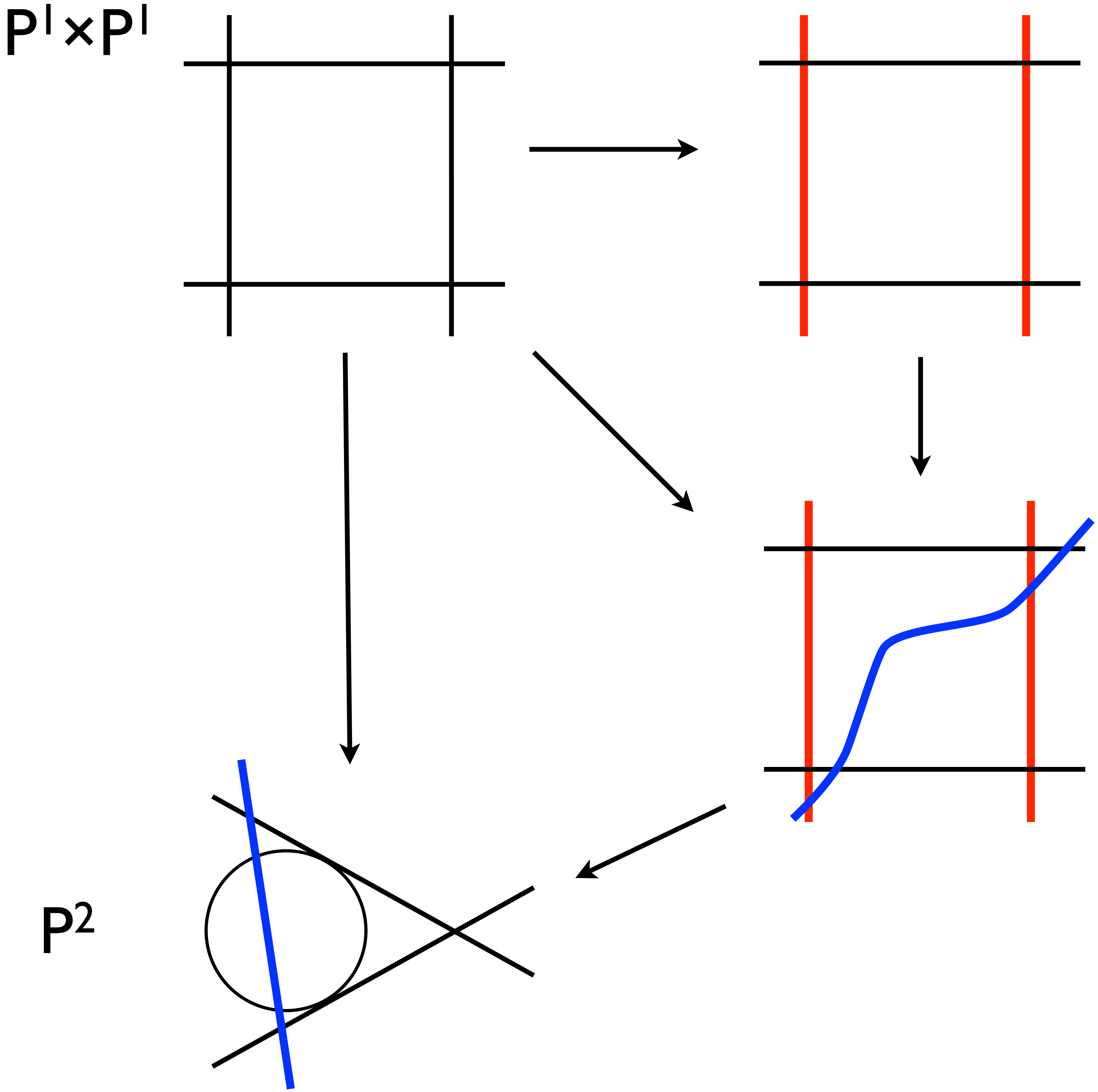

$\mathbf{P}^1 \times \mathbf{P}^1$
$\mathbf{P}^2$

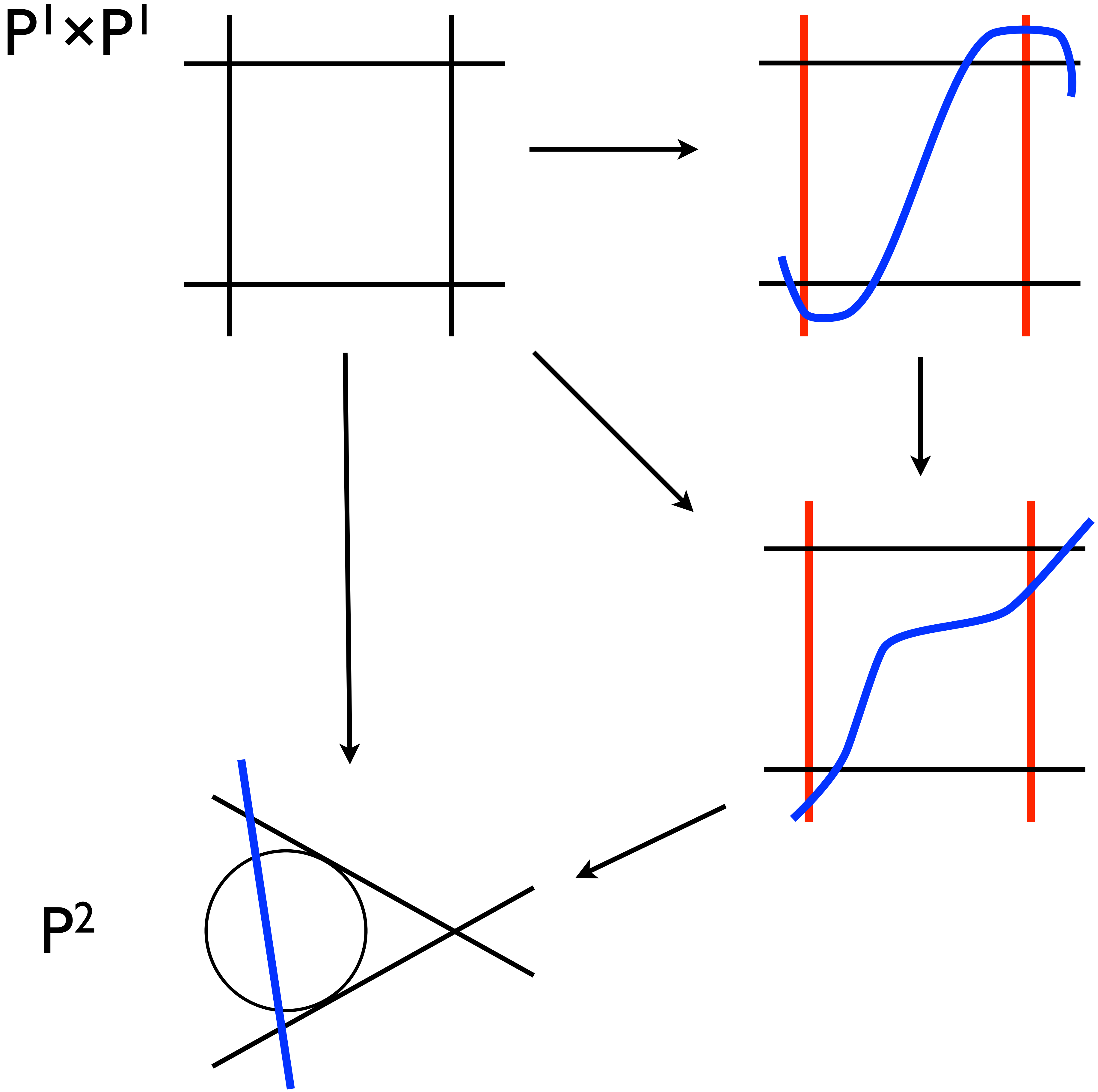

P¹×P¹
P²

$P^1 \times P^1$

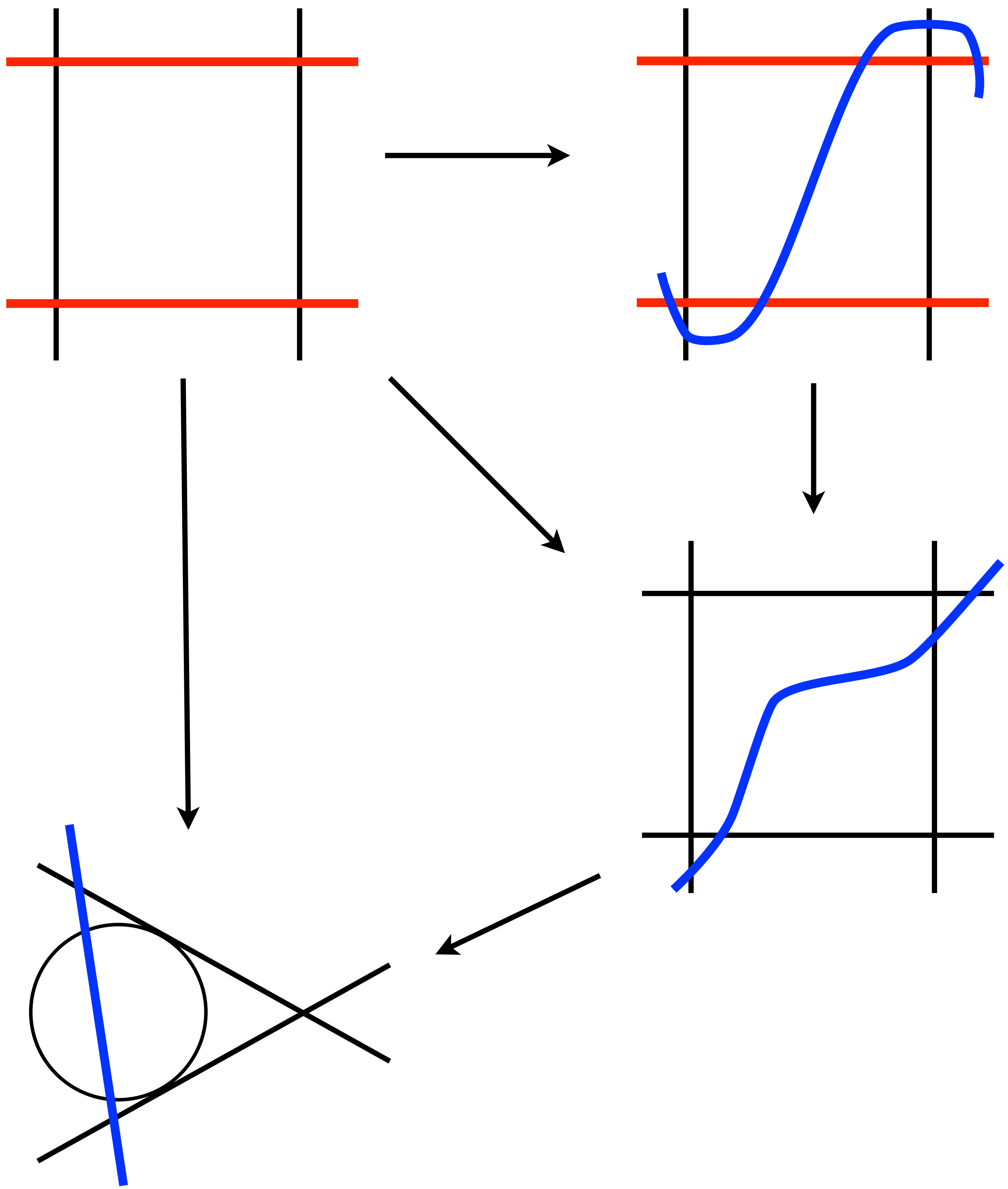


P^2

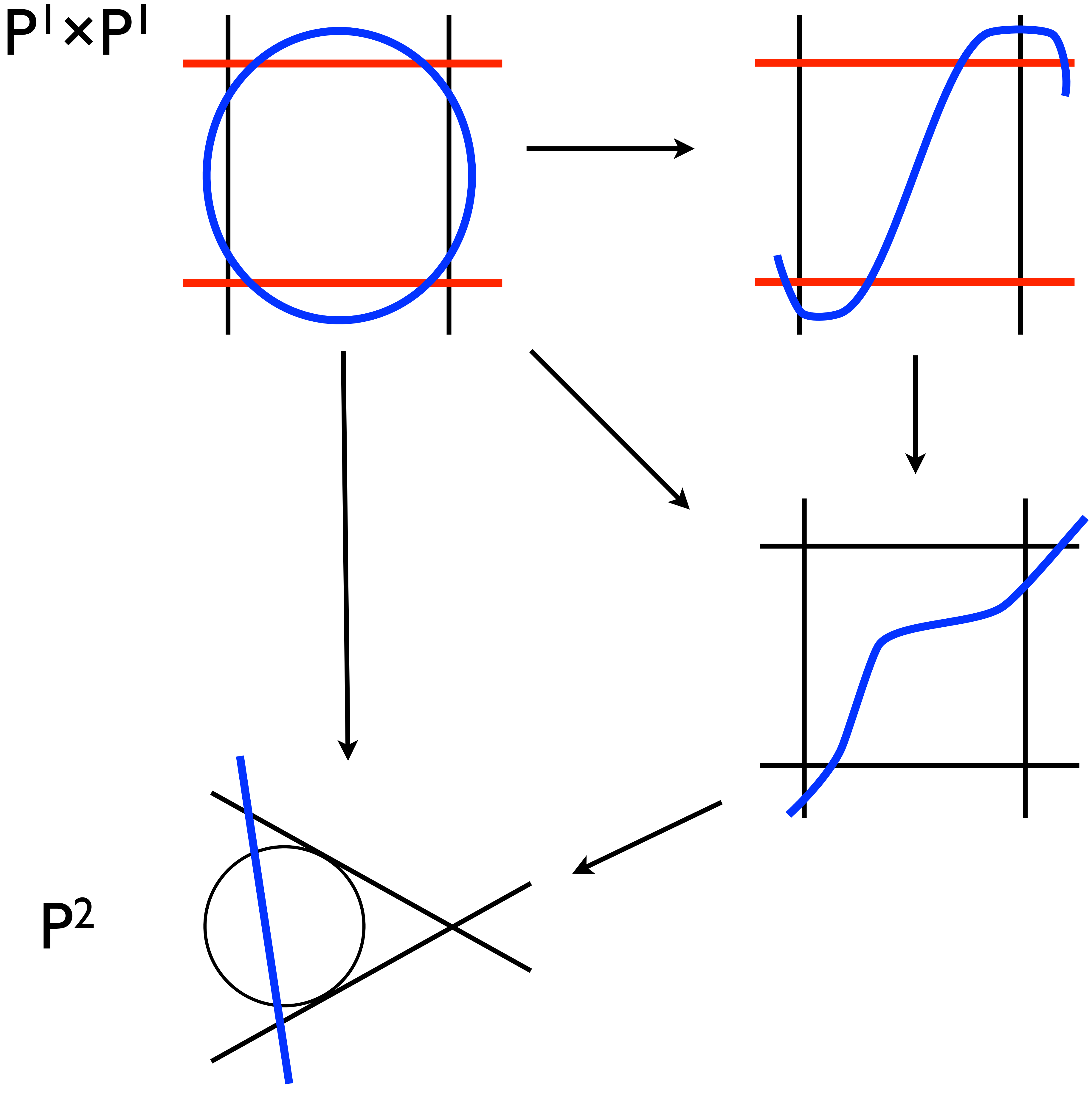

P¹×P¹
P²


$P^1 \times P^1$
P^2

$P^1 \times P^1$

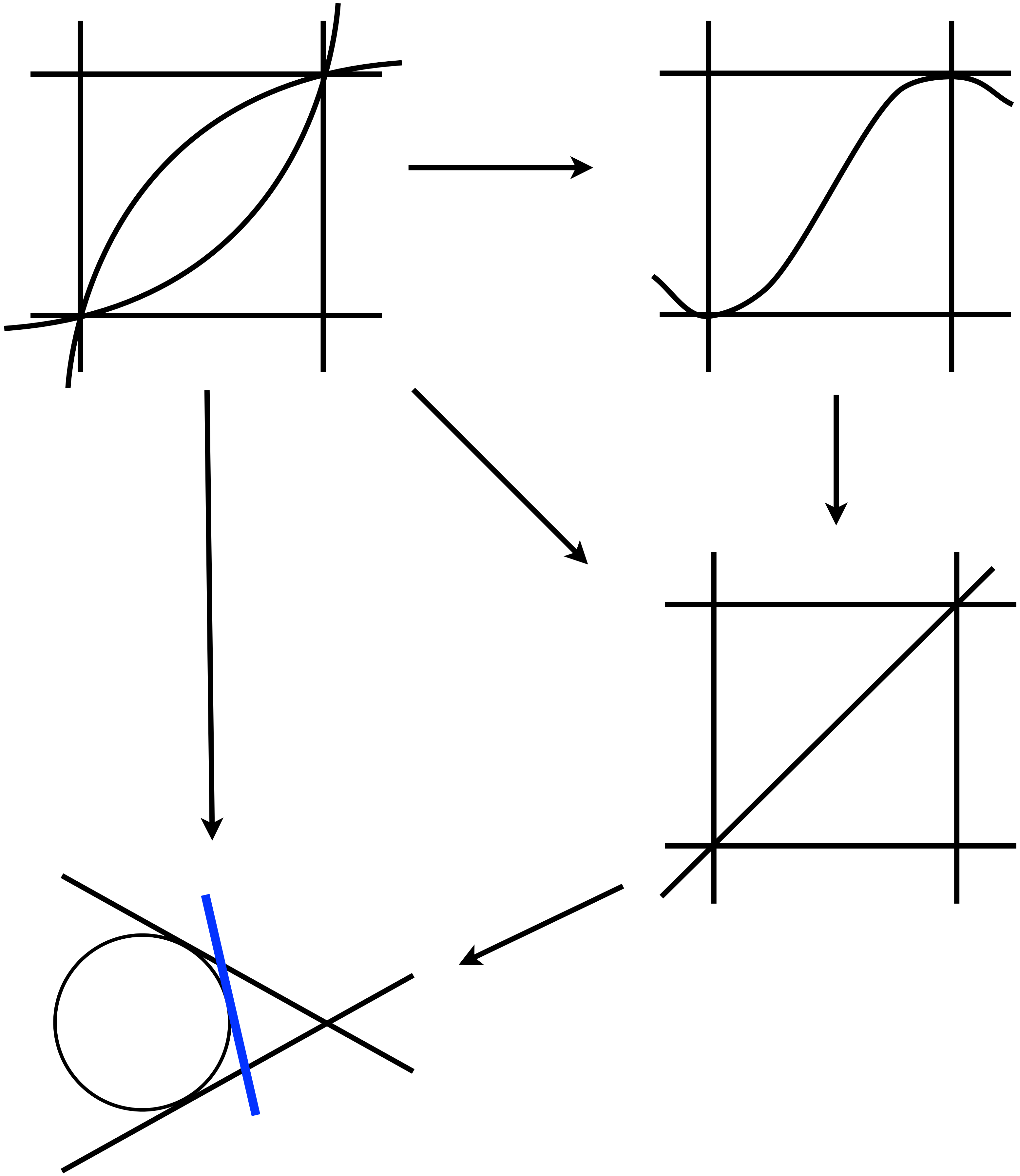


P^2

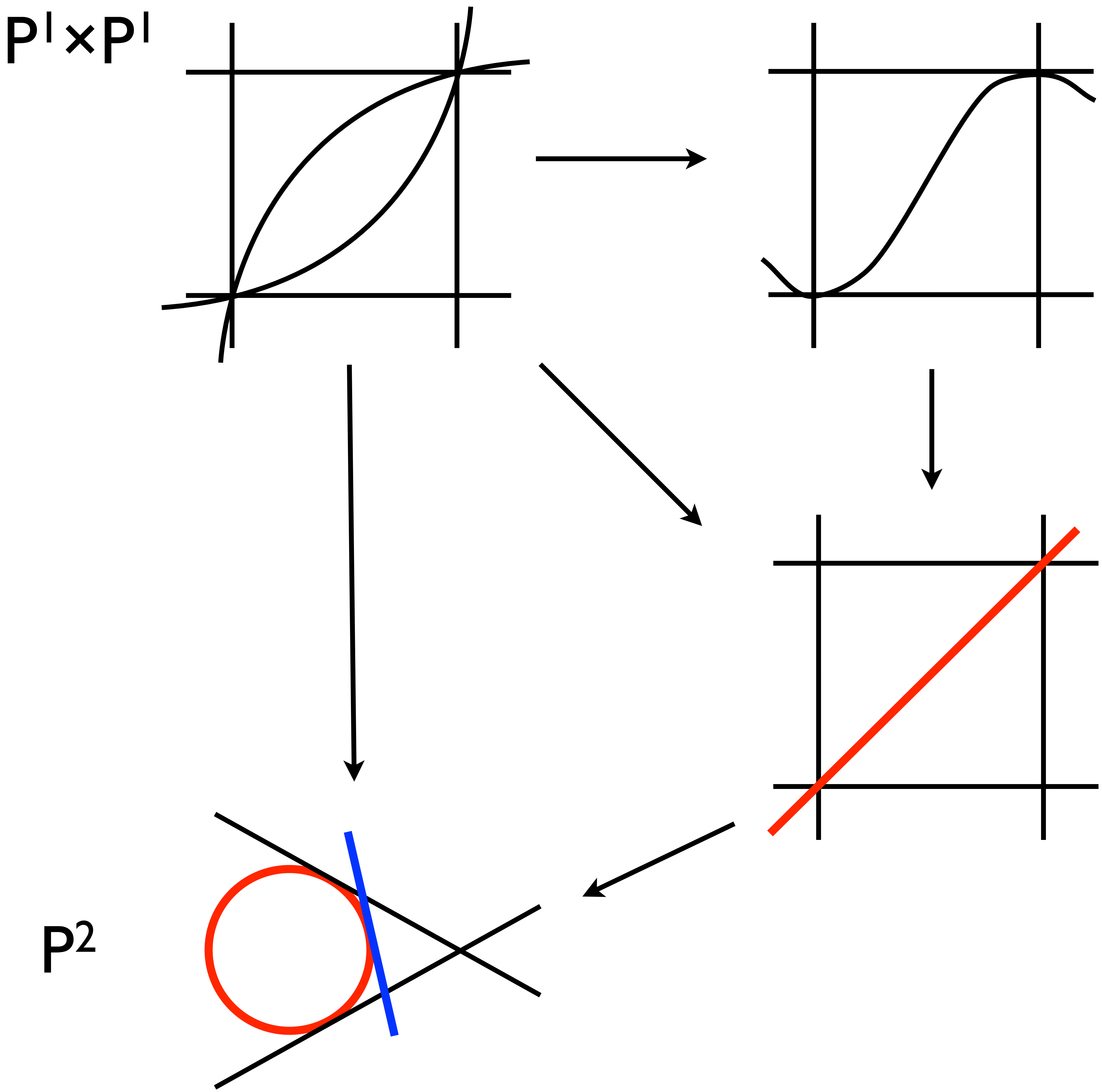


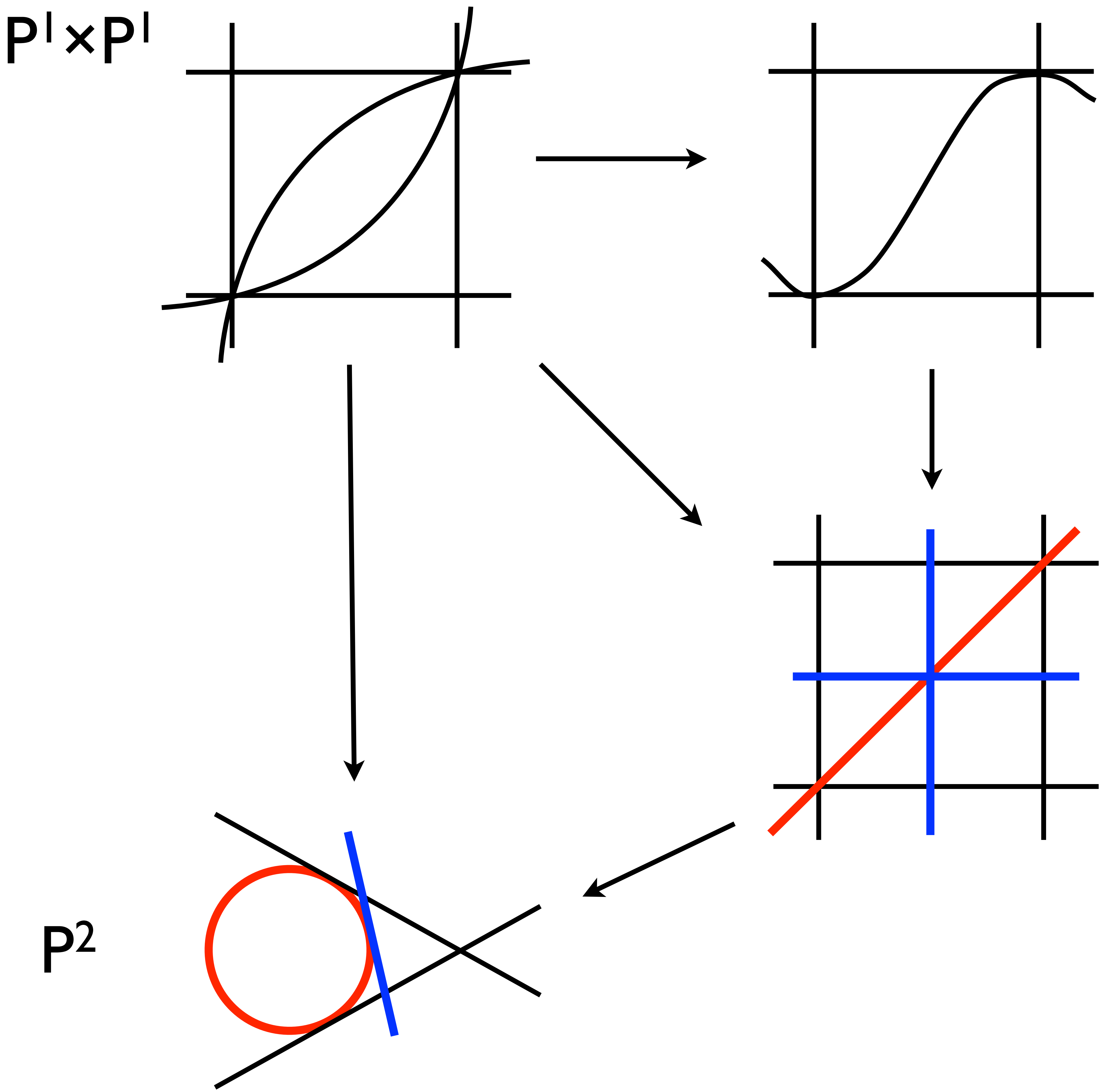
P1×P1
P2

$\mathbf{P}^1 \times \mathbf{P}^1$

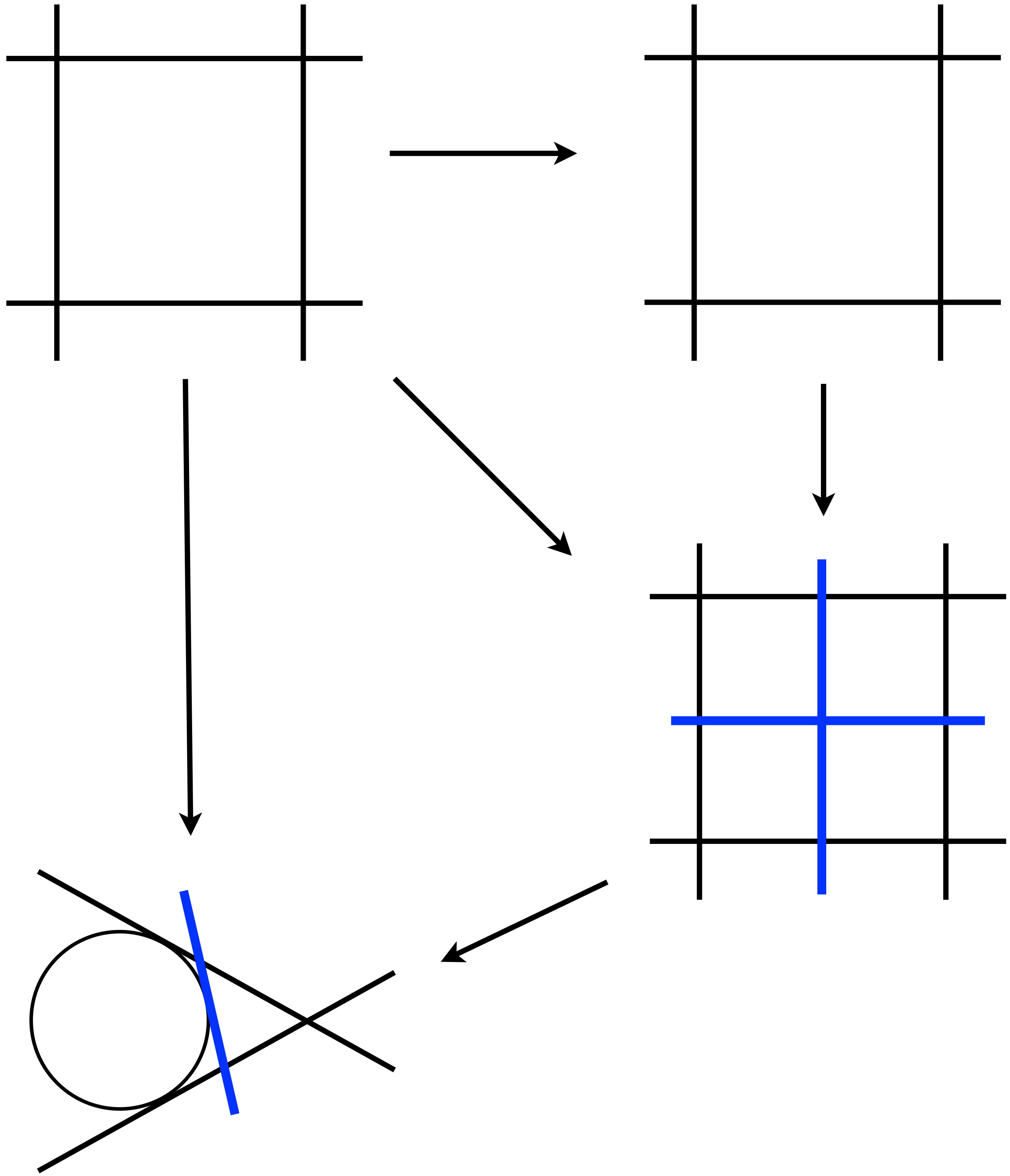


$\mathbf{P}^2$


$P^1 \times P^1$
P^2

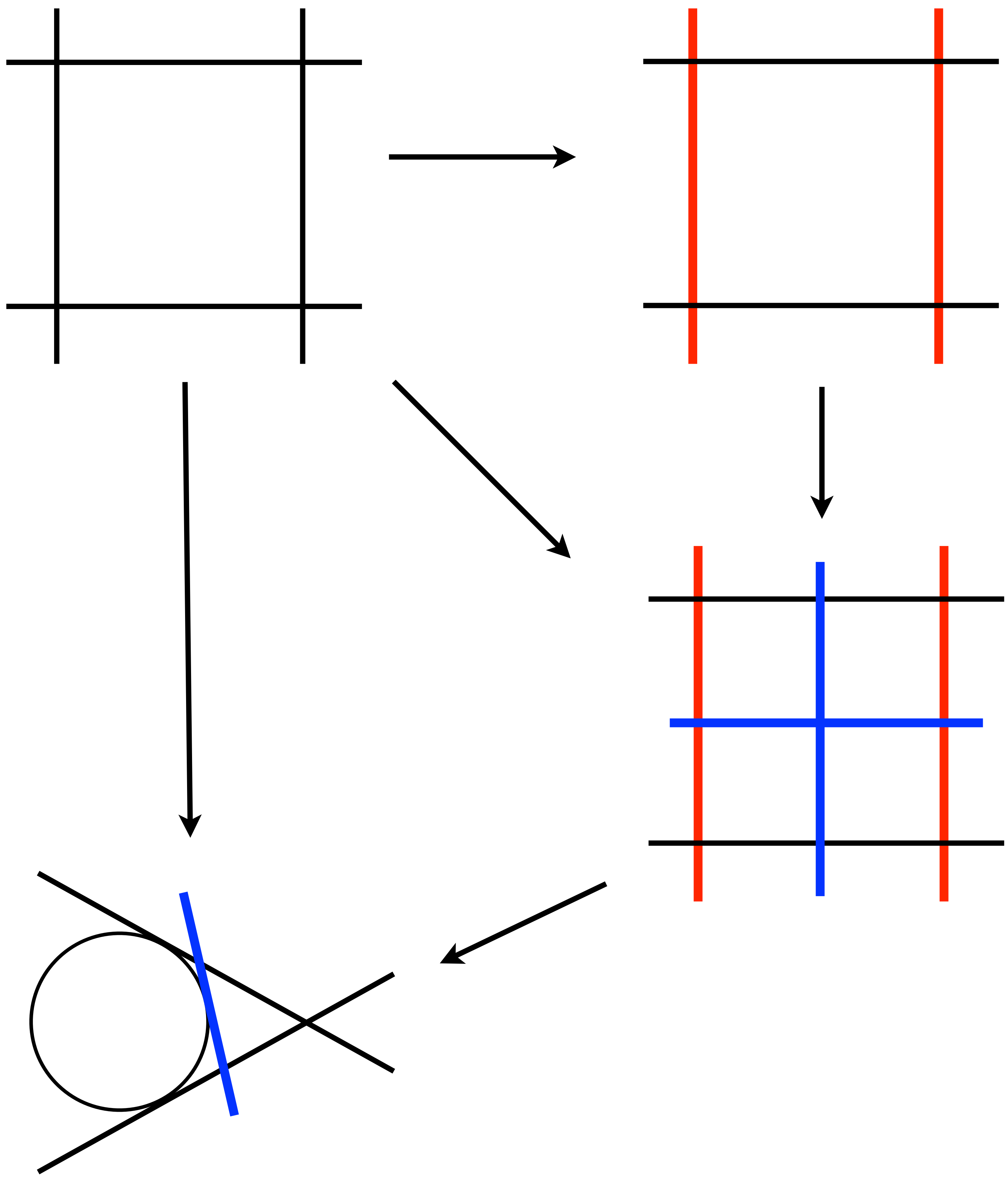

$P^1 \times P^1$
P^2

$P^1 \times P^1$

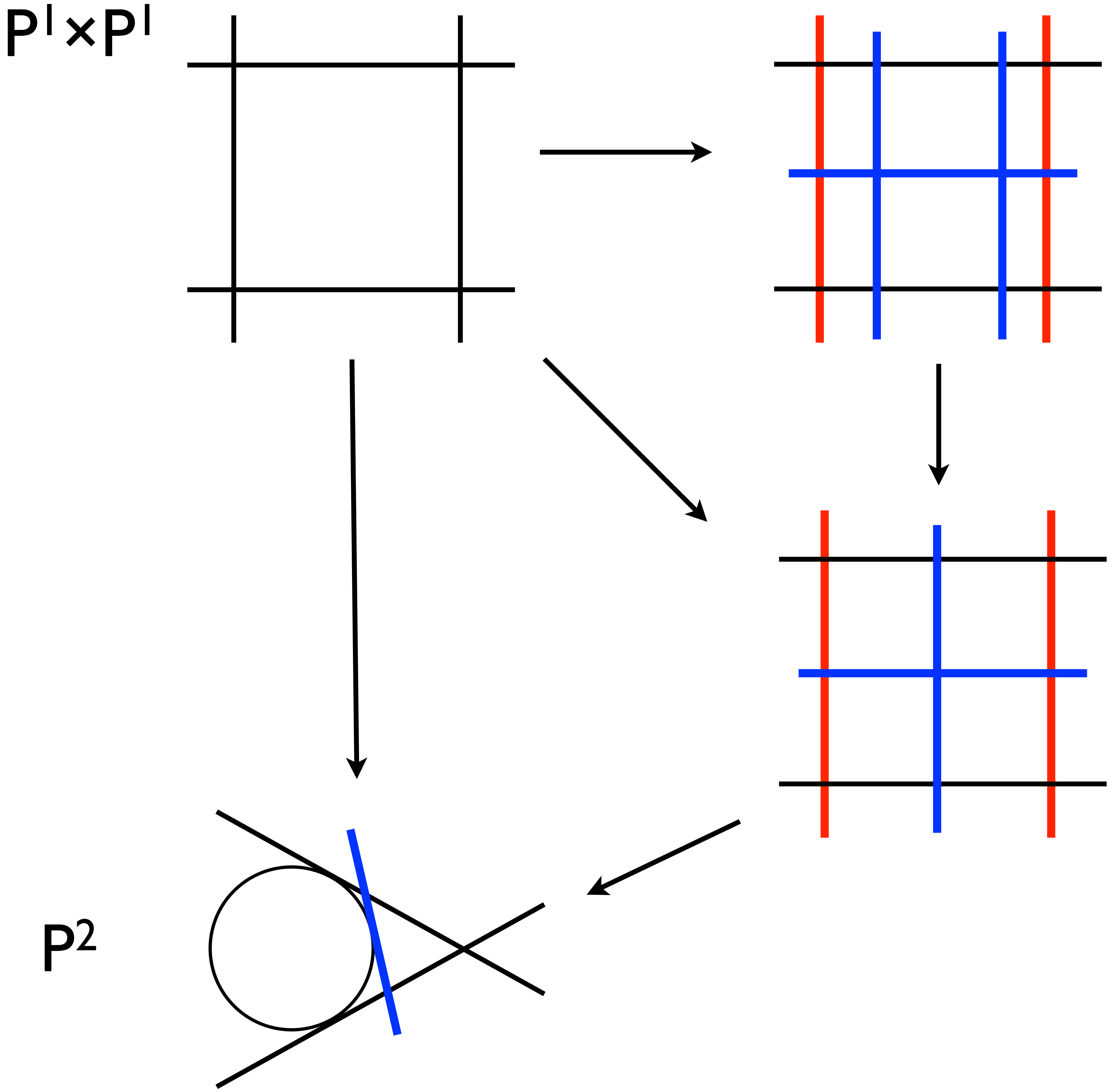


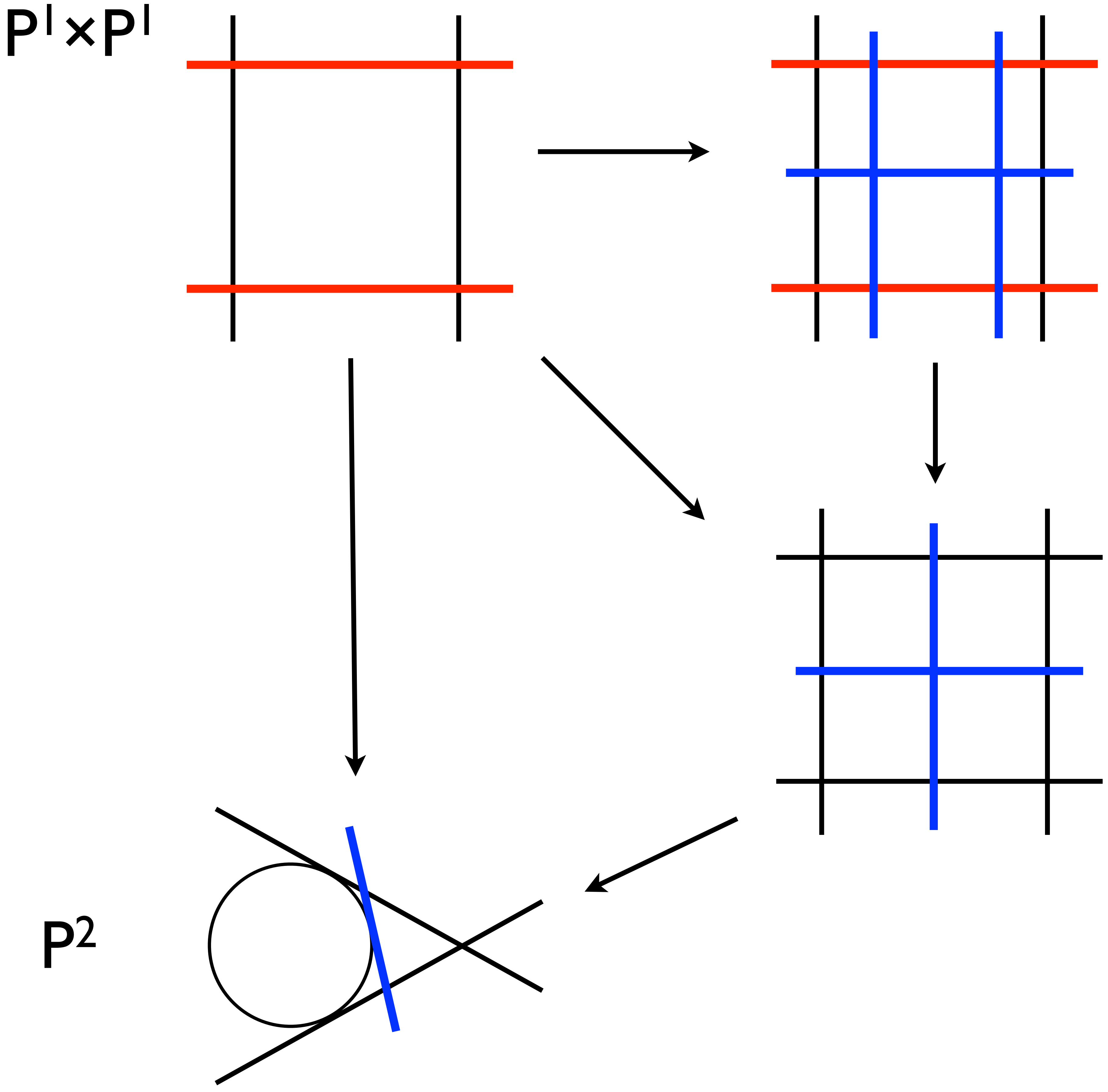
P^2

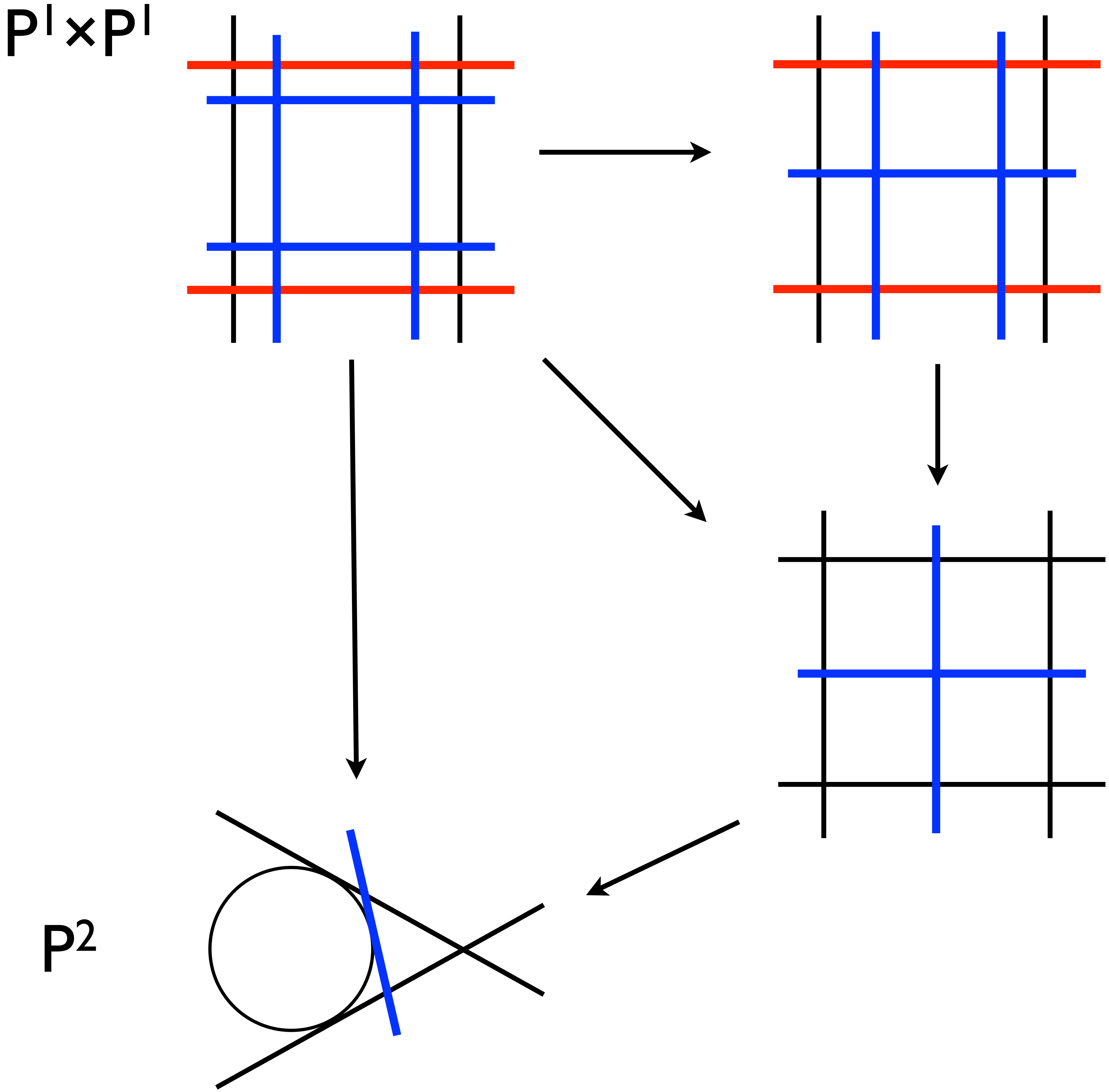
$P^1 \times P^1$



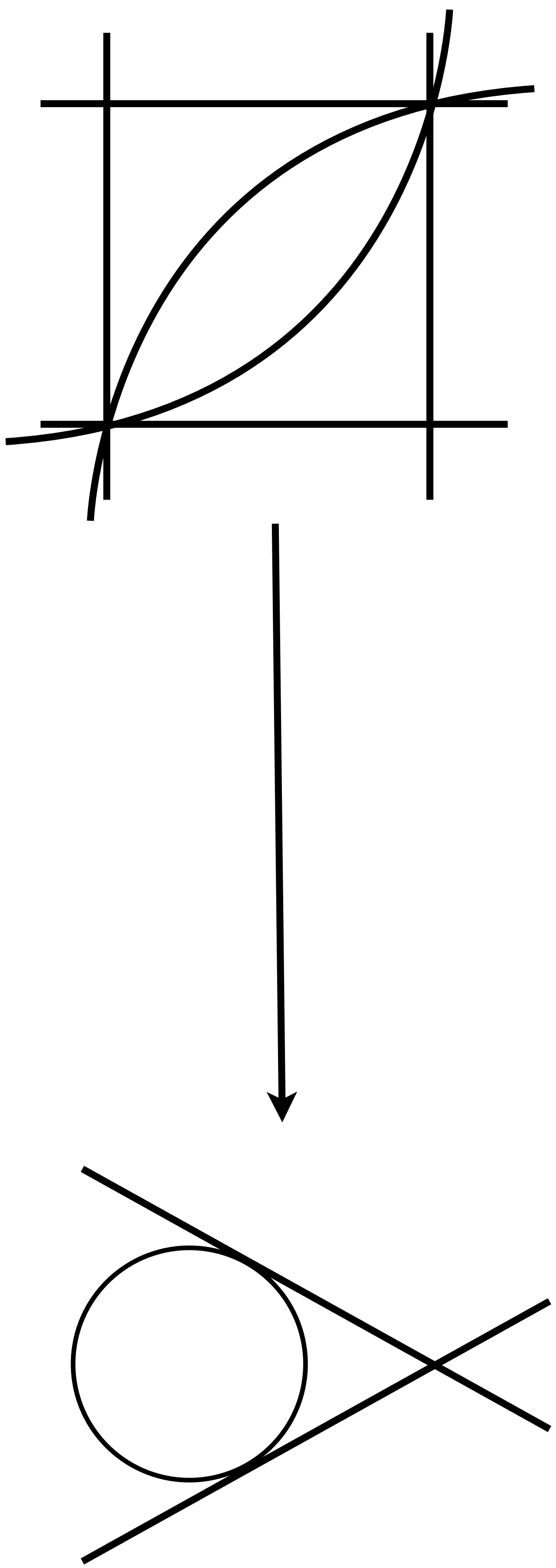
P^2


$P^1 \times P^1$
P^2

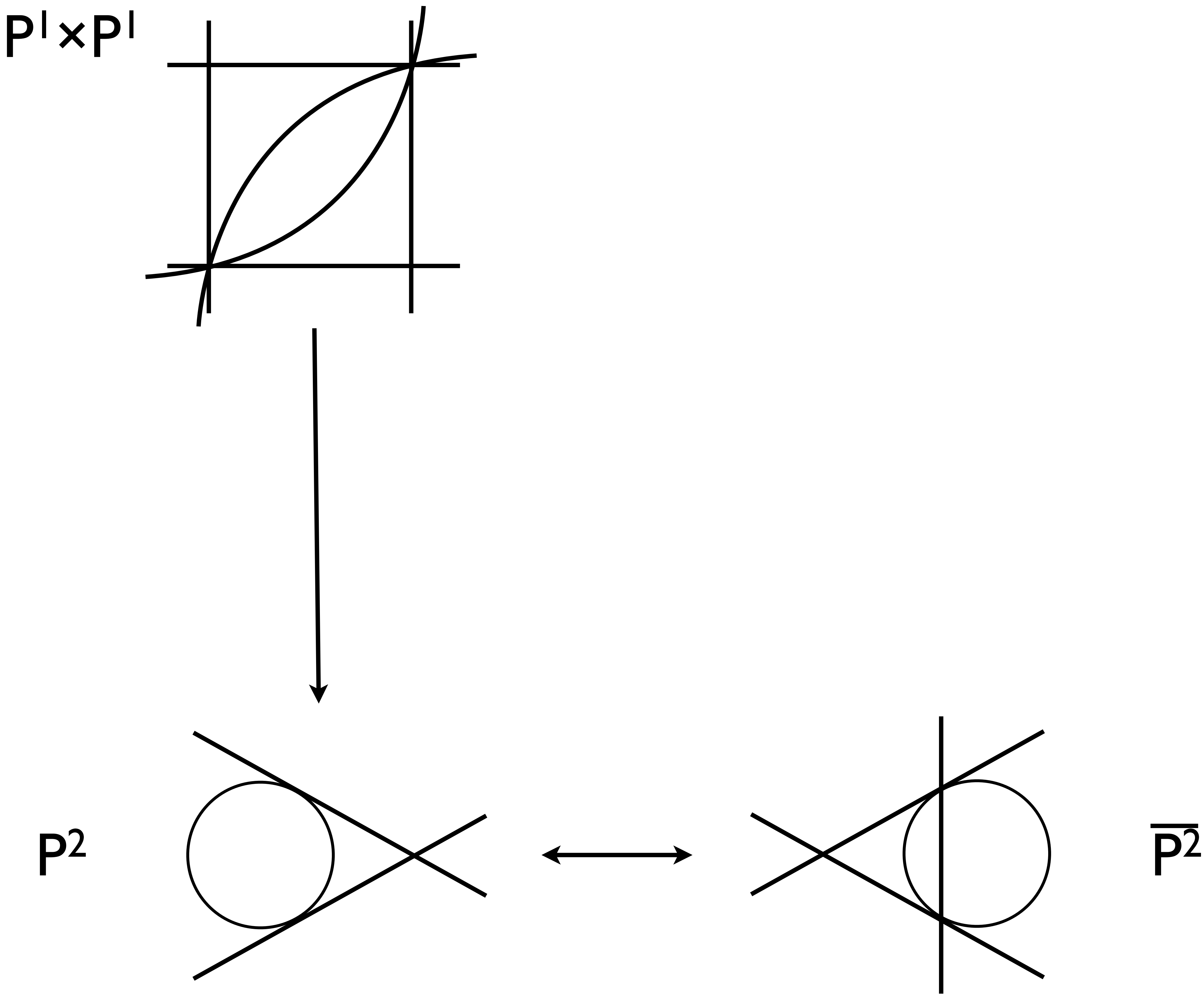

$P^1 \times P^1$
P^2

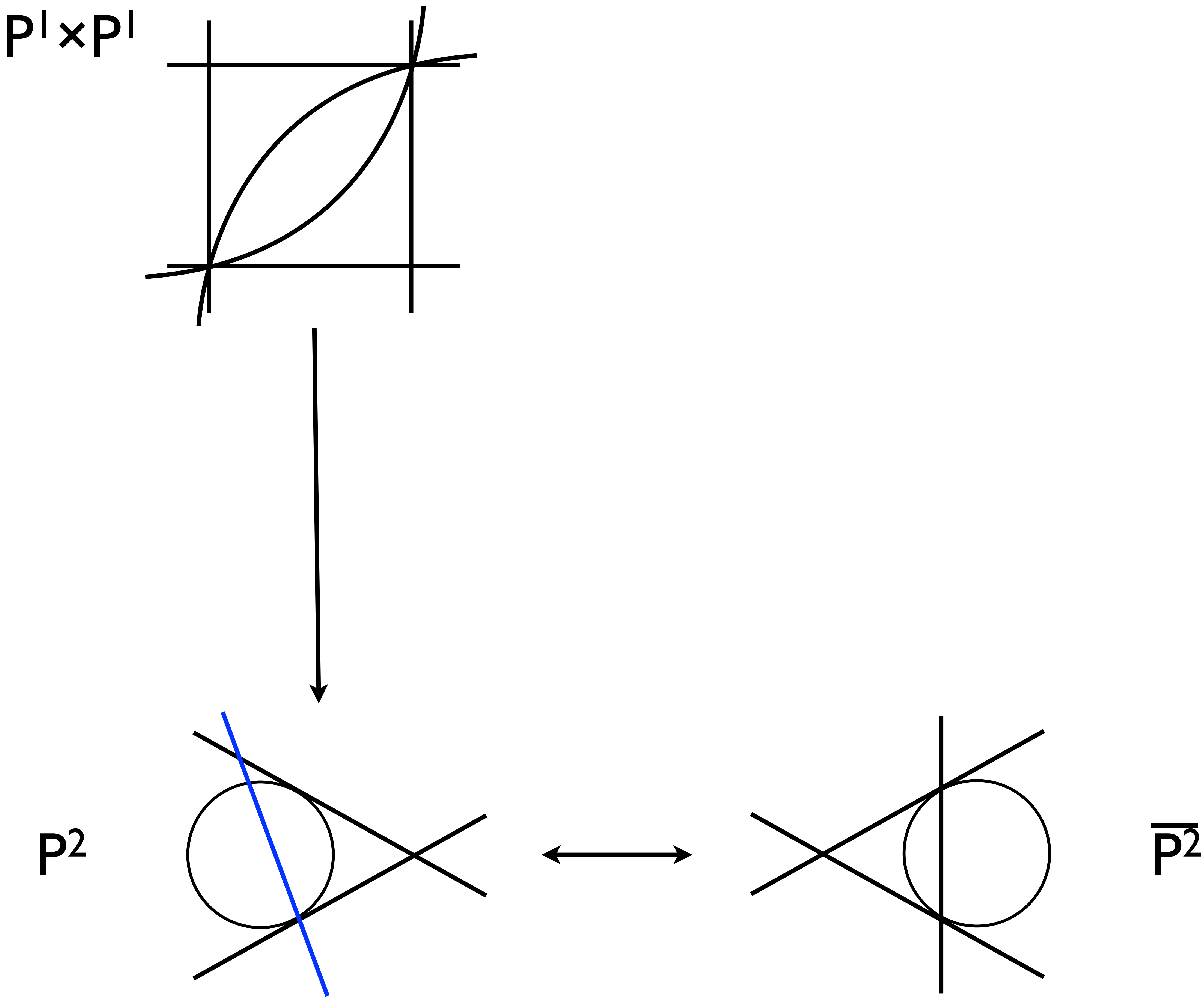

P1×P1
P2

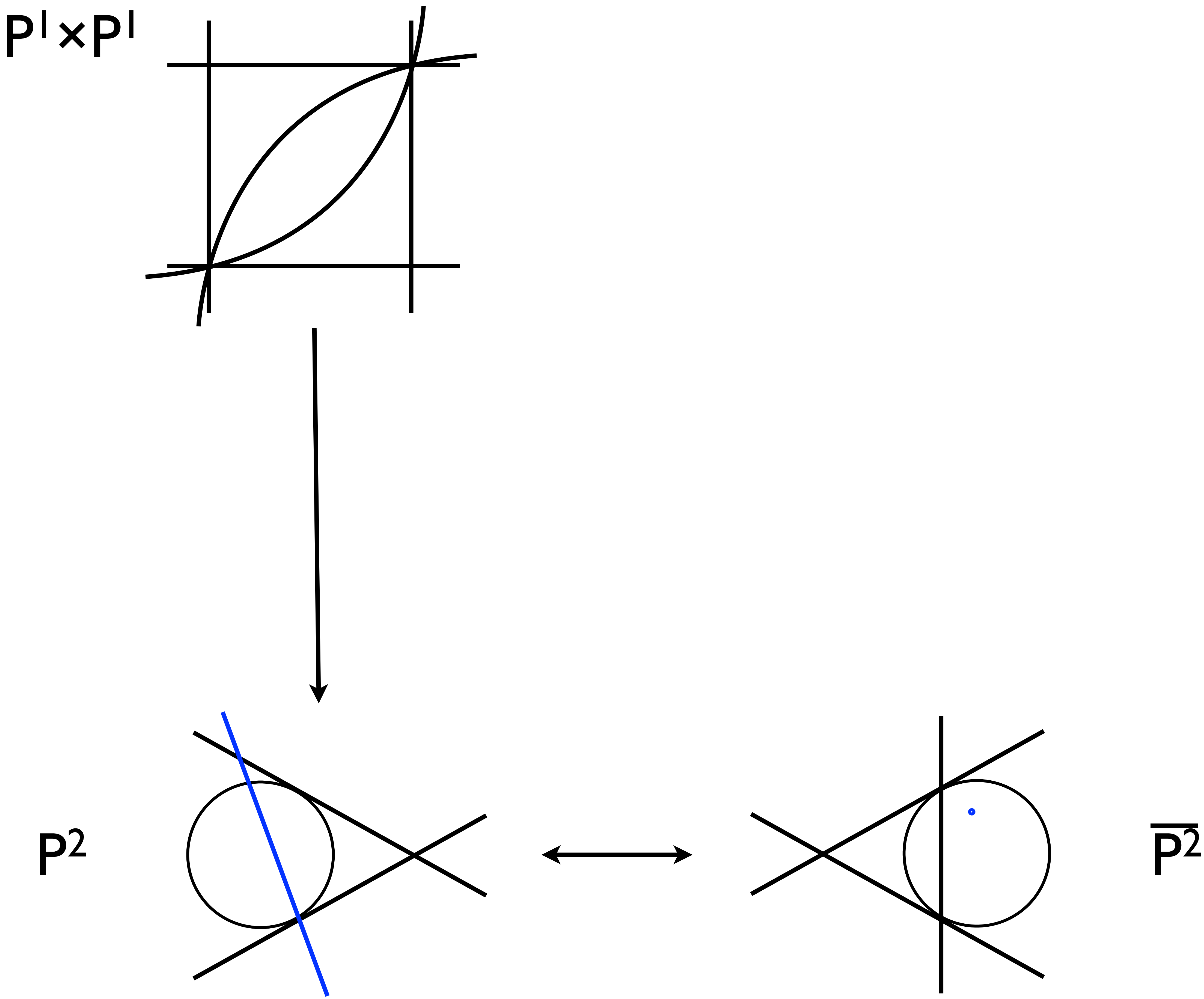
$P^1 \times P^1$

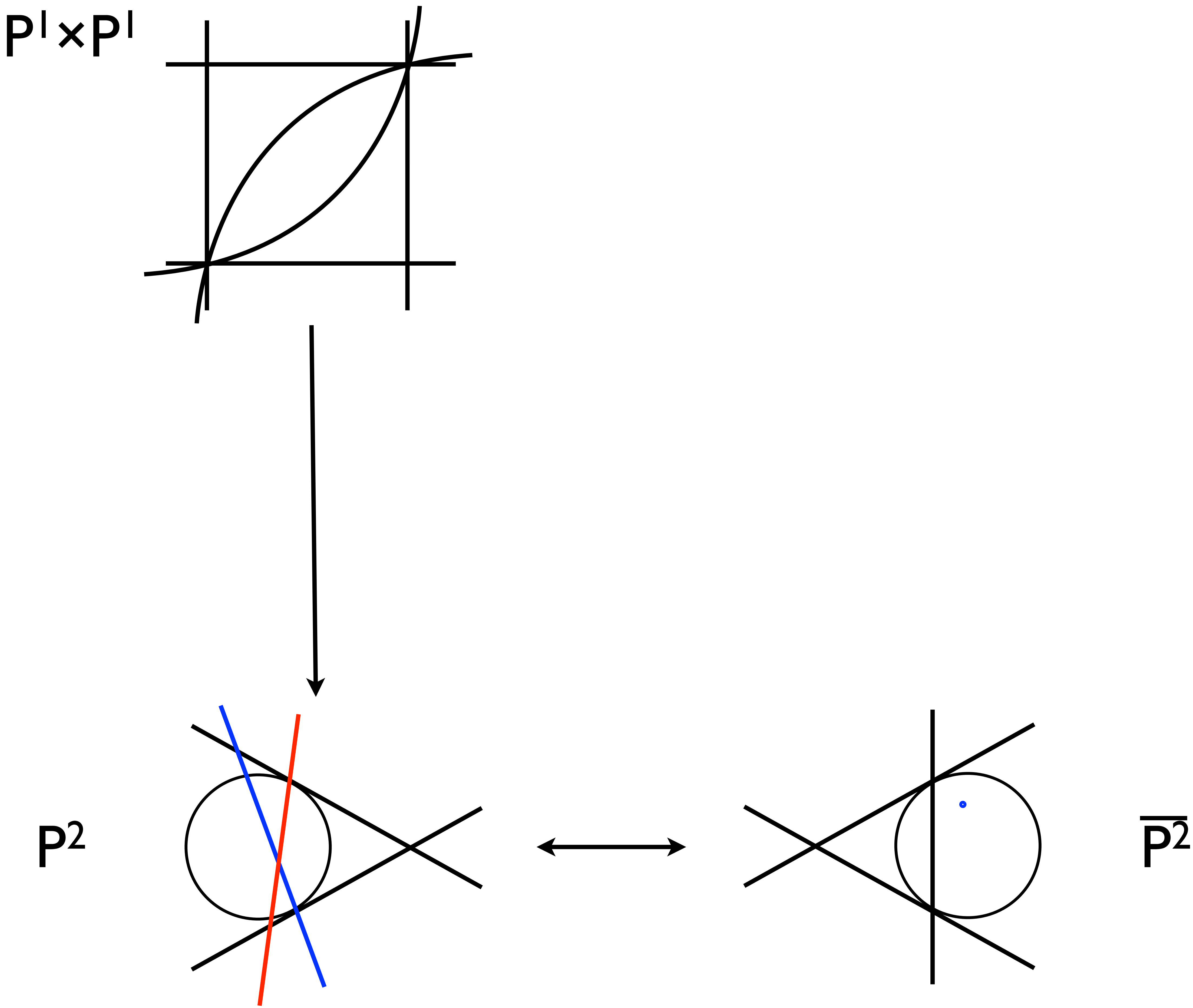


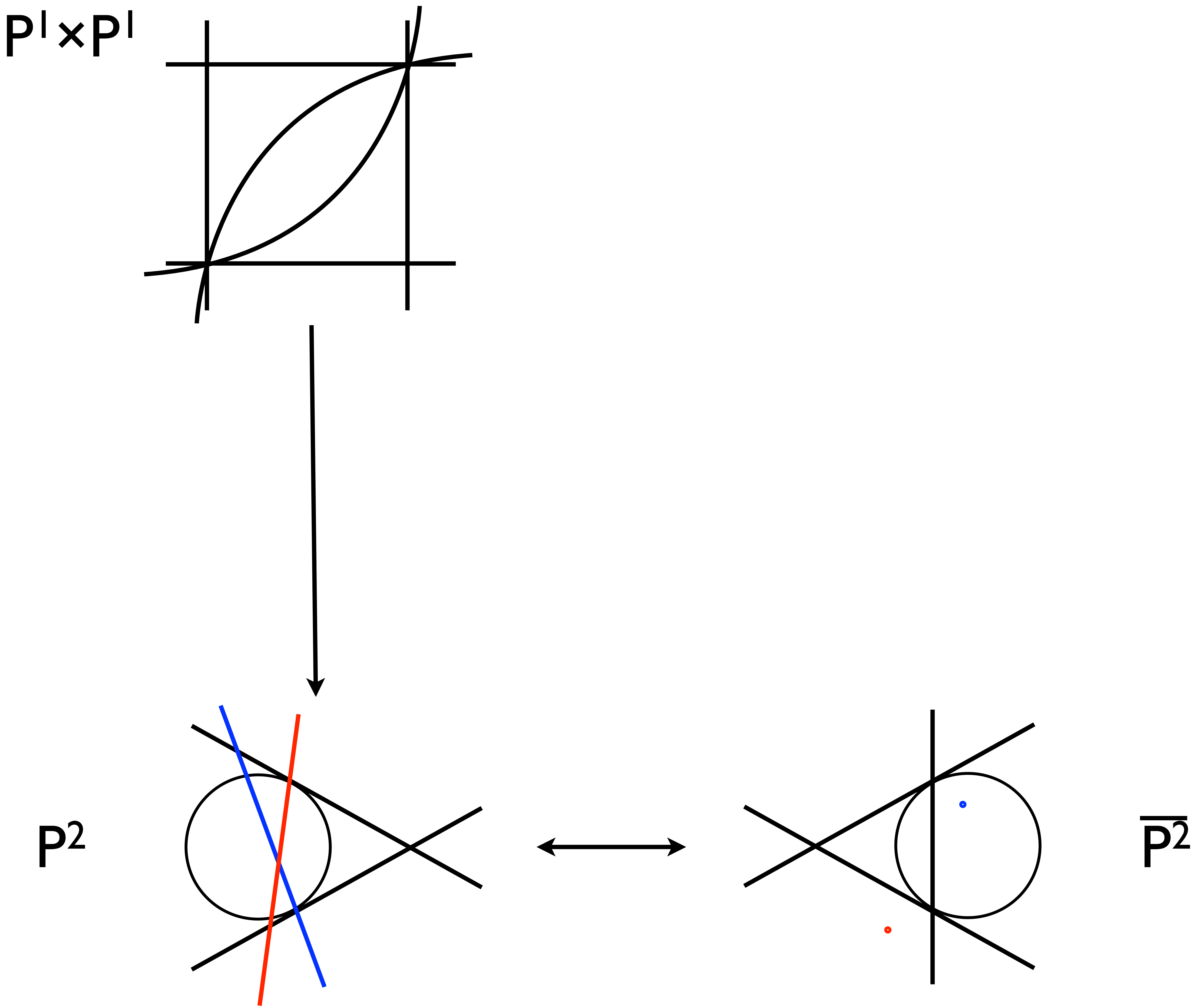
P^2

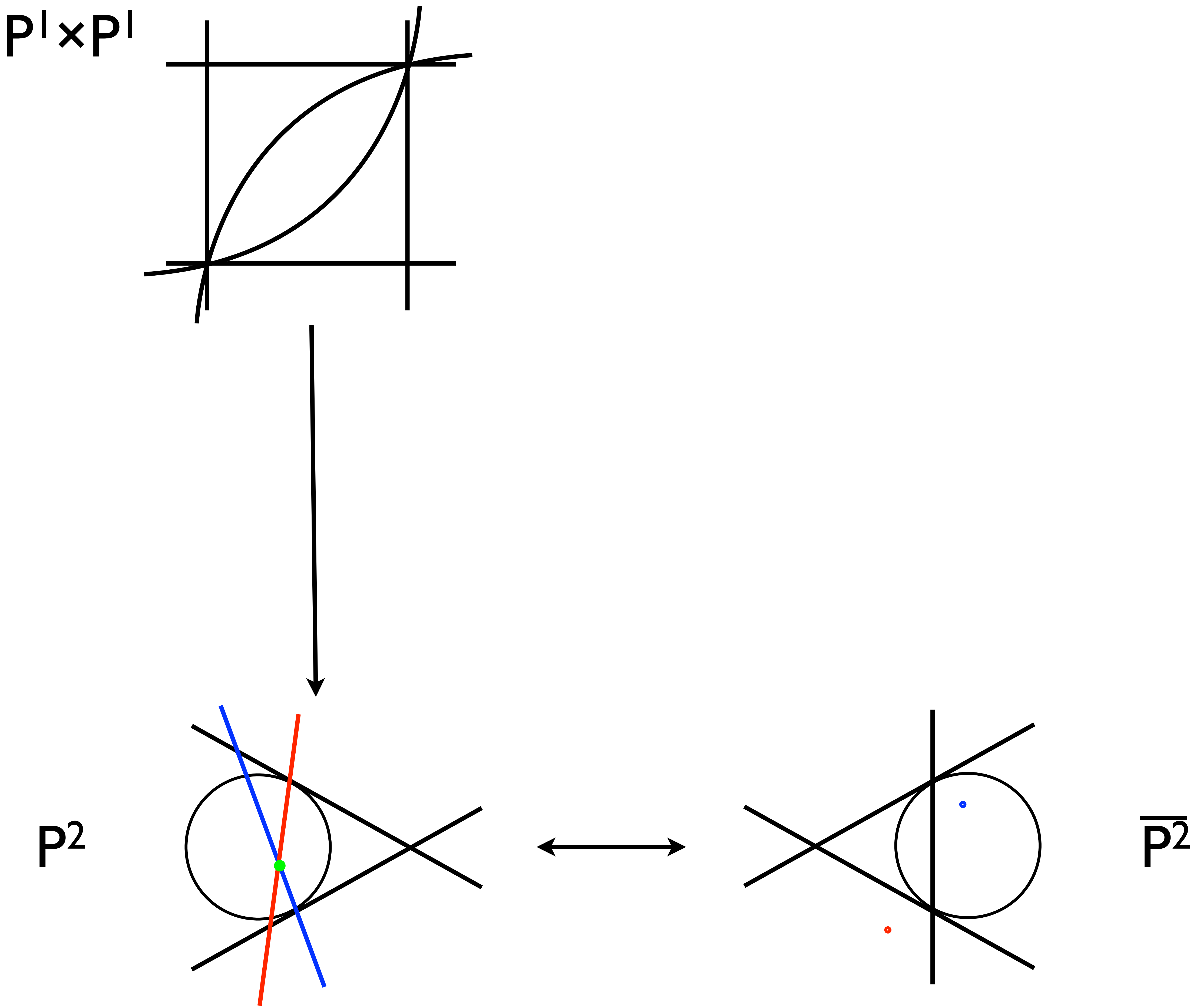

$P^1 \times P^1$
P^2
$\overline{P^2}$

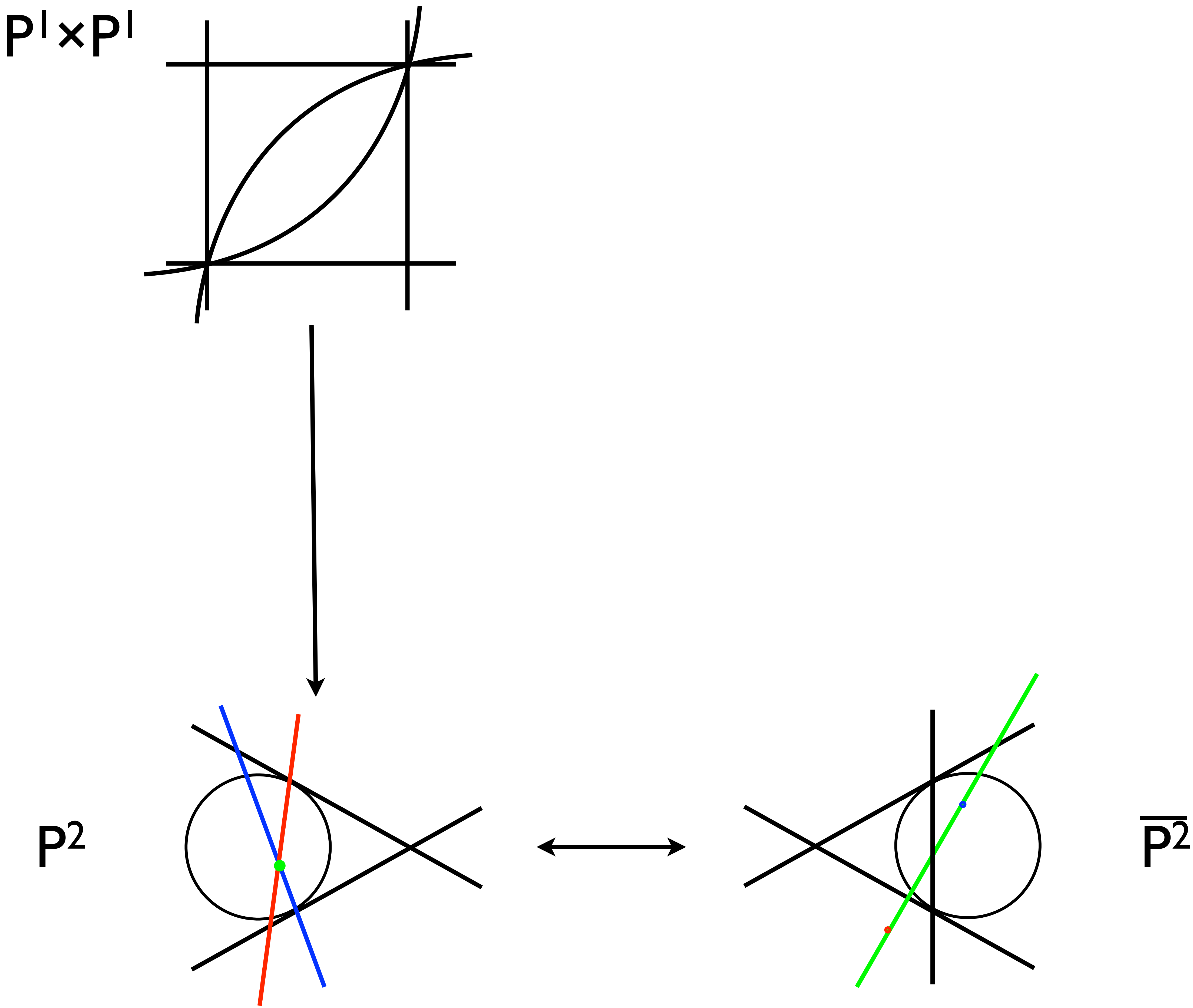

$P^1 \times P^1$
P^2
$\overline{P^2}$

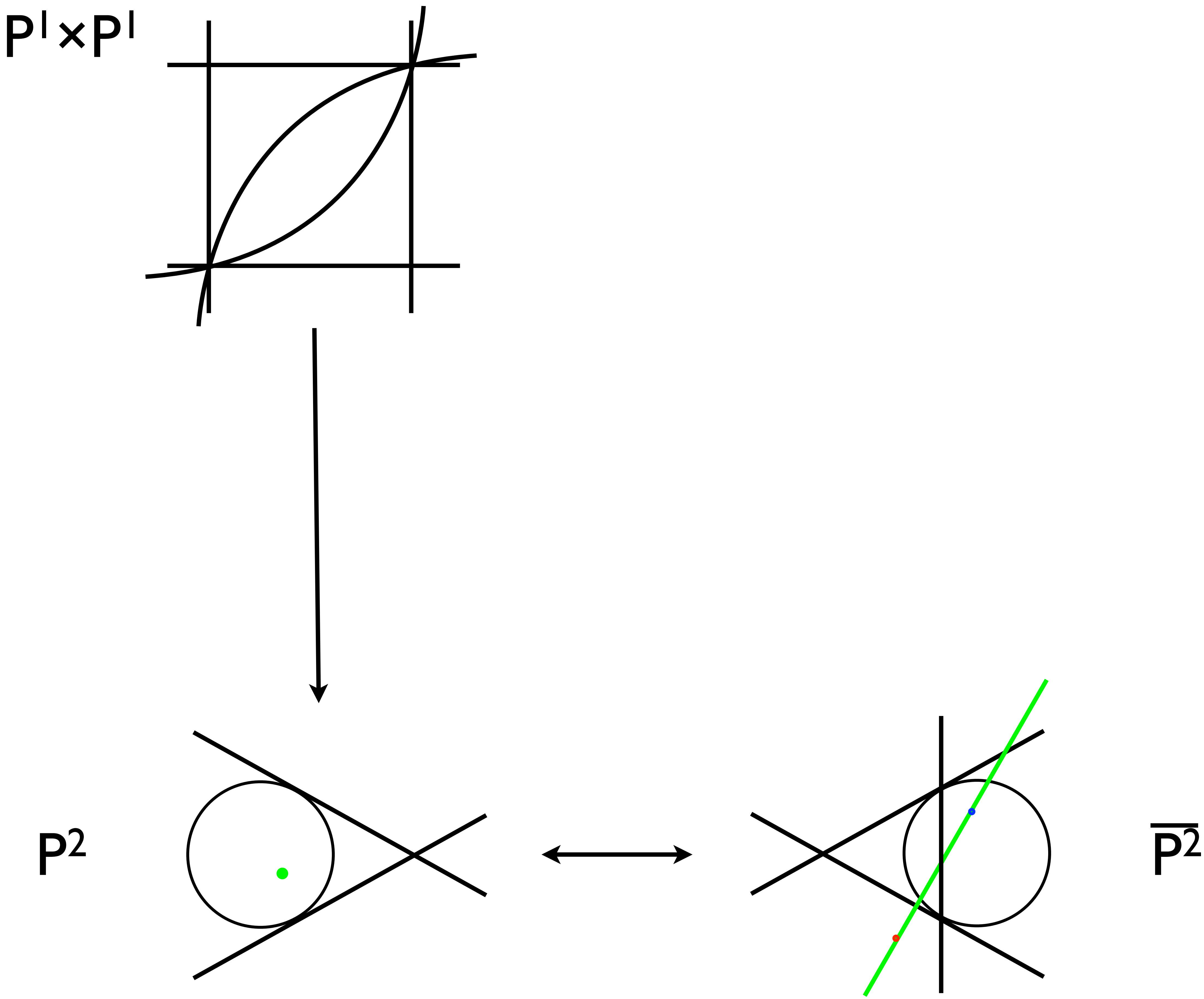

$\mathbf{P}^1 \times \mathbf{P}^1$
$\mathbf{P}^2$
$\overline{\mathbf{P}^2}$


P¹×P¹
P²
P̄²

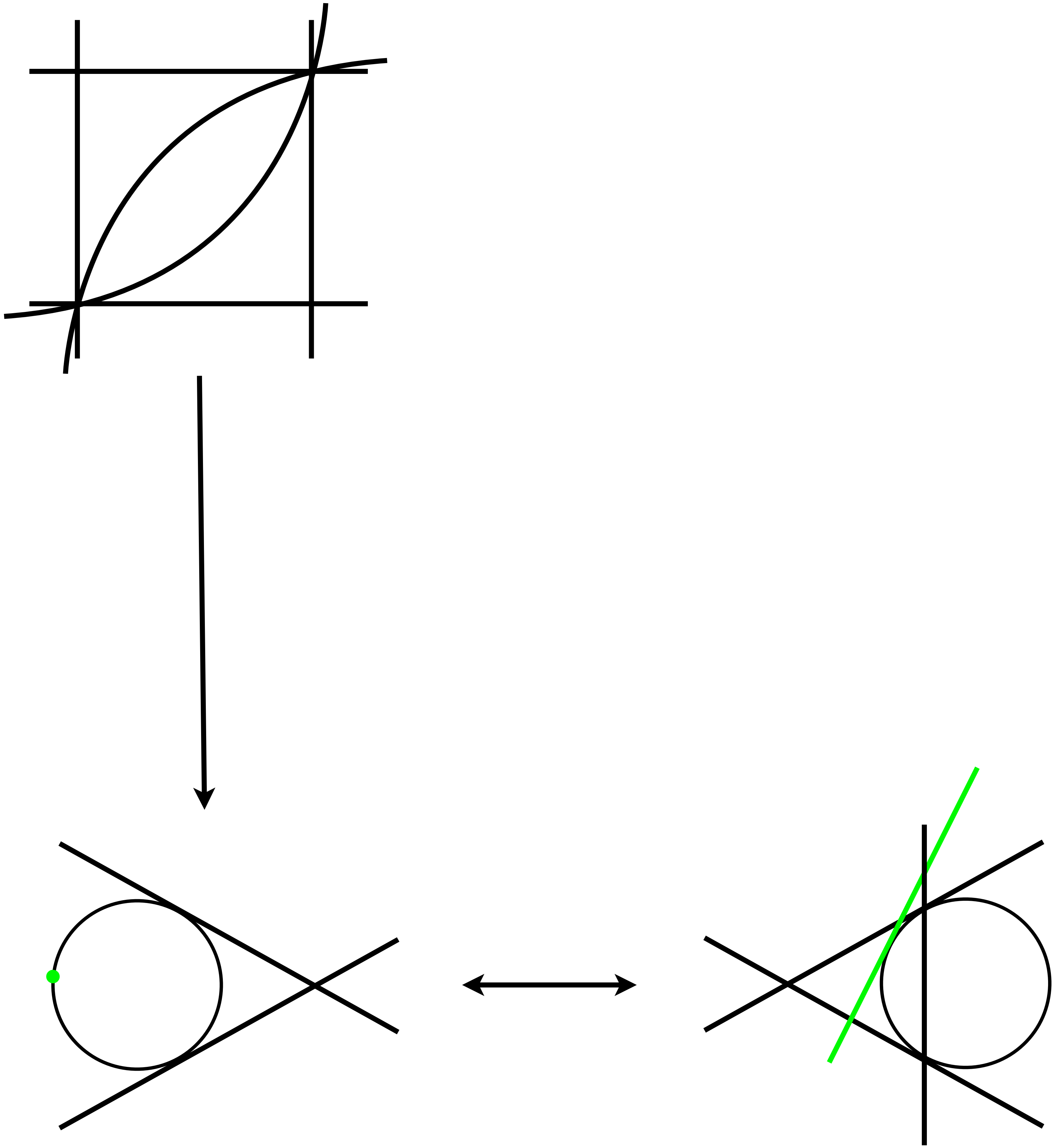

$P^1 \times P^1$
P^2
$\overline{P^2}$


$P^1 \times P^1$
P^2
$\overline{P^2}$


$P^1 \times P^1$
P^2
$\overline{P^2}$


$P^1 \times P^1$
P^2
$\overline{P^2}$

$\mathbb{P}^1 \times \mathbb{P}^1$



$\mathbb{P}^2$

$\overline{\mathbb{P}^2}$