Recent Progress on Topology of Plane Curves: A Quick Trip

Part IV: Other Generalizations of Alexander Polynomials Twisted and Alexander-Oka polynomials

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$$\mathcal{C} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_r \subset \mathbb{P}^2$$

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Definition (Oka, –, Libgober)

The *Alexander polynomial $\Delta_{\mathcal{C},\varepsilon}(t)$ of $\mathcal{C}$ with respect to ε* is the order of $K_\varepsilon/K'_\varepsilon$ as a $\mathbb{Q}[t^{\pm 1}]$-module.

Properties

Remarks

- Oka, Randell. $\Delta_{\mathcal{C},\varepsilon}(t)$ coincides with the characteristic polynomial of the monodromy of the Milnor fiber of $F := \Pi F_i^{\varepsilon_i}$.

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$$\Delta_{\mathcal{C},\varepsilon}(t) \mid \prod_{P\in\mathrm{Sing}(\mathcal{C})} \Delta_{\mathcal{C},\varepsilon,P_i}(t) \prod_i (t^{\varepsilon_i} - 1)^{k_i},$$

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- Artal, $-$, Tokunaga. Consider the evaluation morphism

$$\begin{array}{ccc} \mathbb{Q}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}] & \xrightarrow{\varphi_\varepsilon} & \mathbb{Q}[t^{\pm 1}] \\ t_i & \mapsto & t^{\varepsilon_i} \end{array}$$

then $\tilde{\Delta}_{\mathcal{C},\varepsilon}(t)$ is the generator of $\tilde{\varphi}_\varepsilon(F_1(\mathcal{C}))$.

Alexander Polynomials of a curve

Theorem (Libgober,-)

The Alexander polynomial of $\mathcal{C}$ w.r.t. ε is the first invariant of the colored Burau representation matrix of the braid monodromy of $\mathcal{C}$ w.r.t. ε divided by $(1 - t^{\sum \varepsilon_i})/(1 - t)$.

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Colored Burau Representation:

$$\sigma_1 \mapsto \begin{pmatrix} -t^{\varepsilon_i} & 1 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & \ldots & & \ldots & \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

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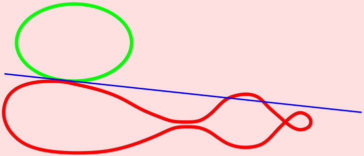
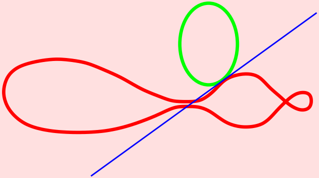
Examples

Consider example from Part I:





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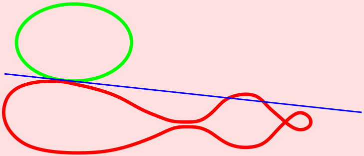
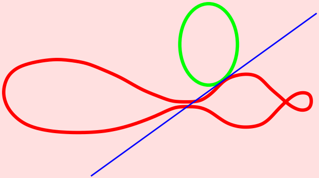


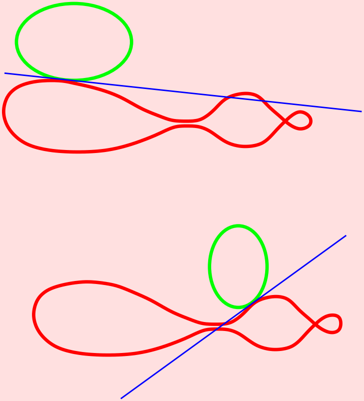
$$\Delta_{\mathcal{C}_1}(t) = (t-1).$$



$$\Delta_{\mathcal{C}_2}(t) = (t-1).$$

Examples

However, if $\varepsilon := (1, 2)$, then



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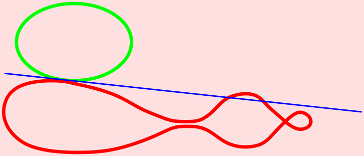
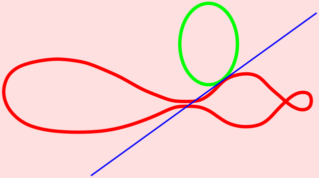


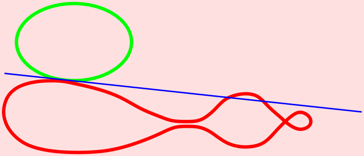
$$\Delta_{\mathcal{C}_1,\varepsilon}(t) = (t-1).$$

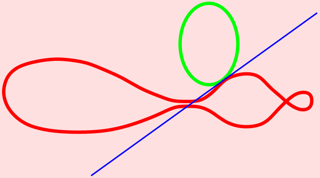


$$\Delta_{\mathcal{C}_2,\varepsilon}(t) = (t^2-1).$$

Examples

Moreover, if $\varepsilon := (2, 1)$, then





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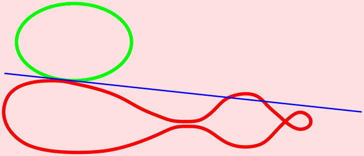
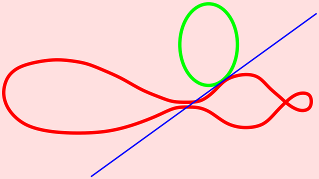


$$\Delta_{\mathcal{C}_1,\varepsilon}(t) = (t-1).$$



$$\Delta_{\mathcal{C}_2,\varepsilon}(t) = (t-1)(t^2+1).$$

Remarks and Open Problems

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- The roots of $\Delta_{\mathcal{C},\varepsilon}(t)$ are d-th roots of unity ($d := \deg F$).
- Find a better (maybe universal) bound on the multiplicities of the roots of $\Delta_{\mathcal{C},\varepsilon}(t)$.

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7. Choose $\widetilde{b}_i$ a lift of b_i in C_{i+1}.
8. $b_i h_i \widetilde{b}_{i-1}$ is a basis of C_i.

Definition

The *Franz-Reidemeister torsion* of $(C_*; c, h)$ is

$$\tau(C_*; c, h) := \prod_{i=0}^{n} [b_i h_i \widetilde{b}_{i-1} | c_i]^{(-1)^{i+1}} \in \mathbb{F}^*/\{\pm 1\}.$$

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Theorem (Kirk, Livingston, Wada)

Let X be a finite CW-complex. If $H_1^{\varepsilon,\rho}(X; \mathbb{F}[t^{\pm 1}])$ is torsion, then

$$\tau_{\varepsilon,\rho}(X) = \Delta_{X,\varepsilon,\rho}(t).$$

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$$\ldots$$

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Note that $\Delta_{G_1}(t) = t^2 - t + 1$ for the classical Alexander polynomial.

Example

- $G_2 := \mathbb{B}_4 = \langle\, x_1, x_2, x_3 : r_1 \equiv x_1 x_2 x_1 (x_2 x_1 x_2)^{-1},\ r_2 \equiv x_1 x_2 x_1 (x_2 x_1 x_2)^{-1},\ r_3 \equiv x_1 x_3 (x_3 x_1)^{-1} \,\rangle$

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- $\det \Phi(x_3 - 1) = (t-1)^2$
- $\Delta_{G_2,\varepsilon,\rho}(t) = (t-1)(t^2+1).$

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- Obtain *geometric* conditions for the existence of certain roots of the twisted Alexander polynomial.
- Find a connection between twisted Alexander polynomials and (maybe non-abelian) coverings.