On the complex volume of hyperbolic knots

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Let M be a closed, oriented, hyperbolic 3-manifold. Then, the *Chern-Simons invariant* of M is defined by

$$\operatorname{cs}(M) = \frac{1}{8\pi^2} \int_{s(M)} \operatorname{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \in \mathbb{R}/\mathbb{Z},$$

where A denotes the connection in the orthonormal frame bundle determined by the metric and $s(M)$ is an orthonormal frame field. In this talk, we define the *complex volume* of M by

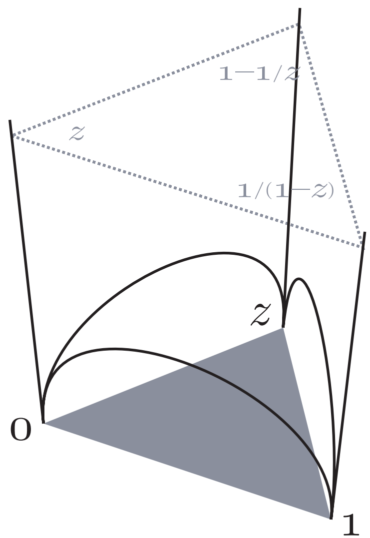
$$\operatorname{cv}(M) = -2\pi^2 \operatorname{cs}(M) + \sqrt{-1}\operatorname{vol}(M) \mod 2\pi^2,$$

which is extended to *cusped* hyperbolic 3-manifolds modulo π^2.

Remark. The 3-dimensional hyperbolic space $\mathbb{H}^3$ is the upper half of $\mathbb{R}^3$ endowed with the metric

$$ds^2 = (dx^2 + dy^2 + dt^2)/t^2.$$

A tetarahedron in $\mathbb{H}^3$ whose 4 vertices are in $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ is called *ideal*. The shape of such a tetrahedron is determined by a complex number called *modulus*.



Conjecture. *Let K be a hyperbolic knot in S^3. If N is large,*

$$J_K(N; e^{2\pi\sqrt{-1}/N}) \sim e^{\frac{N}{2\pi\sqrt{-1}}\left\{-2\pi^2 \mathrm{cs}(S^3\backslash K)+\sqrt{-1}\mathrm{vol}(S^3\backslash K)\right\}},$$

where $J_K(N; q)$ is the N-colored Jones polynomial of K.

This conjecture is still open. However, we can show that

$$J_K(N; e^{2\pi\sqrt{-1}/N}) = \int e^{\frac{N}{2\pi\sqrt{-1}}\{V(x_1,\dots,x_n)+O(1/N)\}} dx_1 \cdots dx_n$$

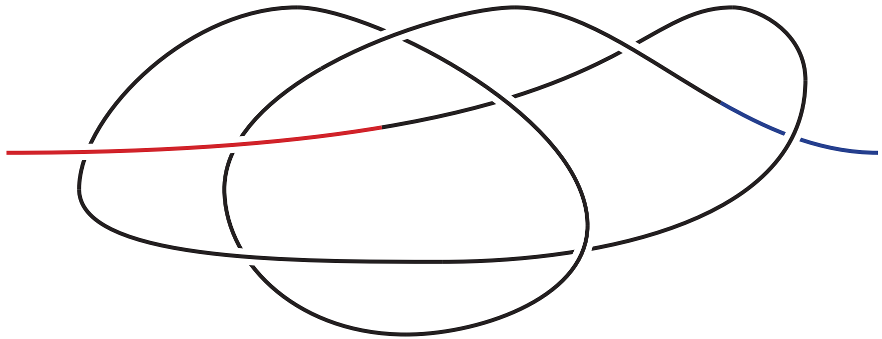
and the *hyperbolicity equations* for $M = S^3 \setminus K$ are given by

$$x_\nu \frac{\partial V}{\partial x_\nu} = 2\pi\sqrt{-1} \cdot r_\nu, \quad r_\nu \in \mathbb{Z}.$$

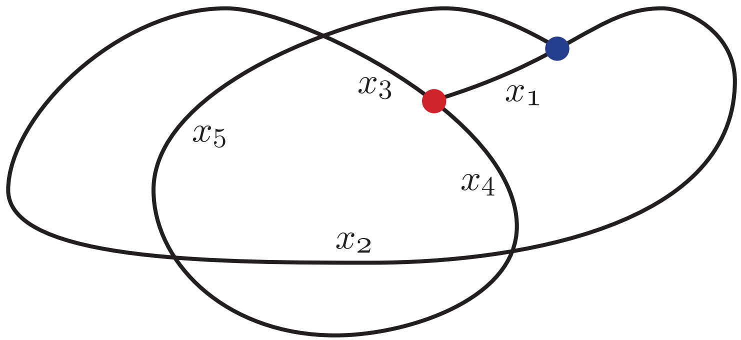
In this talk, we prove that, if $x_\nu = z_\nu$ is the *geometric* solution,

$$\mathrm{cv}(M) = V(z_1, \dots, z_n) - 2\pi\sqrt{-1} \sum_{\nu=1}^{n} r_\nu \log z_\nu \mod \pi^2.$$

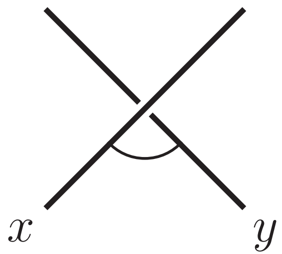
Example. Choose a diagram D of a hyperbolic knot K in S^3, and remove an overpass and an underpass of D which are adjacent.



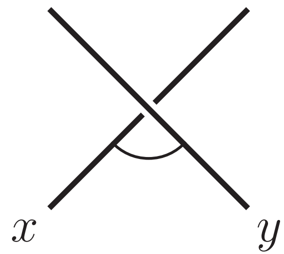
Then, we obtain a subgraph G of D with the *edge variables* x_ν's.



Put *dilogarithm functions* on the interior corners of G.



$$\mathrm{Li}_2(y/x) - \pi^2/6$$



$$\pi^2/6 - \mathrm{Li}_2(x/y)$$

The potential function $V(x_1, x_2, x_3, x_4, x_5)$ is nothing but the sum of these dilogarithm functions, that is,

$$\begin{aligned} &\mathrm{Li}_2(x_1/x_4) - \mathrm{Li}_2(x_1/x_3) + \mathrm{Li}_2(x_1) - \mathrm{Li}_2(1/x_4) \\ &+ \mathrm{Li}_2(x_2/x_4) - \mathrm{Li}_2(x_2) - \mathrm{Li}_2(1/x_2) + \mathrm{Li}_2(x_5/x_2) \\ &- \mathrm{Li}_2(x_5) - \mathrm{Li}_2(1/x_5) + \mathrm{Li}_2(x_3/x_5) - \mathrm{Li}_2(x_3) + \pi^2/3. \end{aligned}$$

Then, there is an ideal triangulation $\mathcal{S}$ of $M = S^3 \setminus K$, such that the hyperbolicity equations for $\mathcal{S}$ are given by

$$\begin{aligned}
0 &\equiv x_1 \frac{\partial V}{\partial x_1} \equiv \ln \frac{1 - x_1/x_3}{(1 - x_1/x_4)(1 - x_1)}, \\
0 &\equiv x_2 \frac{\partial V}{\partial x_2} \equiv \ln \frac{(1 - x_2)(1 - x_5/x_2)}{(1 - x_2/x_4)(1 - 1/x_2)}, \\
0 &\equiv x_3 \frac{\partial V}{\partial x_3} \equiv \ln \frac{(1 - x_1/x_3)(1 - x_3)}{1 - x_3/x_5}, \\
0 &\equiv x_4 \frac{\partial V}{\partial x_4} \equiv \ln \frac{(1 - x_1/x_4)(1 - x_2/x_4)}{1 - 1/x_4}, \\
0 &\equiv x_5 \frac{\partial V}{\partial x_5} \equiv \ln \frac{(1 - x_5)(1 - x_3/x_5)}{(1 - x_5/x_2)(1 - 1/x_5)}
\end{aligned}$$

modulo $2\pi\sqrt{-1}\mathbb{Z}$.

The solutions to the equations above are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} +1.066 \pm 2.484i \\ -1.099 \pm 1.129i \\ -0.812 \mp 0.173i \\ -0.099 \pm 1.129i \\ -1.177 \pm 0.250i \end{pmatrix}, \begin{pmatrix} +1.281 \pm 0.392i \\ -0.317 \pm 0.618i \\ +1.949 \mp 0.441i \\ +0.682 \pm 0.618i \\ +0.487 \mp 0.110i \end{pmatrix}, \begin{pmatrix} 0.304 \\ 0.833 \\ 0.725 \\ 1.833 \\ 1.379 \end{pmatrix}$$

with $r_\nu \equiv 0$. Note that these equations satisfy

$$\frac{x_1}{x_4}, \frac{x_1}{x_3}, x_1, \frac{1}{x_4}, \frac{x_2}{x_4}, x_2, \frac{1}{x_2}, \frac{x_5}{x_2}, x_5, \frac{1}{x_5}, \frac{x_3}{x_5}, x_3 \notin \{0, 1, \infty\}.$$

The critical values of $V(x_1, x_2, x_3, x_4, x_5)$ are given by

$$11.9099 \pm 4.1249i,\ \ 1.85138 \pm 1.10891i,\ \ -1.20365,$$

and so $\mathrm{cv}(M) = 11.9099 \pm 4.1249i$.

- **The knot 6 _ 1**

```
li[x_] := PolyLog[2, x]
```

$$v[x\_, y\_, z\_] := li[x] + li[z] + li\left[\frac{1}{x}\right] + li\left[\frac{1}{z}\right] + li[y] - li\left[\frac{1}{y}\right] - li\left[\frac{z}{x}\right] - li\left[\frac{y}{z}\right] - \frac{\pi^2}{3}$$

$$x = \{x1, x2, x3\};\ dv = Table[Exp[x[[i]]\ \partial_{x[[i]]}\ v[x1, x2, x3]], \{i, 3\}]\ //\ Simplify$$

$$\left\{\frac{1}{-x1 + x3}, \frac{x2\ (x2 - x3)}{(-1 + x2)^2\ x3}, \frac{x1 - x3}{x1\ x2 - x1\ x3}\right\}$$

$$z = NSolve\left[\left\{\frac{1}{-x1 + x3} == 1, \frac{x2\ (x2 - x3)}{(-1 + x2)^2\ x3} == 1, \frac{x1 - x3}{x1\ x2 - x1\ x3} == 1\right\}\right]$$

{{x3 → 1.89923 + 0.400532 ⅈ, x1 → 0.899232 + 0.400532 ⅈ, x2 → 0.971274 + 0.813859 ⅈ},
{x3 → 1.89923 − 0.400532 ⅈ, x1 → 0.899232 − 0.400532 ⅈ, x2 → 0.971274 − 0.813859 ⅈ},
{x3 → −0.399232 + 0.32564 ⅈ, x1 → −1.39923 + 0.32564 ⅈ, x2 → 0.278726 + 0.48342 ⅈ},
{x3 → −0.399232 − 0.32564 ⅈ, x1 → −1.39923 − 0.32564 ⅈ, x2 → 0.278726 − 0.48342 ⅈ}}

$$Table[Im[x[[i]]\ \partial_{x[[i]]}\ v[x1, x2, x3]], \{i, 3\}]\ /.\ z$$

$\{\{3.99339 \times 10^{-16}, 1.50669 \times 10^{-14}, 6.28319\}, \{-3.99339 \times 10^{-16}, -1.50669 \times 10^{-14}, -6.28319\},$
$\{-2.07598 \times 10^{-17}, 1.21614 \times 10^{-14}, 1.38414 \times 10^{-15}\},$
$\{2.07598 \times 10^{-17}, -1.21614 \times 10^{-14}, -1.38414 \times 10^{-15}\}\}$

$$\{v[x1, x2, x3] - 2\pi i\ Log[x3]\ /.\ z[[1]],\ v[x1, x2, x3]\ /.\ z[[3]]\}$$

{0.211005 + 1.4151 ⅈ, −6.79074 + 3.16396 ⅈ}

- **The knot 6 _ 2**

```
li[x_] := PolyLog[2, x]
```

$$v[x\_, y\_, z\_] := li\left[\frac{z}{x}\right] + li\left[\frac{1}{z}\right] + li[y] - li\left[\frac{1}{y}\right] - li[z] - li\left[\frac{1}{x}\right] - li[x] - li\left[\frac{y}{z}\right] + \frac{\pi^2}{3}$$

$$x = \{x1, x2, x3\};\ dv = Table[Exp[x[[i]]\ \partial_{x[[i]]}\ v[x1, x2, x3]], \{i, 3\}] \,//\, Simplify$$

$$\left\{-x1 + x3,\ \frac{x2\ (x2 - x3)}{(-1 + x2)^2\ x3},\ \frac{x1\ (-1 + x3)^2}{(x1 - x3)\ (x2 - x3)}\right\}$$

$$z = NSolve\left[\left\{-x1 + x3 == 1,\ \frac{x2\ (x2 - x3)}{(-1 + x2)^2\ x3} == 1,\ \frac{x1\ (-1 + x3)^2}{(x1 - x3)\ (x2 - x3)} == 1\right\}\right]$$

```
{{x2 → 0.871221 + 1.10766 i, x1 → 1.20635 - 0.340852 i, x3 → 2.20635 - 0.340852 i},
 {x2 → 0.871221 - 1.10766 i, x1 → 1.20635 + 0.340852 i, x3 → 2.20635 + 0.340852 i},
 {x2 → 0.629714, x1 → -0.482881, x3 → 0.517119},
 {x2 → -0.186078 + 0.874646 i, x1 → -0.964913 - 0.621896 i, x3 → 0.0350866 - 0.621896 i},
 {x2 → -0.186078 - 0.874646 i, x1 → -0.964913 + 0.621896 i, x3 → 0.0350866 + 0.621896 i}}
```

$$Table[Im[x[[i]]\ \partial_{x[[i]]}\ v[x1, x2, x3]], \{i, 3\}] \ /.\ z$$

$$\{\{-4.78966\times10^{-16}, -1.71751\times10^{-15}, 6.28319\}, \{4.78966\times10^{-16}, 1.71751\times10^{-15}, -6.28319\}, \{0, 0., 0.\}, \{6.5484\times10^{-17}, 2.71463\times10^{-16}, -3.50586\times10^{-17}\}, \{-6.5484\times10^{-17}, -2.71463\times10^{-16}, 3.50586\times10^{-17}\}\}$$

$$\{v[x1, x2, x3] - 2\pi i\, Log[x3] \ /.\ z[[1]],\ v[x1, x2, x3] \ /.\ z[[3]],\ v[x1, x2, x3] \ /.\ z[[4]]\}$$

$$\{0.3291 + 1.53058\, i,\ 2.40108 + 0.\, i,\ 5.87256 + 4.40083\, i\}$$

- **The knot 6 _ 3**

$$v[x\_, y\_, z\_] := li\left[\frac{y}{x}\right] + li[x] + li[z] + li\left[\frac{1}{y}\right] - li\left[\frac{1}{z}\right] - li[y] - li\left[\frac{1}{x}\right] - li\left[\frac{z}{y}\right]$$

$$x = \{x1, x2, x3\}; dv = Table[Exp[x[[i]]\, \partial_{x[[i]]} v[x1, x2, x3]], \{i, 3\}] // Simplify$$

$$\left\{\frac{-x1 + x2}{(-1 + x1)^2}, -\frac{x1\,(-1 + x2)^2}{(x1 - x2)\,(x2 - x3)}, \frac{x3\,(-x2 + x3)}{x2\,(-1 + x3)^2}\right\}$$

$$z = NSolve\left[\left\{\frac{-x1 + x2}{(-1 + x1)^2} == 1, -\frac{x1\,(-1 + x2)^2}{(x1 - x2)\,(x2 - x3)} == 1, \frac{x3\,(-x2 + x3)}{x2\,(-1 + x3)^2} == 1\right\}\right]$$

$$\{\{x3 \to 0.659772 + 0.298454\, i, x1 \to 0.0829546 - 0.592379\, i, x2 \to 0.573013 + 0.494098\, i\},$$
$$\{x3 \to 0.659772 - 0.298454\, i, x1 \to 0.0829546 + 0.592379\, i, x2 \to 0.573013 - 0.494098\, i\},$$
$$\{x3 \to 0.108378 - 0.818891\, i, x1 \to 0.158836 - 1.20014\, i, x2 \to -0.57395 + 0.818891\, i\},$$
$$\{x3 \to 0.108378 + 0.818891\, i, x1 \to 0.158836 + 1.20014\, i, x2 \to -0.57395 - 0.818891\, i\},$$
$$\{x3 \to 0.23185 + 1.65564\, i, x1 \to 1.25821 - 0.569162\, i, x2 \to 1.00094 - 0.863088\, i\},$$
$$\{x3 \to 0.23185 - 1.65564\, i, x1 \to 1.25821 + 0.569162\, i, x2 \to 1.00094 + 0.863088\, i\}\}$$

$$Table[Im[x[[i]]\, \partial_{x[[i]]} v[x1, x2, x3]], \{i, 3\}] /. z$$

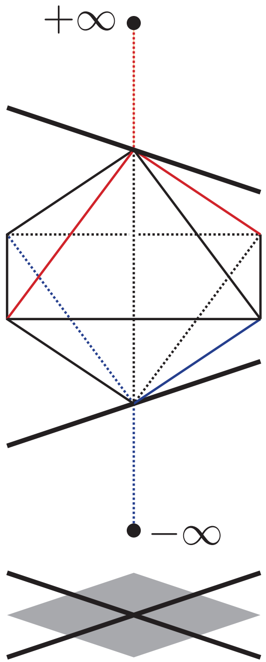
$$\{\{-4.34041\times 10^{-16}, -5.63794\times 10^{-16}, 2.27002\times 10^{-17}\},$$
$$\{4.34041\times 10^{-16}, 5.63794\times 10^{-16}, -2.27002\times 10^{-17}\},$$
$$\{-3.47798\times 10^{-16}, 3.27939\times 10^{-16}, -1.7647\times 10^{-16}\},$$
$$\{3.47798\times 10^{-16}, -3.27939\times 10^{-16}, 1.7647\times 10^{-16}\},$$
$$\{9.4853\times 10^{-15}, -1.22141\times 10^{-15}, -1.55317\times 10^{-15}\},$$
$$\{-9.4853\times 10^{-15}, 1.22141\times 10^{-15}, 1.55317\times 10^{-15}\}\}$$

$$\{v[x1, x2, x3] /. z[[2]], v[x1, x2, x3] /. z[[4]], v[x1, x2, x3] /. z[[5]]\}$$

$$\{-1.89061 + 0.924305\, i, -1.11022\times 10^{-16} + 5.69302\, i, 1.89061 + 0.924305\, i\}$$

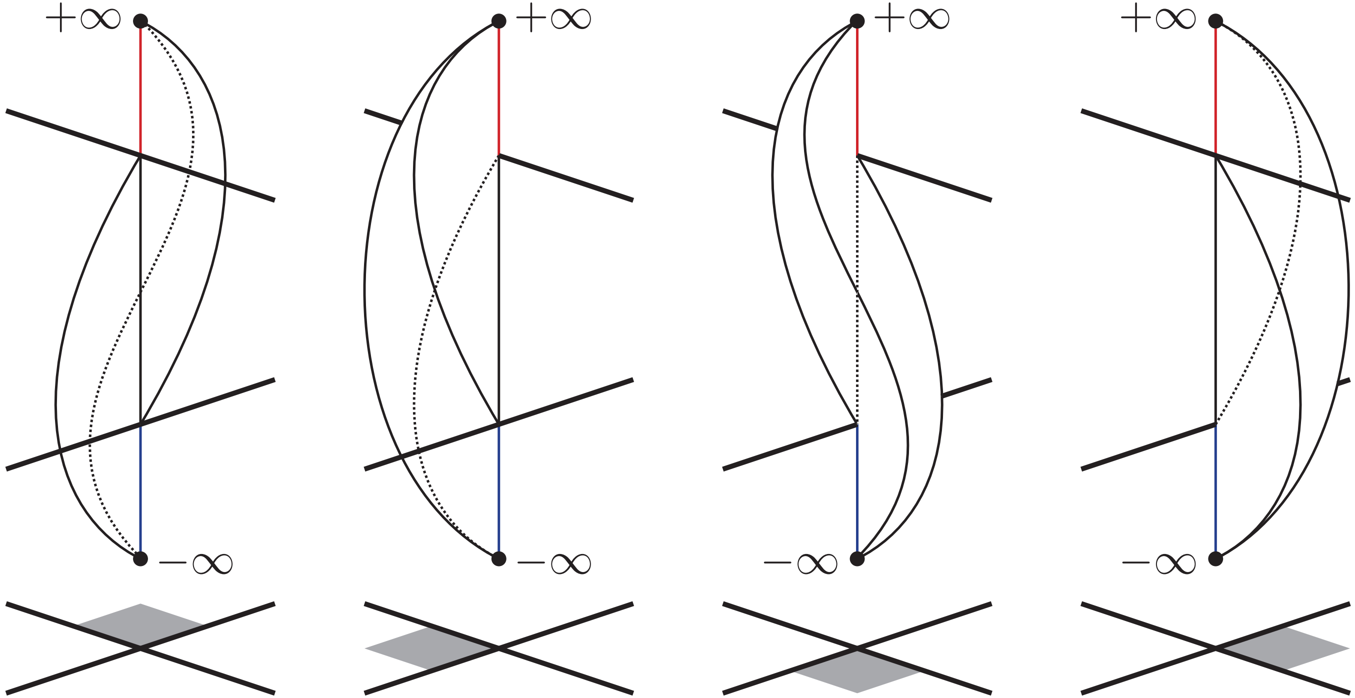
1. An ideal triangulation of M

We first prepare an ideal octahedron at each crossing of D.



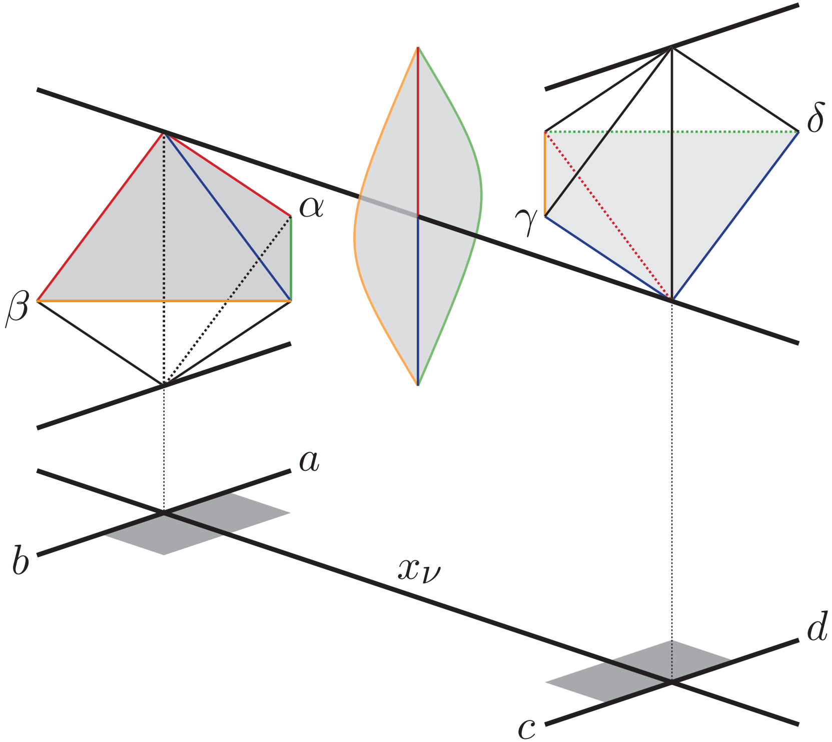
Then, glue the red edges above the crossing, and glue the blue edges below the crossing, where $\pm\infty$ denote the poles of S^3.

This octahedron decomposes into 4 tetrahedra as follows.



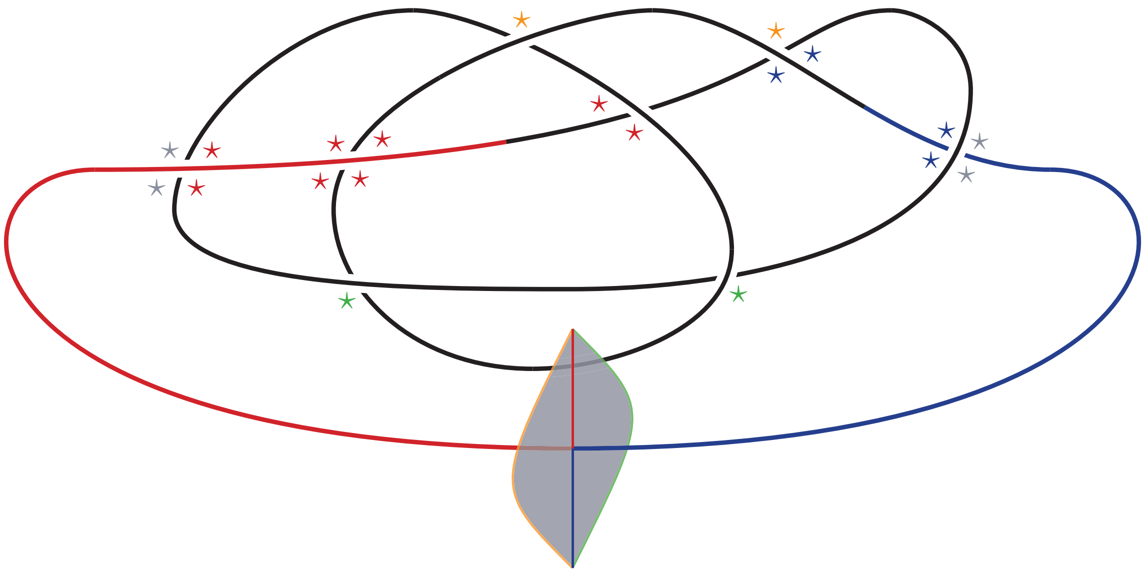
We shall consider the ratios of the edge variables represent the moduli around the vertical edge.

Gluing them along the edges of D, we obtain $S^3 \setminus (K \cup \{\pm\infty\})$.



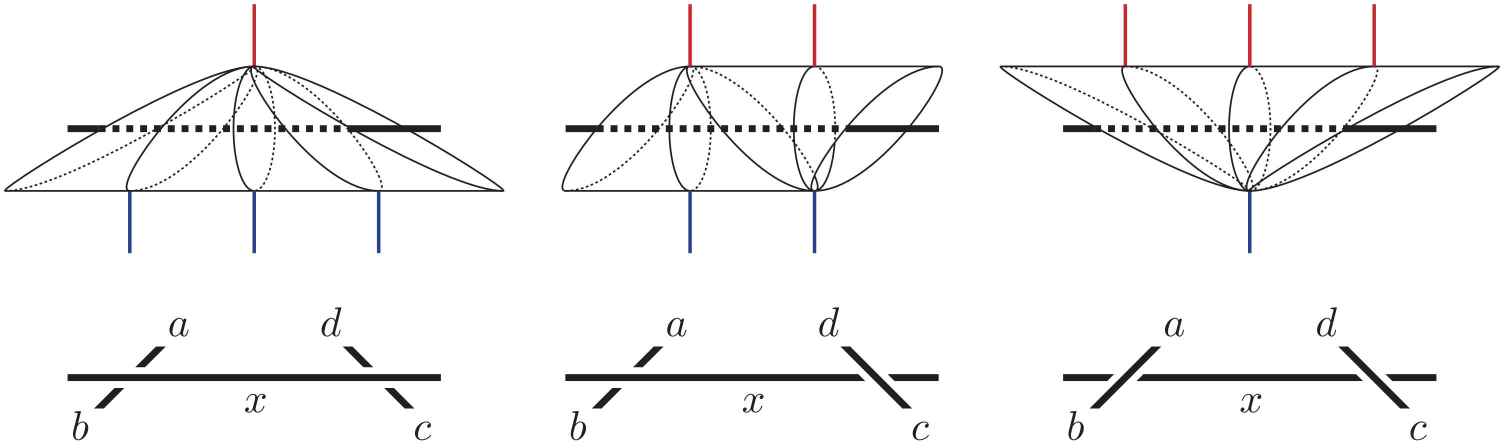
In the picture above, the moduli of the tetrahedra $\alpha, \beta, \gamma, \delta$ are represented by $a/x_\nu, x_\nu/b, c/x_\nu, x_\nu/d$ respectively.

If we collapse the leaf below, the tetrahedra corresponding to the $\star$ corners are collapsed, and we obtain an ideal triangulation $\mathcal{S}$ of M. Note that the tetrahedra in $\mathcal{S}$ correspond to the dilogarithm functions in $V(x_1, \ldots, x_n)$.



Suppose that $x_\nu = z_\nu$ gives the hyperbolic structure of M.

The hyperbolicity equations for $\mathcal{S}$ can be read as follows.



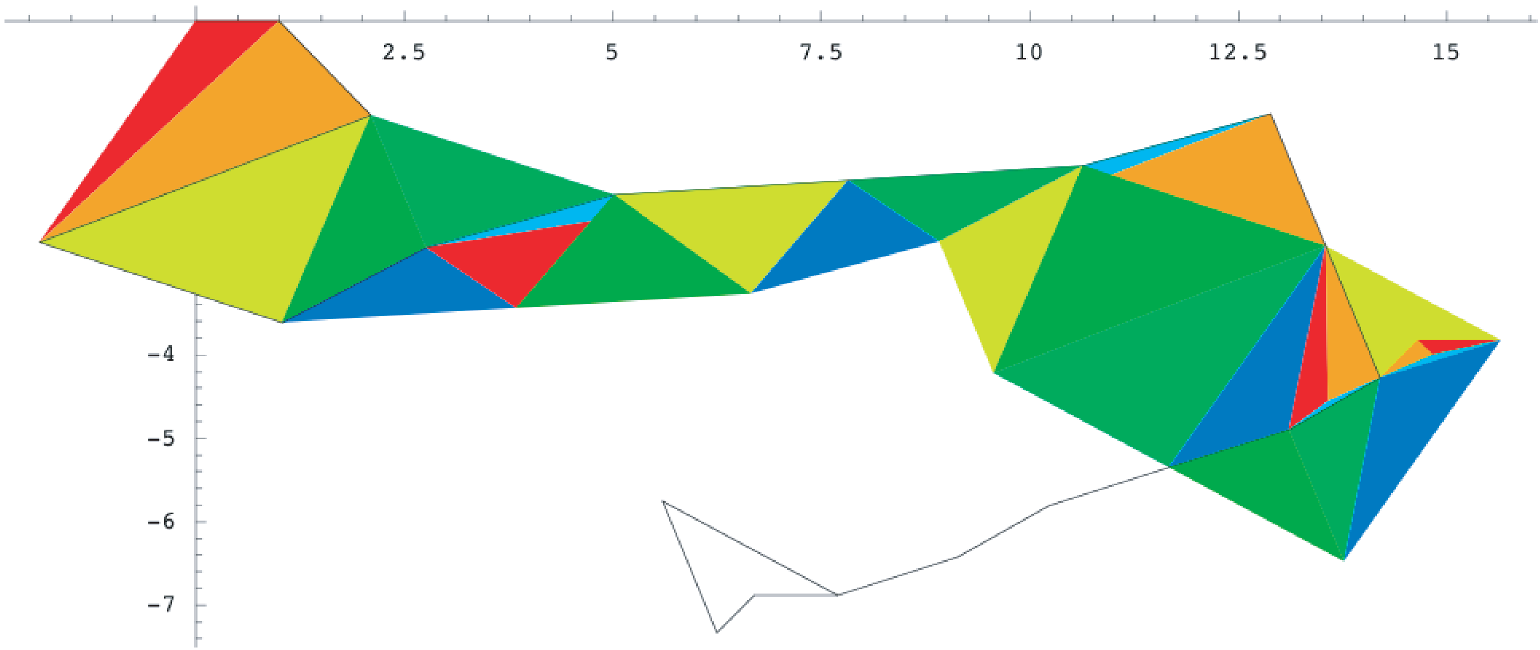
$$\frac{(1-b/x)(1-d/x)}{(1-a/x)(1-c/x)}=1 \qquad \frac{(1-a/x)(1-x/c)}{(1-b/x)(1-x/d)}=1 \qquad \frac{(1-x/a)(1-x/c)}{(1-x/b)(1-x/d)}=1$$

Curiously, these equations coincide with the equations

$$\exp\left(x_\nu \frac{\partial V}{\partial x_\nu}\right) = 1.$$

Let $x_\nu = z_\nu$ be the geometric solution to the equations above.

Remark. The lifts of $\partial N(K)$ form *holospheres* in $\mathbb{H}^3$. It is easy to draw the triangulations of them induced by $\mathcal{S}$ if we know the geometric solution $z_1, \ldots, z_n$ to the hyperbolicity equations.

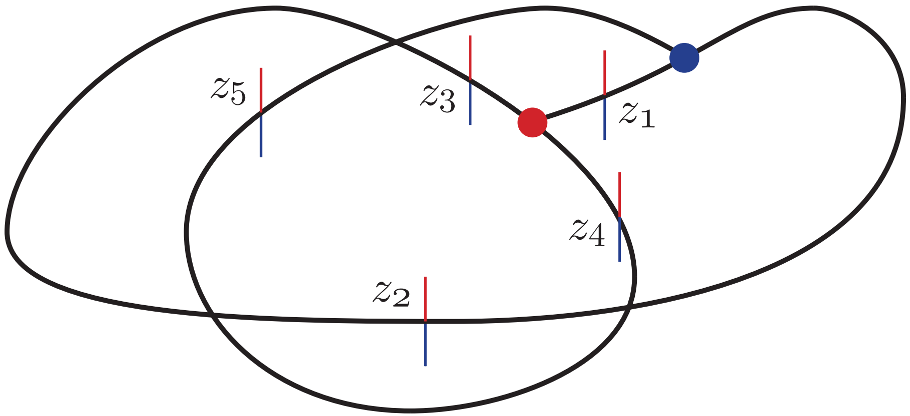


2. Zickert's formula

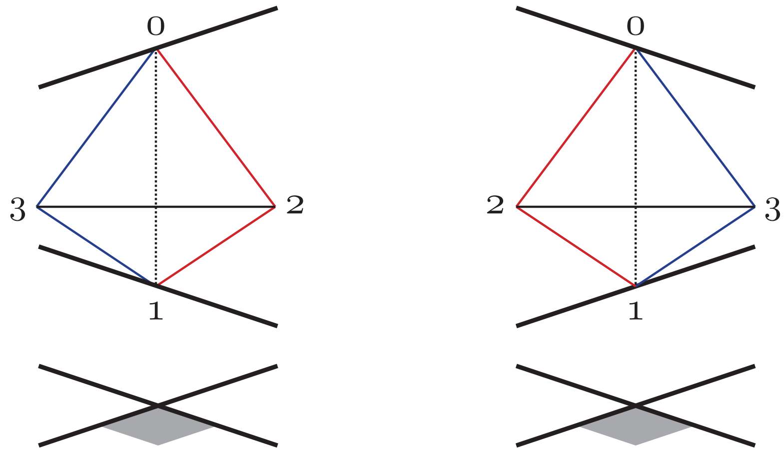
Let E be an edge of $\mathcal{S}$ and $\tilde{E}$ its lift. Then, the holospheres at $\partial\tilde{E}$ are interchanged by an element of $\mathrm{PSL}_2(\mathbb{C})$ conjugate to

$$\begin{pmatrix} 0 & -1/\xi(E) \\ \xi(E) & 0 \end{pmatrix}.$$

We call $\xi(E) \in \mathbb{C}$ the *edge parameter* of E. Note that the ratios of the parameters of the red and blue edges coincides with z_ν's.



Order the vertices of each tetrahedron as follows.



For a tetrahedron τ in $\mathcal{S}$, define $a(\tau), b(\tau) \in \{0, 1, 2, \dots, n\}$ by

$$z_{a(\tau)} = \xi(\tau_{02})/\xi(\tau_{03}),\ \ z_{b(\tau)} = \xi(\tau_{12})/\xi(\tau_{13}),$$

where τ_{ij} is the edge of τ between the vertices i and j, and put

$$u(\tau) = \ln \xi(\tau_{03}) - \ln \xi(\tau_{13}) + \ln \xi(\tau_{12}) - \ln \xi(\tau_{02}),$$

$$v(\tau) = \ln \xi(\tau_{02}) - \ln \xi(\tau_{23}) + \ln \xi(\tau_{13}) - \ln \xi(\tau_{01}).$$

Then, there exist $p_\tau, q_\tau \in \mathbb{Z}$ such that

$$u(\tau) = \ln z_\tau + p_\tau \pi\sqrt{-1},\ v(\tau) = -\ln(1 - z_\tau) + q_\tau \pi\sqrt{-1},$$

where we put $z_\tau = z_{a(\tau)}/z_{b(\tau)}$. We now define $L(\tau)$ by

$$\begin{aligned}\varepsilon(\tau)L(\tau) = \mathrm{Li}_2(z_\tau) + \frac{1}{2}\ln z_\tau \ln(1 - z_\tau) - \frac{\pi^2}{6} \\ + \frac{1}{2}\pi\sqrt{-1}\{q_\tau \ln z_\tau + p_\tau \ln(1 - z_\tau)\},\end{aligned}$$

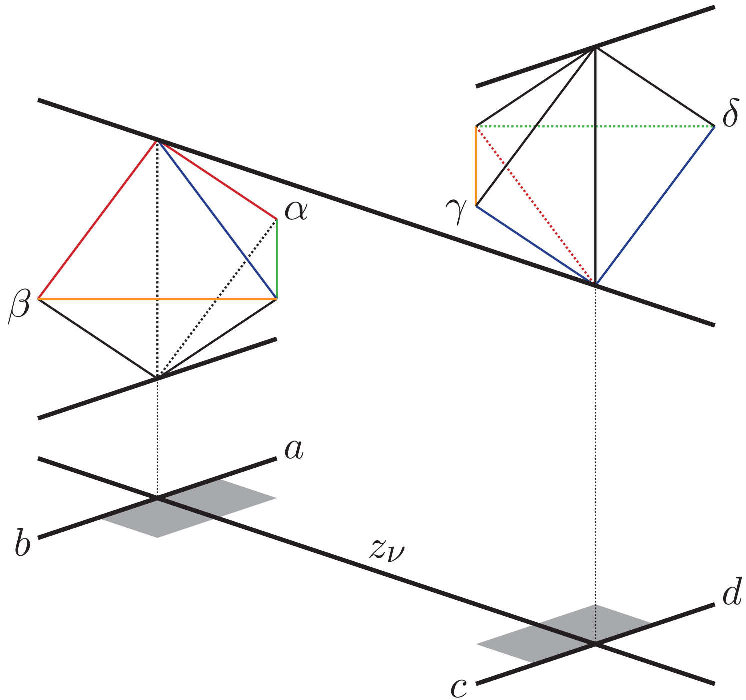
where $\varepsilon(\tau)$ takes 1 or -1 according as τ is right-handed or not. In our case, p_τ is always *even*, and we can write

$$\varepsilon(\tau)L(\tau) \equiv \mathrm{Li}_2(z_\tau) - \frac{\pi^2}{6} + \frac{1}{2}u(\tau)\{v(\tau) + 2\ln(1 - z_\tau)\} \mod \pi^2.$$

Zickert's Theorem. $\mathrm{cv}(M) \equiv \sum_\tau L(\tau) \mod \pi^2$.

3. Proof

Consider 4 tetrahedra around an edge of G. Let A, B, X, Y be the logarithm of the parameters of red, blue, green, orange edges respectively.



Let P, Q, R, S be the logarithms of the parameters of

$$\alpha_{01} = \beta_{01},\ \alpha_{13} = \beta_{13},\ \gamma_{02} = \delta_{02},\ \gamma_{01} = \delta_{01}$$

respectively. Then, $L(\alpha) + L(\beta) + L(\gamma) + L(\delta)$ is equal to

$$\begin{aligned}
&\mathrm{Li}_2(a/z_\nu) - \mathrm{Li}_2(b/z_\nu) - \mathrm{Li}_2(z_\nu/c) + \mathrm{Li}_2(z_\nu/d) \\
&+ \tfrac{1}{2}\{B - Q + \xi(\alpha_{12}) - A\}\{A - X + Q - P + 2\ln(1 - a/z_\nu)\} \\
&- \tfrac{1}{2}\{B - Q + \xi(\beta_{12}) - A\}\{A - Y + Q - P + 2\ln(1 - b/z_\nu)\} \\
&- \tfrac{1}{2}\{\xi(\gamma_{03}) - B + A - R\}\{R - Y + B - S + 2\ln(1 - z_\nu/c)\} \\
&+ \tfrac{1}{2}\{\xi(\delta_{03}) - B + A - R\}\{R - X + B - S + 2\ln(1 - z_\nu/d)\},
\end{aligned}$$

and the coefficient of $A - B$, which is congruent to $\ln z_\nu$, becomes

$$-\ln(1 - a/z_\nu) + \ln(1 - b/z_\nu) - \ln(1 - z_\nu/c) + \ln(1 - z_\nu/d)$$

which should be equal to $-2\pi\sqrt{-1}\cdot r_\nu$.

We put

$$w_+(\tau) = \tfrac{1}{2}\bigl\{\ln\xi(\tau_{03}) - \ln\xi(\tau_{02})\bigr\}\bigl\{v(\tau) + 2\ln(1-z_\tau)\bigr\},$$
$$w_-(\tau) = \tfrac{1}{2}\bigl\{\ln\xi(\tau_{12}) - \ln\xi(\tau_{13})\bigr\}\bigl\{v(\tau) + 2\ln(1-z_\tau)\bigr\},$$

so that

$$\varepsilon(\tau)L(\tau) \equiv \mathrm{Li}_2(z_\tau) - \pi^2/6 + w_+(\tau) + w_-(\tau) \mod \pi^2.$$

Then, the above observation implies that

$$w(\nu) = \sum_{a(\tau)=\nu} \varepsilon(\tau)w_+(\tau) + \sum_{b(\tau)=\nu} \varepsilon(\tau)w_-(\tau)$$

equals $-2\pi\sqrt{-1}\cdot r_\nu \ln z_\nu$ modulo $4\pi^2$, and $\sum_\tau L(\tau)$ is equal to

$$V(z_1,\ldots,z_n) + \sum_{\nu=1}^{n} w(\nu) = V(z_1,\ldots,z_n) - 2\pi\sqrt{-1}\sum_{\nu=1}^{n} r_\nu \ln z_\nu$$

modulo π^2. $\square$