On non-singular stable maps of 3-manifolds with boundary into the plane

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ABSTRACT. Let $M$ be a compact connected orientable 3-manifold with non-empty boundary and $f : M \to \mathbb{R}^2$ a stable map. In this paper we study the existence of an immersion or embedding lift of $f$ to $\mathbb{R}^n$ ($n \geq 3$) with respect to the standard projection $\mathbb{R}^n \to \mathbb{R}^2$. We also characterize the orientable 3-dimensional handlebody in terms of stable maps which have only a restricted class of singularities. Moreover, by using the concept of an embedding lift of a certain map of a 2-dimensional polyhedron into $\mathbb{R}^2$, we give a characterization of $S^3$.

1. Introduction

Let $M$ be a smooth manifold, $f : M \to \mathbb{R}^m$ a smooth map and $\pi : \mathbb{R}^n \to \mathbb{R}^m$ ($n > m$) a standard projection. Then we ask if there exists an immersion or embedding $g : M \to \mathbb{R}^n$ which satisfies $f = \pi \circ g$. Such a map $g$ is called an immersion or embedding lift of $f$.

In this paper, $M$ will be a compact connected orientable 3-manifold with non-empty boundary, of class $C^\infty$. Let $f : M \to \mathbb{R}^3$ be a stable map. We ask if there exists an immersion or embedding lift of $f$ to $\mathbb{R}^n$ ($n \geq 3$) with respect to the standard projection $\pi : \mathbb{R}^n \to \mathbb{R}^2$, $(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2)$. A point $x$ in $M$ is called a singularity if $\text{rank } df_x < 2$. $S(f)$ denotes the set of singularities of $f$. Our main result is the following theorem.

THEOREM 1. Let $M$ be a compact connected orientable 3-manifold with non-empty boundary and $f : M \to \mathbb{R}^2$ a stable map. We consider the condition (1): For any $r \in \mathbb{R}^2$, $f^{-1}(r)$ is either empty or homeomorphic to a finite disjoint union of closed intervals and points. Then the following two conditions are equivalent.

(a) $f$ has an immersion lift to $\mathbb{R}^3$.
(b) $S(f) = \emptyset$ and $f$ satisfies the condition (1).

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By Whitehead [13, there exists an immersion $i : M \to \mathbb{R}^3$ for every compact connected orientable 3-manifold $M$ with non-empty boundary. Thus $f = \pi \circ i$ satisfies $S(f) = \emptyset$ and the condition (I), provided that $f$ is stable. We show that a submersion $f : M \to \mathbb{R}^2$ whose restriction to $\partial M$ is stable, is a stable map in Lemma 2 of §3. Hence, after a slight perturbation of $i$, we may assume that $f = \pi \circ i$ is a stable map. Moreover, it is not difficult to prove that the space of non-singular stable maps is open and dense in the space of submersions of $M$ to $\mathbb{R}^2$ by using Lemma 2.

Based on the arguments in the proof of Theorem 1, we consider the structure of source manifolds of a certain class of stable maps. For a stable map $f : M \to \mathbb{R}^2$ with $S(f) = \emptyset$, the normal forms around points of $\partial M$ consist exactly of four types: regular, $\mathcal{F}_I$, $\mathcal{F}_{II}$ and $\mathcal{C}$ (for details, see §3 and 4). A point of $\partial M$ is of regular type (or of type $\mathcal{C}$) if it is a regular point (resp. a cusp point) of $f|\partial M$. Fold points of $f|\partial M$ are classified into two types: $\mathcal{F}_I$ and $\mathcal{F}_{II}$. We consider a stable map which has only points of regular type or of type $\mathcal{F}_I$ on $\partial M$. Such a map is called a boundary special generic map.

**Theorem 2.** A compact connected orientable 3-manifold $M$ with non-empty boundary is an orientable 3-dimensional handlebody (i.e., $M$ is diffeomorphic to $\mathbb{S}^1 \times D^2$, $k \geq 0$) if and only if there exists a boundary special generic map $f : M \to \mathbb{R}^2$.

The tool for the proof of Theorems 1 and 2 is the Stein factorization which consists of 2-dimensional polyhedron $W_f$, $q_f : M \to W_f$ and $\tilde{f} : W_f \to \mathbb{R}^2$ with $f = \tilde{f} \circ q_f$. Although $W_f$ is not a manifold, we can define an embedding lift of $\tilde{f}$ and get the following theorem.

**Theorem 3.** Let $\hat{M}$ be a closed, connected, orientable 3-manifold. Suppose that there exists a stable map $f : \hat{M} - \text{Int} D^3 \to \mathbb{R}^2$ with $S(f) = \emptyset$ and the condition (I). If there exists an embedding lift $g_e : W_f \to \mathbb{R}^3$ of $\tilde{f}$, then $\hat{M}$ is homeomorphic to $S^3$.

The paper is organized as follows. In §2 we recall some fundamental concepts: stable maps, Stein factorizations and etc. In §3 we clarify the local normal forms of $f$ on the neighborhoods of singular points of $f|\partial M$. In §4 we investigate the semi-local structures of $f$ around simple or non-simple points of $\partial M$ and the Stein factorization. In §5 we prove Theorem 1 using the Stein factorization. In §6 we consider the existence problem of an embedding lift to $\mathbb{R}^n$ and get Proposition 10 which guarantees the existence of an embedding lift for $n \geq 5$. Moreover we give some examples which have no embedding lifts for $n = 3, 4$. In §7, we prove Theorems 2 and 3.

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2. Preliminaries

Let $M$ be a smooth 3- or 2-dimensional manifold with or without boundary. We denote by $C^\infty(M, \mathbb{R}^2)$ the set of the smooth maps of $M$ into $\mathbb{R}^2$ with the Whitney $C^\infty$ topology. For a smooth map $f : M \to \mathbb{R}^2$, $S(f)$ denotes the singular set of $f$, i.e., $S(f)$ is the set of the points in $M$ where the rank of the differential $df$ is strictly less than two. A smooth map $f : M \to \mathbb{R}^2$ is stable if there exists an open neighborhood $N(f)$ of $f$ in $C^\infty(M, \mathbb{R}^2)$ such that every $g$ in $N(f)$ is right-left equivalent to $f$; i.e., there exist diffeomorphisms $\phi : M \to M$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $g = \varphi \circ f \circ \phi^{-1}$.

We quote an explicit description of a stable map from a closed 3-manifold $\tilde{M}$ into $\mathbb{R}^2$.

**Lemma 1.** ([7]) Let $\tilde{M}$ be a closed 3-manifold. Then a smooth map $f : \tilde{M} \to \mathbb{R}^2$ is stable if and only if $f$ satisfies the following local and global conditions. For each point $p \in \tilde{M}$ there exist local coordinates centered at $p$ and $f(p)$ such that $f$ is expressed by one of the following four types:

1. $f(p)$ is a regular point:
   
   $$(u, x, y) \mapsto (u, x),$$

   $p$:
   
   regular point.

2. $f(p)$ is a definite fold point:
   
   $$(u, x, y) \mapsto (u, x^2 + y^2),$$

   $p$:
   
   definite fold point.

3. $f(p)$ is an indefinite fold point:
   
   $$(u, x, y) \mapsto (u, x^2 - y^2),$$

   $p$:
   
   indefinite fold point.

4. $f(p)$ is a cusp point:
   
   $$(u, x, y) \mapsto (u, y^2 + ux - x^3),$$

   $p$:
   
   cusp point.

Also $f$ should satisfy the following global conditions:

1. If $p$ is a cusp point, then $f^{-1}(f(p)) \cap S(f) = \{p\}$, and

2. $f[S(f) \setminus \{cups\}]$ is an immersion with normal crossings.

Let us recall the definition of the Stein factorization. Let $M$ be a compact orientable 3-manifold with or without boundary, and let $f : M \to \mathbb{R}^2$ be a stable map. For $p, p' \in M$, we define $p \sim p'$ if $f(p) = f(p')$ and $p, p'$ are in the same connected component of $f^{-1}(f(p)) = f^{-1}(f(p'))$. Let $W_f$ be the quotient space of $M$ under this equivalence relation and we denote by $q_f : M \to W_f$ the quotient map. By the definition of the equivalence relation, we have a unique map $\tilde{f} : W_f \to \mathbb{R}^2$ such that $f = \tilde{f} \circ q_f$. The quotient space $W_f$ or more precisely the commutative diagram

$$
\begin{array}{c}
M \\
\downarrow q_f \\
W_f \\
\downarrow \tilde{f} \\
\mathbb{R}^2 \\
\end{array}
$$

is called the Stein factorization of $f$. In general, $W_f$ is not a manifold, but is
homeomorphic to a 2-dimensional finite CW complex. This fact has been obtained for the case $\partial M = \emptyset$ in [7] and [9] (see also [6]). In the case where $\partial M \neq \emptyset$ with $S(f) = \emptyset$ and the condition (1), this will be shown in §4.

3. Local normal forms of $f$ around singular points of $f|\partial M$

Our purpose of this section is to investigate the local normal forms of a stable map $f$ around singular points of $f|\partial M$.

Throughout this section, $M$ is a compact orientable 3-manifold with non-empty boundary, and $f : M \to \mathbb{R}^2$ is a stable map with $S(f) = \emptyset$. Since $f$ is stable, $f|\partial M$ is also stable by [10, p. 2564, Lemma].

Recall the theorem of Whitney ([14]): Let $N$ be a closed 2-manifold, and let $h : N \to \mathbb{R}^2$ be a stable map. Then for each point $x$ in $N$, there exist local coordinates $(x_1, x_2)$ centered at $x$ and $(y_1, y_2)$ centered at $h(x)$ such that $h$ is given by one of the following local normal forms:

(i) $(x_1, x_2) \mapsto (y_1, y_2) = (x_1, x_2)$, $x$: regular point,

(ii) $(x_1, x_2) \mapsto (y_1, y_2) = (x_1^2, x_2)$, $x$: fold point,

(iii) $(x_1, x_2) \mapsto (y_1, y_2) = (-x_1^3 + x_1 x_2, x_2)$, $x$: cusp point.

**Proposition 1.** Let $x$ be a fold point of $f|\partial M$. Then there exist local coordinates $(T, X_1, X_2)$ of $M$ centered at $x$ and $(Y_1, Y_2)$ of $\mathbb{R}^2$ centered at $h(x)$ such that $h$ is given by one of the following local normal forms:

where

| $q_M$ corresponds to $f_t \hat{\partial} 0$ | $\text{Int } M$ corresponds to $f_t > 0$ |
---|---|

**Proof.** By the theorem of Whitney, for $x \in \partial M$, we can choose local coordinates $(t, x_1, x_2)$ centered at $x$ and $(y_1, y_2)$ centered at $h(x)$ such that $f|\partial M$ is expressed by $(0, x_1, x_2) \mapsto (x_1^2, x_2)$, where $\partial M$ corresponds to $\{t = 0\}$ and $\text{Int } M$ corresponds to $\{t > 0\}$. Then we put $f(t, x_1, x_2) = (\varphi(t, x_1, x_2), \psi(t, x_1, x_2))$ so that

\[
\varphi(0, x_1, x_2) = x_1^2, \\
\varphi(0, x_1, x_2) = x_2.
\]

Since the Jacobian matrix of $f$ at $x = (0, 0, 0)$ is

\[
Jf(0) = \begin{pmatrix}
\frac{\partial \varphi}{\partial t}(0) & 0 & 0 \\
\frac{\partial \varphi}{\partial t}(0) & 0 & 1
\end{pmatrix}
\]

and rank $Jf(0) = 2$ by our assumption that $S(f) = \emptyset$, we obtain $(\partial \varphi / \partial t)(0) \neq 0$. 

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Then, we define the map \( \Phi : (t,x_1,x_2) \mapsto (T,X_1,X_2) \) by
\[
\begin{align*}
T &= \varphi(t,x_1,x_2) - x_1^2, \\
X_1 &= x_1, \\
X_2 &= \psi(t,x_1,x_2).
\end{align*}
\]

By the condition \( (\partial \varphi/\partial t)(0) \neq 0 \), we see that the determinant of the Jacobian matrix of \( \Phi \) at \((0,0,0)\), \(|J\Phi(0)|\), is not equal to 0, since
\[
J\Phi(0) = \begin{pmatrix}
\frac{\partial \varphi}{\partial t}(0) & 0 & 0 \\
0 & 1 & 0 \\
\frac{\partial \psi}{\partial t}(0) & 0 & 1
\end{pmatrix}.
\]

Hence, \((T,X_1,X_2)\) forms local coordinates. Then we get \( f(T,X_1,X_2) = (\varphi(t,x_1,x_2),\psi(t,x_1,x_2)) = (X_1^2 + T,X_2) \). Moreover, \( \{t = 0\} \) corresponds to \( \{T = 0\} \) by this coordinate change, since \( \Phi(0,x_1,x_2) = (\varphi(0,x_1,x_2) - x_1^2,x_1,\psi(0,x_1,x_2)) = (0,x_1,x_2) \).

Then on a neighborhood of \( x \), \( \{t \geq 0\} \) corresponds to \( \{T \geq 0\} \) or to \( \{T \leq 0\} \) by \( \Phi \). By replacing \( T \) with \(-T\) if necessary, we may always assume that \( \{T > 0\} \) corresponds to \( \text{Int} M \) and \( \{T = 0\} \) corresponds to \( \partial M \).

According to this change of coordinates, \( f \) is expressed either by \( (T,X_1,X_2) \mapsto (X_1^2 + T,X_2) \) or by \( (T,X_1,X_2) \mapsto (X_1^2 - T,X_2) \). This completes the proof.

**Proposition 2.** Let \( x \) be a cusp point of \( f|\partial M \). Then there exist local coordinates \((T,X_1,X_2)\) of \( M \) centered at \( x \) and \((Y_1,Y_2)\) of \( \mathbb{R}^2 \) centered at \( f(x) \) such that \( f \) is given by the local normal form \((Y_1,Y_2) = (-X_1^3 + X_1X_2 + T,X_2)\), where \( \partial M \) corresponds to \( \{T = 0\} \) and \( \text{Int} M \) corresponds to \( \{T > 0\} \).

**Proof.** By the theorem of Whitney, for \( x \in \partial M \), we can choose local coordinates \((t,x_1,x_2)\) centered at \( x \) and \((y_1,y_2)\) centered at \( f(x) \) such that \( f|\partial M \) is expressed by \((0,x_1,x_2) \mapsto (-x_1^3 + x_1y_2,x_2)\), where \( \partial M \) corresponds to \( \{t = 0\} \) and \( \text{Int} M \) corresponds to \( \{t > 0\} \). Then we put \( f(t,x_1,x_2) = (\varphi(t,x_1,x_2),\psi(t,x_1,x_2)) \) so that
\[
\varphi(0,x_1,x_2) = -x_1^3 + x_1x_2, \\
\psi(0,x_1,x_2) = x_2.
\]

In this case, we consider the map \( \Phi : (t,x_1,x_2) \mapsto (T,X_1,X_2) \) defined by
\[
\begin{align*}
T &= \varphi(t,x_1,x_2) + x_1^2 - x_1\psi(t,x_1,x_2), \\
X_1 &= x_1, \\
X_2 &= (t,x_1,x_2).
\end{align*}
\]
Then, by an argument similar to that in the proof of Proposition 1, we see that $(T, X_1, X_2)$ forms local coordinates. So, by the same reason, we get the local normal form $f(T, X_1, X_2) = (-X_1^3 + X_1X_2 + T, X_2)$. However, these two types of normal forms coincide with each other through the changes of coordinates $(T, X_1, X_2) \mapsto (T, -X_1, X_2)$ and $(T, X_1, X_2) \mapsto (-T, Y_1, Y_2)$. This completes the proof.

We can also obtain the following proposition.

**Proposition 3.** Let $x$ be a regular point of $f|\mathcal{M}$. Then there exist local coordinates $(T, X_1, X_2)$ of $M$ centered at $x$ and $(Y_1, Y_2)$ of $R^2$ centered at $f(x)$ such that $f$ is given by the local normal form $(Y_1, Y_2) = (X_1, X_2)$, where $\mathcal{M}$ corresponds to $\{T = 0\}$ and $\text{Int} M$ corresponds to $\{T > 0\}$.

Now, we show the following Lemma 2. This lemma guarantees the existence of a stable map which satisfies the condition (b) of Theorem 1 as explained in §1.

**Lemma 2.** Let $M$ be a compact 3-manifold with non-empty boundary and $f : M \to R^2$ a submersion such that $f|\mathcal{M}$ is a stable map. Then $f$ is also stable.

**Proof.** Let us prepare a notion of the infinitesimal stability of Mather ([4, p. 73] and [11]) modified for the case $\mathcal{M} \neq \emptyset$ as follows. Let $\pi : M \to R^2$ be a smooth map and $\pi_R : TR^2 \to R^2$ the canonical projection. A smooth map $w : M \to TR^2$ is called a vector field along $x$ if $w$ satisfies $w = \pi_R \circ w$. Then we say that $x$ is strongly infinitesimally stable if for every $w$, a vector field along $x$, there always exist a vector field $s$ on $M$ whose restriction to $\mathcal{M}$ is a vector field on $\mathcal{M}$ (i.e., each vector of $s$ on $\mathcal{M}$ is tangent to $\mathcal{M}$) and a vector field $t$ on $R^2$ such that

$$w = (dx) \circ s + t \circ x,$$

where $dx : TM \to TR^2$ is the differential of $x$.

By using an argument similar to that of Mather [11], we can show that a strongly infinitesimally stable map is stable. Thus, it is sufficient to prove that $f$ is strongly infinitesimally stable.

Since $f|\mathcal{M}$ is stable and hence infinitesimally stable, for any $w$, $w|\mathcal{M}$ is expressed by $w|\mathcal{M} = d(f|\mathcal{M}) \circ s + t \circ (f|\mathcal{M})$, where $s$ is a vector field on $\mathcal{M}$ and $t$ is a vector field on $R^2$. It is easy to see that there exists a vector field $\overline{s}$ on $M$ such that $\overline{s}|\mathcal{M} = s$. If we define the new vector field $w'$ along $f$ by $w' = w - (df) \circ \overline{s} - t \circ f$, then $w'$ satisfies $w'|\mathcal{M} = 0$. By the argument in the proof of [4, p. 78, Proposition 2.1], we see that there exists a smooth subbundle $H$ complementary to $\text{Ker}(df)$ in $TM$ and that the isomorphism $df_x : H_x \to T_{f(x)}R^2$ $(x \in M)$ induces an isomorphism on sections, $C^\infty(H) \to C^\infty(TM)$. Therefore, $f$ is strongly infinitesimally stable.
\[ C^\infty(T^* \mathbb{R}^2). \] Here, \( C^\infty(H) \) denotes the set of sections of \( H \subset TM \) over \( M \) and \( C^\infty(T^* \mathbb{R}^2) \) denotes the set of vector fields along \( f \). Hence we can construct a vector field \( s' : M \to H \subset TM \) such that \( w' = (df') \circ s' \). Obviously we have \( s'|\partial M = 0 \), since \( w'|\partial M = 0 \), and \( w \) is expressed by \( w = (df) \circ (\tilde{s} + s') + t_0 \circ f \). Note that the vector field \( \tilde{s} + s' \) is tangent to \( \partial M \) on \( \partial M \). This completes the proof.

\[ \Box \]

4. Stein factorization

In §3, we gave the local normal forms of a stable map \( f : M \to \mathbb{R}^2 \) with \( S(f) = \emptyset \) around singular points of \( f|\partial M \). In this section, we investigate the structure of the Stein factorization of a stable map \( f : M \to \mathbb{R}^2 \). Our purpose is to show that (b) implies (a) in Theorem 1. So, throughout this section we assume \( S(f) = \emptyset \) and the condition (I).

**Definition 1.** Let \( M \) be a compact orientable 3-manifold with non-empty boundary, and \( f : M \to \mathbb{R}^2 \) a stable map with \( S(f) = \emptyset \). Then \( p \in S(f|\partial M) \) is a simple point if the connected component of \( f^{-1}(f(p)) \) containing \( p \) intersects \( S(f|\partial M) \) only at \( p \).

Let \( \mathcal{F}_f \) (or \( \mathcal{F}_H \)) be the set of fold points of \( S(f|\partial M) \) around which \( f \) is expressed by the local normal form \( (Y_1, Y_2) = (X_1^2 + T, X_2) \) (resp. \( (X_1^2 - T, X_2) \)) as in Proposition 1. Note that a point in \( \mathcal{F}_f \) is always simple and that \( \mathcal{F}_H \) may contain non-simple points. We denote the set of non-simple points by \( \mathcal{F} \).

Let \( \mathcal{C} \) be the set of cusp points of \( f|\partial M \). Note that a cusp point is always simple, since \( f|\partial M \) is a stable map. We denote the images of \( \mathcal{F}_f, \mathcal{F}_H, \mathcal{C} \) and \( \mathcal{F} \) by \( \mathcal{q}f \) in \( W_f \) by \( W\mathcal{F}_f, W\mathcal{F}_H, W\mathcal{C} \) and \( W\mathcal{F} \), respectively. Furthermore, we put \( \Sigma = \mathcal{q}(S(f|\partial M)) \). Note that, \( \Sigma = W\mathcal{F}_f \cup W\mathcal{F}_H \cup W\mathcal{C} \). For \( p \in W_f \), we define as follows:

- **\( p \): regular point** \( \Leftrightarrow p \in W_f - \Sigma \),
- **\( p \): fold point of type I** \( \Leftrightarrow p \in W\mathcal{F}_f \),
- **\( p \): fold point of type II** \( \Leftrightarrow p \in W\mathcal{F}_H \),
- **\( p \): cuspidal point** \( \Leftrightarrow p \in W\mathcal{C} \),
- **\( p \): tridental point** \( \Leftrightarrow p \in W\mathcal{F} \).

**Definition 2.** Let \( M \) be a compact orientable 3-manifold with non-empty boundary, and \( f : M \to \mathbb{R}^2 \) a stable map with \( S(f) = \emptyset \). For any \( y \in \mathbb{R}^2 \), an embedding of a closed interval \( \alpha : J \to \mathbb{R}^2 \) is called a transverse arc at \( y \) if \( y \) is in \( \alpha(\text{Int} J) \), \( \alpha \) is transverse to \( f|\partial M \), and \( \alpha(J) \cap f(S(f|\partial M)) = \{ y \} \cap f(S(f|\partial M)) \).

For \( x \in M \), if \( \alpha : J \to \mathbb{R}^2 \) is a transverse arc at \( f(x) \), then the component of \( f^{-1}(\alpha(J)) \) containing \( x \) is called a transverse manifold at \( x \) and is denoted by \( T(x) \).
fibers of $f$ in $T(x)$
described by vertical lines
Let us first consider simple singular points of $f|\partial M$. By using local normal forms obtained in §3 and by repeating Levine’s argument as described in [9, Chapter I], we obtain the following propositions, the proofs of which are easy exercises. In [9], Levine considers compact 3-dimensional manifolds without boundary, while we treat the case with boundary. Thus a main difference from the argument of [9] is the structures of the transverse manifolds. But, we can easily obtain the structures of transverse manifolds based on the local normal forms near singularities of $f|\partial M$ as described in Propositions 1, 2 and 3.

**Proposition 4.** Let $x$ be a simple point in $\mathcal{F}_1$ (or $\mathcal{F}_H$). Then the transverse manifold, $T(x)$, of $f$ at $x$ is as in Figure 1 (i) (resp. Figure 1 (ii)), and the Stein factorization $W_f$ and the map $\tilde{f}$ near $q_f(x)$ are as in Figure 1 (i)$'$ (resp. Figure 1 (ii)$'$).

**Proposition 5.** Let $x$ be a cusp point in $\mathcal{C}$. Then the transverse manifold, $T(x)$, of $f$ at $x$, the Stein factorization $W_f$ and the map $\tilde{f}$ near $q_f(x)$ are as in Figure 2.
Let us now consider a non-simple singular point of $f|\partial M$.

**Proposition 6.** Let $x$ be a non-simple point in $S(f|\partial M)$. Then there exists a neighborhood of $q_f(x)$ in the Stein factorization $W_f$ as in Figure 3.

![Figure 3](image)

**Proof.** Since $f|\partial M$ is stable, $f(S(f|\partial M))$ forms a normal crossing around $f(x)$. Furthermore, non-simple points must belong to $\mathcal{F}_H$. By the condition (I), a component of $f^{-1}(f(x))$ containing $x$ is homeomorphic to a closed interval, and it contains two singular points of $f|\partial M$.

As in Levine [9, p. 15, 1.4] we investigate how the fibers are situated around a non-simple point. Then we see that the connected component of $f^{-1}(U)$ containing $x$ is as in Figure 4, where $U$ is a certain compact neighborhood of $f(x)$ in $\mathbb{R}^2$. Thus, the corresponding Stein factorization is easily seen to be as in Figure 3.

![Figure 4](image)
Summarizing the above results, we obtain the following proposition.

**Proposition 7.** Let $M$ be a compact orientable 3-manifold with non-empty boundary, and let $f : M \to \mathbb{R}^2$ be a stable map with $S(f) = \emptyset$ and the condition (1). For each $x \in M$, there exists a neighborhood of $q_f(x)$ in $W_f$ which is homeomorphic to one of the polyhedrons as in Figure 5. Moreover, $W_f$ is a 2-dimensional polyhedron.

**Remark 1.** Note that $W_f - \Sigma$ has a natural structure of a $C^\infty$-manifold of dimension two which is induced from $\mathbb{R}^2$ by the local homeomorphism $f$, and that $\Sigma - (W^C \cup W^F)$ also has a natural structure of a $C^\infty$-manifold of dimension one.

### 5. Immersion lift from $M$ to $\mathbb{R}^3$

In this section, we prove Theorem 1. We may suppose that $M$ and $\mathbb{R}^2$ are oriented. Then each connected component of fibers of $f$ which is homeo-
morphic to a closed interval has the induced orientation.

We first prove the implication (a) ⇒ (b) in Theorem 1. Since \( f = \pi \circ F \) for an immersion \( F \) and a submersion \( \pi \), we have \( S(f) = \emptyset \). Let \( r \) be a point of \( f(M) \). Then by Propositions 1, 2 and 3, for every \( x \in f^{-1}(r) \), there exists an open neighborhood \( U \) of \( x \) in \( M \) such that \( U \) satisfies one of the following:

1. \( U \cap f^{-1}(r) \approx (-1, 1) \) \( \begin{array}{c} (x \in \text{Int } M \cup \mathcal{F}_U) \end{array} \)
2. \( U \cap f^{-1}(r) \approx [0, 1) \) \( \begin{array}{c} (x \in (\partial M \cap (M \setminus S(f | \partial M))) \cup \emptyset) \end{array} \)
3. \( U \cap f^{-1}(r) \) is a point \( (x \in \mathcal{F}_f) \),

where “≈” denotes a homeomorphism. Thus, \( f^{-1}(r) \) is a disjoint union of 1-dimensional manifolds with or without boundary and discrete points. By the compactness of \( f^{-1}(r) \), its image must be homeomorphic to a finite disjoint union of circles, closed intervals and points. However, since \( f^{-1}(r) \subset \{r\} \times \mathbb{R} \), \( f^{-1}(r) \) cannot contain circles. This implies the condition (I) and hence (b).

The remainder of this section is devoted to the proof of the implication (b) ⇒ (a) in Theorem 1 or its restatement, Proposition 9.

Set \( Y = \{r \in C \mid 0 \leq r \leq 1, 0 = \pi \}, Y_0 = \{r \in C \mid r \neq 0, 0 = \pi \}, Y_1 = \{r \in C \mid r \neq 0, 0 = \pi \} \) and \( Y_2 = \{r \in C \mid r \neq 0, 0 = \pi \} \). Define \( \sigma : [0, 1] \to \mathbb{R} \cdot \) Assume that \( x \in \mathcal{F}_U \). Then, there exist homeomorphisms \( A : q_f(T(x)) \to Y \) and \( \lambda : f(T(x)) \to [0, 1] \) such that \( \sigma \circ A = \lambda \circ \tilde{q}_f(T(x)) \). We say that \( A^{-1}(Y_0) \) is the stem and \( A^{-1}(Y_1) \) and \( A^{-1}(Y_2) \) are the arms of \( q_f(T(x)) \). The transverse manifold \( T(x) \), its image \( q_f(T(x)) \) in \( W_f \) and their images in \( \mathbb{R}^2 \) are described in Figure 6.

The fibers of \( f \) in \( T(x) \) are described by vertical lines with arrows consistent with their orientations. The two arms in \( q_f(T(x)) \) are classified into the upper arm \( \alpha_+ \), and the lower arm \( \alpha_- \) by the images of the upper branch \( \tilde{\alpha}_+ \) and the lower branch \( \tilde{\alpha}_- \) respectively in \( T(x) \). The upper branch \( \tilde{\alpha}_+ \) contains the upper part of the fiber passing through the point \( x \) as in Figure 6, and the lower branch \( \tilde{\alpha}_- \) contains the lower part.

---

Fig. 6
Since $W_f$ is a polyhedron by Proposition 7, we can take sufficiently small regular neighborhoods $N(p)$ of $p \in W' \cup W \bar{\mathcal{T}}$ so that $N(p) \cap N(p') = \emptyset$ if $p \neq p'$, and that $N(p)$ coincides with a component of $\bar{f}^{-1}(D)$ for some $D \subset \mathbb{R}^2$, where $D$ is homeomorphic to $I \times I$, $I = [0,1]$. Moreover, if $c$ is a connected component of $W \bar{\mathcal{T}}_I - \bigcup_p \operatorname{Int} N(p)$ (or $W \bar{\mathcal{T}}_I - \bigcup_p \operatorname{Int} N(p)$), then $c$ has a regular neighborhood $N(c)$ relative boundary in $W_f$ which is homeomorphic to $I \times c$ (or $Y \times c$ resp.). In fact, since $\bar{f}$ is an immersion on $W_f - \Sigma$, a regular neighborhood $N(c)$ is homeomorphic to an $I$-bundle (or $Y$-bundle resp.) over $c$. When $c \subset W \bar{\mathcal{T}}_I - \bigcup_p \operatorname{Int} N(p)$, this $I$-bundle is immersed in $\mathbb{R}^2$ and hence trivial. Furthermore, suppose that $c \subset W \bar{\mathcal{T}}_I - \bigcup_p \operatorname{Int} N(p)$ and $N(c)$ contains a non-trivial $Y$-bundle over a circle $c_1$ in $c$ which exchanges the arms along $c_1$. Then for a section $s$ of the sub $I$-bundle consisting of the stems along $c_1$, $q_f^{-1}(s)$ forms a non-orientable $I$-bundle, i.e., M"obius band. This contradicts the induced orientations of fibers.

We may assume that $N(c) \cap N(c') = \emptyset$ if $c \neq c'$. We may also assume $(\bigcup_p N(p)) \cup (\bigcup_c N(c)) = N(\Sigma)$, the regular neighborhood of $\Sigma$.

**Definition 3.** Let $M$ be a compact orientable 3-manifold with non-empty boundary, and let $f : M \to \mathbb{R}^2$ be a stable map with $S(f) = \emptyset$ and the condition (1). Then a continuous map $g : W_f \to \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ is said to be an immersion lift of $\bar{f}$ to $\mathbb{R}^3$ if $\bar{f} = \pi \circ g$ and the following conditions (1), (2), (3) and (4) are satisfied.

1. $g(W_f - \Sigma)$ is a smooth immersion with normal crossings.
2. $g|\Sigma$ is an injection, and $g|((\Sigma - (W' \cup W \bar{\mathcal{T}}))$ is a smooth embedding.
3. $g|N(\Sigma)$ is an injection, and $g|((N(\Sigma) - \Sigma)$ is a smooth embedding.
4. For each $x \in \mathcal{Q}_H - \mathcal{Q}$, we have $\pi' \circ g(a) > \pi' \circ g(b)$ for any point $a$ of the upper arm and any point $b$ of the lower arm of $q_f(T(x))$, where $\pi' : \mathbb{R}^2 \to \mathbb{R}$ is the projection to the last coordinate.

**Proposition 8.** Let $M$ be a compact orientable 3-manifold with non-empty boundary, and let $f : M \to \mathbb{R}^2$ be a stable map with $S(f) = \emptyset$ and the condition (1). Then there exists an immersion lift $g : W_f \to \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ of the form $g(x) = (\bar{f}(x), h_0(x))$.

**Proof.** Let $p$ be a point of $W' \cup W \bar{\mathcal{T}}$. Then we define $g|((N(p) \cap \Sigma) : N(p) \cap \Sigma \to \mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ by $g|((N(p) \cap \Sigma) = \bar{f}|(N(p) \cap \Sigma)$. Then $g|((N(p) \cap \Sigma)$ is injective. Moreover, $g$ can be extended all over $\Sigma$ by separating normal crossing points of $\bar{f}|((\Sigma - (W' \cup W \bar{\mathcal{T}}))$ into extra dimension. Thus we can define $g|\Sigma$ so that $g|\Sigma$ satisfies the above condition (2).

Let us extend $g$ over $N(\Sigma)$. First, we lift the neighborhoods $N(p)$, $p \in W' \cup W \bar{\mathcal{T}}$, to $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ so that $g|N(p)$ satisfies the condition (4), and so that the angle between the images of two arms contained in $N(p) - \operatorname{Int} N(p)$
is $\delta \ (0 < \delta < \pi)$ and that the image of each stem contained in $N(p) - \text{Int} N(p)$ is horizontal. To extend $g$ all over $N(\Sigma)$, let $\mathcal{S}$ be the set of the connected components of $\Sigma - \bigcup_p \text{Int} N(p)$, $p \in W \cup W'$. We consider lifts on each $N(c)$, $c \in \mathcal{S}$. Let $\Pi : N(c) \to c$ be the natural bundle projection whose fibers are homeomorphic to $I = [0,1]$ if $c \in W', I$ or to $Y$ if $c \in W''.$

First, for $c \in W'$, define $g : N(c) \to \mathbb{R}^3$ by $x \mapsto (\tilde{f}(x), h_0(\Pi(x)))$, where $h_0 : c \to \mathbb{R}$ is the smooth function which gives the third coordinate. Second, for $c \in W'', N(c)$ is homeomorphic to $Y \times c$. Then define $g : N(c) \to \mathbb{R}^3$ by $x \mapsto (\tilde{f}(x), h_0(\Pi(x)) + Z(x))$, where $h_0 : c \to \mathbb{R}$ is the smooth function which gives the third coordinate and $Z : N(c) \to \mathbb{R}$ is defined as follows: if $x$ belongs to a stem, then we define $Z(x) = 0$, and if $x$ belongs to an upper (resp. lower) arm, then we define $Z(x) = \| \tilde{f}(x) - \tilde{f}(\Pi(x)) \| \tan \delta / 2$ (resp. $-\| \tilde{f}(x) - \tilde{f}(\Pi(x)) \| \tan \delta / 2$). Here note that our construction of the lifts on $N(p)$ and on $N(c)$ are consistent, and then we may assume that $g|N(\Sigma)$ is an injection and that $g(N(\Sigma) - \Sigma)$ is a smooth embedding by choosing a sufficiently small $\delta$. Thus a lift on $N(\Sigma)$ which satisfies the conditions (3) and (4) has been constructed.

Finally, we can extend the lift to whole $W_f$ by using an argument similar to that of [7, pp. 26–27] and complete the proof. \hfill \Box

**Proposition 9.** Let $M$ be a compact orientable 3-manifold with non-empty boundary, and $f : M \to \mathbb{R}^2$ a stable map with $S(f) = \emptyset$ and the condition (1). Then there exists an immersion $F : M \to \mathbb{R}^3$ which makes the following diagram commutative.

\[
\begin{array}{ccc}
\mathbb{R}^3 & \xrightarrow{\pi} & \mathbb{R}^2 \\
\downarrow{g_f} & & \downarrow{p} \\
M & \xrightarrow{f} & \mathbb{R}^2
\end{array}
\]

**Proof.** We use the same notations as in the proof of Proposition 8, and construct an immersion lift $F : M \to \mathbb{R}^3$ based on $g : W_f \to \mathbb{R}^3$.

First, let us construct a lift on $q_f^{-1}(N(\Sigma))$ to $\mathbb{R}^3$. We lift $q_f^{-1}(N(p)) \times (p \in W \cup W')$ as the top figure in Figure 2 and Figure 4, and then we lift the other part of $q_f^{-1}(N(\Sigma))$ as the top figures in (i)', (ii)' of Figure 1 so that $F|q_f^{-1}(N(\Sigma))$ is expressed by $x \mapsto g(q_f(x)) + (0,0,h_0(x))$, where $h_0 : q_f^{-1}(N(\Sigma)) \to \mathbb{R}$ is an orientation preserving embedding on each $q_f$-fiber. In the construction, we can arrange so that the orientation of the $F$-image of each oriented fiber of $q_f$ contained in $\{r\} \times \mathbb{R}$ ($r \in \mathbb{R}^2$) coincides with that of the last coordinate of $\mathbb{R}^3$. By (3) of Definition 3, we can construct the lift $F|q_f^{-1}(N(\Sigma))$ as an embedding.

Similarly, for $q_f^{-1}(W_f - N(\Sigma))$, we can construct a smooth function
$h_1 : q_f^{-1}(W_f - N(\Sigma)) \to \mathbb{R}$, where $h_1 = h_0$ on $q_f^{-1}(W_f - N(\Sigma)) \cap q_f^{-1}(N(\Sigma))$, and define $F|q_f^{-1}(W_f - N(\Sigma))$ by $x \mapsto q(x) + (0, 0, h_1(x))$ so that the restriction of $h_1$ to each $q_f$-fiber (which is homeomorphic to a closed interval by the condition (I)) is an orientation preserving embedding, and that $F|q_f^{-1}(W_f - N(\Sigma))$ is an immersion. This completes the proof of Proposition 9.

Now we have completed the proof of Theorem 1 by proving (b) ⇒ (a) by Proposition 9 and (a) ⇒ (b) at the beginning of this section. We give some remarks before closing the section.

**Remark 2.** The condition $S(f) = \emptyset$ does not imply the condition (I) in Theorem 1 as follows. Let $N$ be an annulus, and consider $M = N \times S^1$. Let $\rho : N \to \mathbb{R}$ be a height function as in Figure 7 such that $\rho$ is non-singular, while $\rho|\partial M$ is a Morse function with exactly four critical points, and that $\rho$ contains a fiber homeomorphic to $S^1$. Then define $\rho \times \text{id} : N \times S^1 \to \mathbb{R} \times S^1$ by $(x, t) \mapsto (\rho(x), t)$. Finally, consider an embedding $\eta : \mathbb{R} \times S^1 \to \mathbb{R}^2$ and we define $f = \eta \circ (\rho \times S^1) : M \to \mathbb{R}^2$. This $f$ is stable, $S(f) = \emptyset$, and we can find a point $r \in \mathbb{R}^2$ such that $f^{-1}(r)$ is homeomorphic to $S^1$.

![Fig. 7](image_url)

However, the condition (I) does imply $S(f) = \emptyset$ under the condition that $S(f) \cap \partial M = \emptyset$. To show this, suppose $S(f) \neq \emptyset$. Then there exists a definite fold or an indefinite fold point as a singularity of $f$. If $M$ contains a definite fold point $p \in \text{Int } M$, then there must exist a fiber near $p$ which contains a connected component homeomorphic to $S^1$. If $M$ contains an indefinite fold point $p' \in \text{Int } M$, then the connected component of the fiber containing $p'$ cannot be diffeomorphic to a closed interval or a point. Hence, if $S(f) \neq \emptyset$, then $f$ does not satisfy the condition (I). Thus the condition (I) implies $S(f) = \emptyset$, provided that $S(f) \cap \partial M = \emptyset$. 

...
3. Haefliger [5, Théorème 1] showed that for a stable map from a closed 2-manifold $N$ into $\mathbb{R}^2$, there exists an immersion lift to $\mathbb{R}^3$ with respect to the standard projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ if and only if each connected component of its singular set has an orientable (or non-orientable) neighborhood if the number of cusps on the connected component is even (resp. odd).

Let $F$ be an immersion lift of a stable map $f: M \to \mathbb{R}^2$ as in Theorem 1. Then the stable map $f|\partial M: \partial M \to \mathbb{R}^2$ is also lifted to $\mathbb{R}^3$ by $F|\partial M$. Then, by Haefliger [5], each connected component of $S(f|\partial M)$ must have an even number of cusps, since $\partial M$ is an orientable closed surface.

In fact, cusps of $f|\partial M$ correspond exactly to cuspidal points of $W_f$ by $q_f$. From the structure of $W_f$ obtained in Proposition 7, the connected components of $\mathcal{F}_I$ and those of $\mathcal{F}_{II}$ must connect one after the other alternately at cusp points of $f|\partial M$ as their connecting points, and all of them must form circles. Hence, the number of cusps on each circle is even. Therefore, the stable map $f|\partial M$ automatically satisfies the condition of Haefliger.

4. Kushner-Levine-Porto [7] have given a sufficient condition for the existence of an immersion lift to $\mathbb{R}^4$ with respect to the projection $p: \mathbb{R}^4 \to \mathbb{R}^2, (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2)$, for a stable map from a closed orientable 3-manifold to $\mathbb{R}^2$. Of course, there is no immersion lift to $\mathbb{R}^3$ for a closed 3-manifold.

6. Embedding lift from $M$ to $\mathbb{R}^n$

In §5, we considered the existence problem of an immersion lift $F$ to $\mathbb{R}^3$ for a stable map from $M$ into $\mathbb{R}^2$. We will consider the embedding lift to $\mathbb{R}^n$, $n = 3, 4$ and $n \geq 5$.

5. There is a stable map $f$ which satisfies the condition (b) in Theorem 1 but has no embedding lifts to $\mathbb{R}^3$.

We take the compact orientable 3-manifold with boundary $S^2 \times S^1 - \text{Int} D^3$ for $M$. No stable map from $M$ into $\mathbb{R}^2$ can have an embedding lift $F$ to $\mathbb{R}^3$. In fact, if $M$ is embedded into $\mathbb{R}^3$, then $\partial M = S^2$ bounds an embedded 3-ball in $\mathbb{R}^3$ by the theorem of Schönhflies. This means that $M$ itself is homeomorphic to $D^3$; a contradiction. We identify $M = S^2 \times S^1 - \text{Int} D^3$ with $D^2 \times I \cup_{\varphi} S^2 \times I$ and give an immersion $i: M \to \mathbb{R}^3$ as in Figure 8, where $\varphi: D^2 \times \partial I \to S^2 \times \partial I$ is a handle attaching map. We can see that the map $f = \pi \circ i$ is stable by Lemma 2. Moreover, $S(f) = \emptyset$ and $f$ satisfies the condition (1).

In this example, two cusps appear around each component of $\varphi(D^2 \times \partial I)$. The upper and lower arms in $q_f(T(x)) \subset W_f$ at the fold points $x \in \partial M$ of type $\mathcal{F}_{II}$ are drawn in the figure so as to satisfy the condition (4) of Definition...
3. We understand that it is difficult to modify the immersion lift of \( f \) to an embedding keeping this condition.

\[ \text{Remark 6. There is a stable map } f \text{ which satisfies the condition (b) in Theorem 1 but has no embedding lifts to } \mathbb{R}^4. \]

Let \( M \) be a punctured lens space \( L(2n,q)^\circ \). It is a compact orientable 3-manifold with boundary \( S^2 \). Then we can construct a stable map \( f : M \to \mathbb{R}^2 \) with \( S(f) = \emptyset \) and our condition (1) by Lemma 2. However, it has been shown in [3] that a punctured lens space \( L(2n,q)^\circ \) cannot be embedded in \( \mathbb{R}^4 \). Hence \( f \) cannot have an embedding lift to \( \mathbb{R}^4 \).

**Definition 4.** Let \( M \) be a compact orientable 3-manifold with non-empty boundary, and let \( f : M \to \mathbb{R}^2 \) be a stable map with \( S(f) = \emptyset \) and the condition (1). Then, a continuous map \( g_c : W_f \to \mathbb{R}^n \) is said to be an embedding lift of \( \tilde{f} \) to \( \mathbb{R}^n \) if \( g_c \) satisfies \( \tilde{f} = \pi \circ g_c \) with respect to the projection \( \pi : \mathbb{R}^n \to \mathbb{R}^2 \), \((x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2)\), and the following.

1. \( g_c \) is a topological embedding.
2. \( g_c | (W_f - \Sigma) \) is a smooth embedding.
3. \( g_c | (\Sigma - (W\mathcal{F} \cup W\mathcal{S})) \) is a smooth embedding.
4. \( g_c | N(\Sigma) \subset \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^n \), and \( g_c | N(\Sigma) \) satisfies the condition (4) of
Definition 3 as a map into $\mathbb{R}^3$.

Remark 7. In the example given in Remark 5 (see Figure 8), we can see that $\tilde{f}$ has a lift to $\mathbb{R}^3$ which is a topological embedding. But we have no embedding lift of $f$ as defined in Definition 4, because it contradicts the following proposition.

Proposition 10. Let $M$ be a compact orientable 3-manifold with non-empty boundary, and let $f : M \to \mathbb{R}^2$ be a stable map with $S(f) = 0$ and the condition (I). If there exists an embedding lift $g_e : W_f \to \mathbb{R}^n$ of $\tilde{f}$ with respect to $\pi : \mathbb{R}^n \to \mathbb{R}^2$, $(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2)$, then there exists an embedding lift $F_e : M \to \mathbb{R}^n$ of $f$. In particular, for $n \geq 5$, there always exists an embedding lift $F_e$ of $f$.

Proof. By virtue of the condition (4) of Definition 4, we can construct an embedding lift on $q_f^{-1}(N(\Sigma))$ so that $F_e(q_f^{-1}(N(\Sigma))) \subset \mathbb{R}^3 \times \{0\}$ by using an argument similar to that in the proof of Proposition 9.

Then, we construct the lift on $q_f^{-1}(W_f - N(\Sigma))$ as follows. By the construction of $F_e(q_f^{-1}(N(\Sigma)))$, we have $F_e(q_f^{-1}(p)) \subset \tilde{f}(p) \times \mathbb{R} \times \{0\} \subset \mathbb{R}^3 \{0\} \subset \mathbb{R}^n$ for any $p \in N(\Sigma)$. Hence, we can construct $F_e$ on $q_f^{-1}(W_f - N(\Sigma))$ by $x \mapsto g_e(q_f(x)) + (0, 0, h_0(x), 0, \ldots, 0)$, where $h_0$ is an orientation preserving embedding on each $q_f$-fiber. Since $g_e(q_f(W_f - N(\Sigma)))$ is a smooth embedding, we can arrange so that $F_e(x) \neq F_e(x')$ if $q_f(x) \neq q_f(x')$. Thus an embedding lift $F_e$ of $f$ has been constructed.

The existence of an immersion lift $g : W_f \to \mathbb{R}^3$ is guaranteed by our Proposition 8. In general, the lift $g(W_f - N(\Sigma))$ has normal crossings. However, if $n \geq 5$, then we can separate the normal crossings into extra dimensions in $\mathbb{R}^n$ by Thom’s transversality theorem so that $g$ satisfies $\pi \circ g = \tilde{f}$. Therefore, for $n \geq 5$, we can always construct an embedding lift from $W_f$ to $\mathbb{R}^n$ and hence from $M$ to $\mathbb{R}^n$. This completes the proof.

7. Applications

In this section, first we prove Theorem 2 as an application of the results obtained in §4. For a closed orientable 3-manifold $M$, Burlet-de Rham [1] have proved that there exists a special generic map $f : M \to \mathbb{R}^2$ if and only if $M$ is diffeomorphic to $S^3$ or to a connected sum $\sharp^k(S^2 \times S^1)$, where a special generic map is a stable map which has only definite fold points as its singularities. Saeki [12] has obtained a characterization of graph manifolds by using simple stable maps (defined in [12]), where a graph manifold is defined to be a 3-manifold built up of $S^1$-bundles over surfaces attached along their torus boundaries. As an analogy, we consider the structure of source manifolds of
the boundary special generic maps defined as follows.

**Definition 5.** Let $M$ be a compact orientable 3-manifold with non-empty boundary, and $f : M \to \mathbb{R}^2$ a stable map with $S(f) = \emptyset$. Then $f$ is called a boundary special generic map if $S(f|\partial M) = \mathcal{F}_I$.

**Lemma 3.** Let $M$ be a compact orientable 3-manifold with non-empty boundary. Then any boundary special generic map $f : M \to \mathbb{R}^2$ satisfies the condition (1).

**Proof.** Let $r$ be a point in $f(M)$ and $r'$ a point such that $r' \not\in f(M)$. Consider a smooth embedding $C : [0,1] \to \mathbb{R}^2$ such that $C(0) = r$, $C(1) = r$ and $C$ is transverse to $f|\partial M$. Then $f|f^{-1}(C([0,1])) : f^{-1}(C([0,1])) \to C([0,1])$ is a non-singular function on a surface with boundary, and each singularity of $f|\partial M$ in $f^{-1}(C([0,1]))$ belongs to $\mathcal{F}_I$ so that only arcs appear or disappear in the inverse image. Set

$$A = \{ t \in [0,1] \mid f^{-1}(C(t)) \neq S^1 \}.$$  

Then we have (1) $A \ni 0$, in particular, $A \neq \emptyset$, (2) $A$ is open, and (3) the complement of $A$ is open. Since $[0,1]$ is connected, we see $A = [0,1]$. Hence $f^{-1}(r)$ does not contain a circle component. Then the result follows as in the proof of (a) ⇒ (b) in Theorem 1 given at the beginning of § 5.

**Proof of Theorem 2.** Suppose that $M$ is a compact orientable 3-dimensional handlebody. Then, we can construct a boundary special generic map $f$ from $M$ into $\mathbb{R}^2$ as in Figure 9, where $i$ is an embedding so that $\pi \circ i$ has only singularities of type $\mathcal{F}_I$ at $\partial M$.

![Fig. 9](image-url)
Conversely, suppose that $f: M \to \mathbb{R}^2$ is a boundary special generic map. Then $W_f$ must be a connected surface with non-empty boundary by Lemma 3 and Propositions 4 and 7. Since $M$ is compact, so is $W_f$. By the smooth structure of $W_f - N(\Sigma)$ defined in Remark 1, the continuous map $q_f^{-1}(W_f - N(\Sigma))$ is a differentiable map, and moreover a submersion. Here, note that rank $\mathbf{d}(f|_{\partial M})_x = \dim \mathbb{R}^2$ for all $x \in \partial M \cap q_f^{-1}(W_f - N(\Sigma))$. So, by applying Lemma 3 and Ehresmann's fibration theorem ([2] and [8, p. 23]), $q_f^{-1}(W_f - N(\Sigma))$ has a structure of an I-bundle over $W_f - N(\Sigma)$.

Let us prove Theorem 3 as an application of the arguments in §5 and 6.

**Proof of Theorem 3.** If there exists an embedding lift $g_e: W_f \to \mathbb{R}^3$, then there also exists an embedding lift $F_e: M - \text{Int} D^3 \to \mathbb{R}^3$ by Proposition 10. Since $\partial (M - \text{Int} D^3) = S^2$, $S^2$ is embedded in $\mathbb{R}^3$ by $F_e$. By the theorem of Schönflies, $S^2 = \partial (M - \text{Int} D^3)$ bounds a 3-ball in $\mathbb{R}^3$; i.e., $M - \text{Int} D^3$ must be homeomorphic to $D^3$. Hence $\hat{M} = (M - \text{Int} D^3) \cup D^3 \approx D^3 \cup D^3 \approx S^3$, where each "≈" denotes a homeomorphism. This completes the proof. □

**References**

Non-singular stable maps


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