HIROSHIMA MATH. J.
30 (2000), 537–541

On orbit closure decompositions of tiling spaces by the generalized projection method

Kazushi KOMATSU

(Received April 3, 2000)
(Revised May 25, 2000)

ABSTRACT. Let $\mathcal{T}(E)$ be the tiling space on a p-dimensional subspace E of $\mathbf{R}^d$ with a fixed lattice L by the generalized projection method. By using the dual of the lattice L we will construst explicitly a parameter family of the orbit closure decomposition of $\mathcal{T}(E)$ and characterize its dimension. As its application we obtain that the parameters of the orbit closure decomposition of $\mathcal{T}(E)$ correspond to the periods of $\mathcal{T}(E^\perp)$, provided that $\mathcal{T}(E)$ and $\mathcal{T}(E^\perp)$ are given by the generalized projection method from an integral lattice L.

1. Introduction

In 1981 de Bruijn introduced the multigrid and projection methods to construct aperiodic tilings such as Penrose tilings. The multigrid method was generalized by Kramer and Neri (1984). The projection method was generalized to a higher dimensional lattice $\mathbf{Z}^d$ by Duneau and Katz (1985). Gähler and Rhyner (1986) extended the projection method to general lattices and showed that these generalized multigrid and projection methods are equivalent.

First, we recall the definitions of tilings and tiling spaces by the generalized projection method (cf. [3], [4]). Let L be a lattice in $\mathbf{R}^d$ with a basis $\{b_i | i = 1, 2, \dots, d\}$. Let E be a p-dimensional subspace of $\mathbf{R}^d$, and $E^\perp$ be its orthogonal complement with respect to the standard innner product. Let $\pi : \mathbf{R}^d \to E$ and $\pi^\perp : \mathbf{R}^d \to E^\perp$ be the orthogonal projections. We put $A = \{\sum_{i=1}^d r_i b_i | 0 \le r_i \le 1\}$. For any $x \in E^\perp$ we put $x + \pi^\perp(A) = \{x + u | u \in \pi^\perp(A)\}$, which is a compact set with a nonempty interior. For p vectors $b_{k_i} \in \{b_i\}$ such that $\{\pi(b_{k_i})\}$ is linearly independent we put $T(k_1, k_2, \dots, k_p) = \{\sum_{i=1}^p r_i b_{k_i} | 0 \le r_i \le 1\}$. Taking such b_{k_i} we define $F_p(L) = \{v + T(k_1, k_2, \dots, k_p) \mid v \in L, k_i$ such that $\{\pi(b_{k_i})\}$ is linearly independent$\}$ and $T(x) = \{\pi(S) | S \subset (x + \pi^\perp(A)) \times E, S \in F_p(L)\}$. Note that $T(k_1, k_2, \dots, k_p)$ is a p-dimensional parallelotope. $T(x)$ is a tiling on a p-dimensional subspace E of $\mathbf{R}^d$ by the

2000 *Mathematics Subject Classification*. 05B45, 37B50, 52C20, 52C22, 52C23
Key words and phrases. tiling, tiling space, generalized projection method, orbit closure, period

generalized projection method. A tiling space $\mathscr{T}(E) = \{u + T(x) | u \in E, x \in E^{\perp}\}$ is defined by a space of tilings consisting of all translates by $E = \mathbf{R}^p$ of the tilings $T(x)$ for all $x \in E^{\perp}$. In the case $L = \mathbf{Z}^p$ the tiling space $\mathscr{T}(E)$ is defined by any of three equivalent methods, the multigrid method, the projection method, or the oblique tiling method of Oguey, Duneau, and Katz (1988). Tiling spaces are topological dynamical systems with continuous $\mathbf{R}^p$ translation action and with topology defined by a tiling metric on tilings of $\mathbf{R}^p$ (see for example [12]).

In order to state theorems we remind of several definitions. The dual lattice L^* is defined by the set of vectors $y \in \mathbf{R}^d$ such that $\langle y, x\rangle \in \mathbf{Z}$ for all $x \in L$, where $\langle\, ,\rangle$ denotes standard inner product. A lattice L is called integral if $\langle x, y\rangle \in \mathbf{Z}$ for all $x, y \in L$. The standard lattice is both integral and self dual.

Let $\mathrm{Orb}(T(x))$ denote the orbit of $T(x)$ in $\mathscr{T}(E)$ by the $\mathbf{R}^p$ translation action and $\mathrm{span}(A)$ denote the $\mathbf{R}$-linear span of a set A.

The purpose of this paper is to show the following theorem:

THEOREM 1. *Let $\mathscr{T}(E)$ be the tiling space on a p-dimensional subspace E of $\mathbf{R}^d$ by the generalized projection method and $p' : E^{\perp} \to \mathrm{span}(L^* \cap E^{\perp})$ be the orthogonal projection. Define $p : L \to \mathrm{span}(L^* \cap E^{\perp})$ by $p = p' \circ (\pi^{\perp}|L)$. We take a basis $x_1, \dots, x_k$ of any direct summand K such that $L = p^{-1}(\{0\}) \oplus K$. Then $\mathscr{T}(E)$ decomposes into a k parameter family of orbit closures $\overline{\mathrm{Orb}(T(t_1x_1 + \cdots + t_kx_k))}$ for $t_1, \dots, t_k \in \mathbf{R}$.*

In particular, we obtain that k is equal to $\mathrm{rank}(L^* \cap E^{\perp})$.

Note that any two orbit closures are either coincident or disjoint. A. Hof (1988) proved that $E^{\perp} \cap L^* = \{0\}$ if and only if $\mathscr{T}(E) = \overline{\mathrm{Orb}(T(0))}$.

Assume that L is integral. Then we see that $\mathrm{rank}(L^* \cap E^{\perp}) = \mathrm{rank}(L \cap E^{\perp})$ because $L \subset L^*$ and L^*/L is finite. In [7] we proved that the number of independent periods of the tiling space $\mathscr{T}(E^{\perp})$ is equal to $\mathrm{rank}(L \cap E^{\perp})$. By Theorem 1 we immediately obtain the following theorem in the case that L is integral:

THEOREM 2. *Let $\mathscr{T}(E)$ (resp. $\mathscr{T}(E^{\perp})$) be the tiling space on a p-dimensional subspace E (resp. $(d-p)$-dimensional subspace $E^{\perp}$) of $\mathbf{R}^d$ by the generalized projection method and assume that L is an integral lattice. Then $\mathscr{T}(E)$ decomposes into a k parameter family of orbit closures, where k is equal to the number of independent periods of the tiling space $\mathscr{T}(E^{\perp})$.*

According to the unpublished thesis of C. Hillman (1988) the tiling space $\mathscr{T}(E)$ decomposes into a k parameter family of orbit closures, where k is equal to the number of independent periods of the tiling space $\mathscr{T}(E^{\perp})$ in the case that $L = \mathbf{Z}^d$.

2. Proof of Theorem 1.

We define $F : \mathbf{R}^d/L \to \mathscr{T}(E)$ by $F([x]) = T(x)$ for $x \in \mathbf{R}^d$. Note that F is well-defined by the construction of $T(x)$. Since $T(x+u) = u + T(x)$, we have $F([x+u]) = u + T(x)$ for any $x \in \mathbf{R}^d$ and $u \in E = \mathbf{R}^p$, where the tiling $u + T(x)$ is translation of the tiling $T(x)$ by $u \in E = \mathbf{R}^p$. F is called a factor mapping (see for example [12]). Tiling spaces are compact due to [11, p. 357, Lemma 2]. So F satisfies that $F(\bar{A}) = \overline{F(A)}$ for any subset $A \subset \mathbf{R}^d/L$. Thus a parameter family of the orbit closure decomposition of $\mathscr{T}(E)$ is obtained from a parameter family of the orbit closure decomposition of $\mathbf{R}^d/L$ by F.

By a similar argument to [13, p. 52, Theorem 2.3] we can construct linear subspaces $V, W \subset E^\perp$ such that $E^\perp = V \oplus W$, $\pi^\perp(L) \cap V$ is dense in V, $\pi^\perp(L) \cap W$ is a lattice in W and $\pi^\perp(L) = \pi^\perp(L) \cap V + \pi^\perp(L) \cap W$. In particular, V is given by $V = \bigcap_{r>0} \mathrm{span}(U_r(0) \cap \pi^\perp(L))$, where $\mathrm{span}(A)$ denotes the $\mathbf{R}$-linear span of a set A and $U_r(v)$ denotes the open ball of radius r and center v. Let $\mathrm{Orb}([x])$ denote the orbit of $[x]$ in $\mathbf{R}^d/L$ by the $\mathbf{R}^p$ translation action. We take a basis $x_1, \dots, x_k$ of $\pi^\perp(L) \cap W$. We have that the orbit closures of $\mathbf{R}^d/L$ are the cosets of the closed connected subgroup $\overline{\mathrm{Orb}([0])}$. By the properties of V and W we get that $\mathbf{R}^d/L$ decomposes into a k parameter family of the orbit closures $\overline{\mathrm{Orb}([t_1x_1 + \cdots + t_kx_k])}$ for $t_1, \dots, t_k \in \mathbf{R}$. Therefore $\mathscr{T}(E)$ also decomposes into a k parameter family of orbit closures $\overline{\mathrm{Orb}(T(t_1x_1 + \cdots + t_kx_k))}$ for $t_1, \dots, t_k \in \mathbf{R}$.

We can easily see that $k = \mathrm{rank}(L^* \cap E^\perp)$. In fact, we take a basis $c_1, \dots, c_k \in \pi^\perp(L) \cap W$ and a basis $c_{k+1}, \dots, c_{d-p} \in V$. By the definition of L^* we get that $\langle x, y \rangle \in \mathbf{Z}$ for any $x \in \pi^\perp(L)$ if and only if $y \in L^* \cap E^\perp = L^* \cap (V^\perp \cap E^\perp)$ for $y \in E^\perp$. Since $\pi^\perp(L) \cap V$ is dense in V, we obtain that $L^* \cap E^\perp$ is orthogonal to V. Hence, $\mathrm{rank}(L^* \cap (V^\perp \cap E^\perp)) \le \dim V^\perp = k$. For any i with $1 \le i \le k$ we can take $a_i \in E^\perp$ such that a_i is orthogonal to $\{c_j \mid 1 \le j \le d-p,\ j \ne i\}$. Since $\left\langle x, \frac{1}{\langle a_i, c_i \rangle} a_i \right\rangle \in \mathbf{Z}$ for any $x \in \pi^\perp(L)$, we see that $\frac{1}{\langle a_i, c_i \rangle} a_i \in E^\perp \cap L^*$. Now the $\frac{1}{\langle a_i, c_i \rangle} a_i$ are linearly independent in $E^\perp \cap L^*$. Therefore we get $k = \mathrm{rank}(L^* \cap E^\perp)$.

We will construct explicitly the linear subspaces $V, W \subset E^\perp$ mentioned above from the dual of the lattice L. Let $p' : E^\perp \to \mathrm{span}(L^* \cap E^\perp)$ be the orthogonal projection and define $p : L \to \mathrm{span}(L^* \cap E^\perp)$ by $p = p' \circ (\pi^\perp | L)$. We take a direct summand K such that $L = p^{-1}(\{0\}) \oplus K$. By the definition of p we have that $\mathrm{span}(\pi^\perp(p^{-1}(\{0\})))$ is orthogonal to $\mathrm{span}(L^* \cap E^\perp)$. Because $V \subset \mathrm{span}(\pi^\perp(p^{-1}(\{0\})))$ and $k = \mathrm{rank}(L^* \cap E^\perp)$, we see that $\mathrm{span}(\pi^\perp(p^{-1}(\{0\})))$ is equal to V given above. Then $\pi^\perp(L) \cap V$ is dense in V. We put $W = \mathrm{span}(\pi^\perp(K))$.

First, we will show that $p(K)$ is discrete in $\mathrm{span}(L^* \cap E^\perp)$, which is enough to prove that $\pi^\perp(L) \cap W$ is a lattice in W. Suppose that $\{u_i\}$ is a sequence of elements of K such that $\{p(u_i)\}$ converges in $L^* \cap E^\perp$. We may assume that $\{p(u_i)\}$ converges to 0. Write $\pi^\perp(u_i) = v_i + w_i$ with $v_i \in V$, $w_i \in W$ for each i. By the hypothesis we see that $\{w_i\}$ converges to 0. Since $\pi^\perp(K)$ is dense in V we can choose $y_i \in K$ such that $|w_i - \pi^\perp(y_i)| < 1/(2i)$. Then $|\pi^\perp(u_i - y_i)| = |\pi^\perp(u_i) - \pi^\perp(y_i)| = |v_i + w_i - \pi^\perp(y_i)| \le |v_i| + |w_i - \pi^\perp(y_i)|$. So $\pi^\perp(u_i - y_i)$ converges to 0. Thus for sufficiently large i, we have $\pi^\perp(u_i - y_i) \in V$ and $u_i - y_i \in p^{-1}(\{0\})$. Then $u_i \in p^{-1}(\{0\}) \cap K = 0$ and $p(u_i) = 0$. This implies that $p(K)$ is discrete in $\mathrm{span}(L^* \cap E^\perp)$.

The rest of the proof is devoted to show that the two subspaces V, $W \subset E^\perp$ satisfy the following properties: $E^\perp = V \oplus W$, $\pi^\perp(L) \cap W$ is a lattice in W and $\pi^\perp(L) = \pi^\perp(L) \cap V + \pi^\perp(L) \cap W$. Since $p(K)$ spans $\mathrm{span}(L^* \cap E^\perp)$ and $p(K)$ is discrete, $p(K)$ is a lattice of $\mathrm{rank}\, p(K) = \dim E^\perp - \dim V$. Because the restriction $p|K$ is injective, we obtain $k = \mathrm{rank}\, K = \mathrm{rank}\, p(K) = (d-p) - \dim V$. Since $\dim W \le k = (d-p) - \dim V$ and $E^\perp = W + V$ we have $\dim W = (d-p) - \dim V$ and $E^\perp = W \oplus V$. Since $p(K)$ is discrete and p' is continuous, $\pi^\perp(K)$ is discrete. Because $\pi^\perp(K)$ has the same rank as K, we can see that K is isomorphic to $\pi^\perp(K)$ and $\pi^\perp(K)$ is a lattice in W. Since $\pi^\perp(L) = \pi^\perp(K) + \pi^\perp(p^{-1}(\{0\}))$, we obtain $\pi^\perp(K) = \pi^\perp(L) \cap W$ and $\pi^\perp(p^{-1}(\{0\})) = \pi^\perp(L) \cap V$. q.e.d.

References

[1] N. G. de Bruijn, Algebraic theory of Penrose's non-periodic tilings of the plane. I, Kon. Nederl. Akad. Wetesch. Proc Ser. A. (= Indag. Math.) (1981), 39–52.

[2] N. G. de Bruijn, Algebraic theory of Penrose's non-periodic tilings of the plane. II, Kon. Nederl. Akad. Wetesch. Proc Ser. A. (= Indag. Math.) (1981), 53–66.

[3] M. Duneau and A. Katz, Quasiperiodic patterns, Phys Rev. Lett. **54** (1985), 2688–91.

[4] F. Gähler and J. Rhyner, Equivalence of the generalized grid and projection methods for the construction of quasiperiodic tilings, J. Phys. A: Math. Gen. **19** (1986), 267–277.

[5] C. Hillman, Sturmian dynamical systems, PhD. thesis, University of Washington, Seattle, 1998.

[6] A. Hof, Uniform distribution and the projection method, in Quasicrystals and Discrete geometry, edited by Patera J., Fields Institute Monographs vol. 10, 1988, Amer. Math. Soc., 201–206.

[7] K. Komatsu, Aperiodicity criterion for the tiling spaces given by the generalized projection method, Preprint.

[8] K. Komatsu and K. Nakahira, Some remarks on tilings by the projection method, Mem. Fac. Sci. Kochi Univ. (Math.) **18** (1997), 117–122.

[9] P. Kramer and R. Neri, On periodic and Non-periodic Space fillings of E^m obtained by projection, Acta Cryst. **A40** (1984), 580–587.

[10] C. Oguey, M. Duneau and A. Katz, A geometrical approach of quasiperiodic tilings, Comm. Math. Phys. **118** (1988), 99–118.

[11] C. Radin and M. Wolff, Space tilings and local isomorphism, Geometriae Dedicata **42** (1992), 355–360.

[12] E. A. Robinson, The dynamical theory of tilings and quasicrystallography, in Ergodic theory of $\mathbf{Z}^d$-actions, London Math. Soc. Lecture Note Ser. 228, Cambridge University Press, 1996, 451–473.

[13] M. Senechal, "Quasicrystals and Geometry", Cambridge University Press, 1995.

Department of Mathematics,
Faculty of Science,
Kochi University,
Kochi 780-8520, Japan